L-Groups of Sheaves on Stratified Spaces

Markus Zetto

UHH Hamburg

Master's Thesis under the supervision of
Markus Banagl
at Heidelberg University

September 19, 2023

Outline

What are Higher Sheaves?

Ordinary Sheaves

Definition

Given a topological space X and a commutative ring R, a *sheaf of R-modules* on X is a functor $F : \mathrm{Open}(X)^{op} \to \mathrm{Mod}_R$ such that for $\mathfrak{U} = (U_i \subseteq U)_{i \in I}$ an open cover of $U \underset{\text{open}}{\subseteq} X$, the canonical map

$$F(U) \to \mathrm{equ}\left(\prod_i F(U_i) \rightrightarrows \prod_{ij} F(U_i \cap U_j)\right)$$

is an isomorphism. Denote their category by $\mathrm{Sh}(X; R)$.

Remark

Surjectivity of this map allows for gluing, while injectivity ensures uniqueness.

Čech Cohomology

This is equivalent to $F(U) = \check{H}^0(\mathfrak{U}, F)$, where *Čech cohomology* is defined as the cohomology of the *Čech complex*

$$\check{C}(\mathfrak{U}, F) := \left(0 \to \prod_i F(U_i) \xrightarrow{d_0 - d_1} \prod_{ij} F(U_{ij}) \to \dots\right) =$$

$$= \left(\prod_{(i_1,\dots,i_n)\in I^{\times n}} F(U_{i_1,\dots,i_n}), \sum (-1)^i d_i\right)_n$$

with $U_{ij} = U_i \cap U_j$ etc.

Derived Sections

In topology, we are often interested in the sheaf cohomology $H^i(U,F)$ extending $H^0(U,F) = F(U)$ to find (co)homological invariants of a space. In other words, we work in $D(\mathrm{Sh}(X;R))$, replacing F by a resolution $R\Gamma(-,F)$, its image under

$$\mathrm{Ch}(\mathrm{Sh}(X;R)) \to \mathrm{Ch}(\mathrm{Sh}(X,R))[\mathrm{qis}^{-1}] = D(\mathrm{Sh}(X;R))$$

so $H^i(U,F) = H^i(R\Gamma(U,F))$.

But what kind of mathematical object is $R\Gamma(-,F) : \mathrm{Open}(X)^{op} \to D(R)$? **Not a sheaf!**

Comparing Čech- and sheaf cohomology

Theorem

If the open cover (U_i) *is acyclic in the sense that* $R\Gamma^m(U_{i_1\dots i_n}, F) = 0$ *for* $m \neq 0$ *and any tuple* $i_1, \dots, i_n$*, then Čech cohomology with respect to* (U_i) *and sheaf cohomology agree, i.e. the canonical map*

$$R\Gamma(X, F) \xrightarrow{\text{qis}} \left(\prod_i F(U_i) \to \prod_{ij} F(U_{ij}) \to \dots \right)$$

is a quasi-isomorphism. Note that H^0 *always agree.*

Čech-to-sheaf-cohomology spectral sequence

Theorem (Čech-to-sheaf-cohomology spectral sequence)

Let X be a topological space, and $(U_i \subseteq X)$ an open cover. For a sheaf F the complex $R\Gamma(U, F)$ is quasi-isomorphic to the total complex (Čech hypercohomology) of the double complex

$$\check{C}((U_i), F)_{mn} := R\Gamma^m(U_{i_0} \cap \cdots \cap U_{i_n}, F)$$

where the vertical differentials are induced by the differentials in F, and the horizontal ones are alternating sums of restriction maps as usual in the Čech complex.

Čech-to-sheaf-cohomology spectral sequence

$$
\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
\prod_{i_0} R^2\Gamma(U_{i_0}, F) & \longrightarrow & \prod_{i_0 i_1} R^2\Gamma(U_{i_0 i_1}, F) & \longrightarrow & \prod_{i_0 i_1 i_2} R^2\Gamma(U_{i_0 i_1 i_2}, F) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\prod_{i_0} R^1\Gamma(U_{i_0}, F) & \longrightarrow & \prod_{i_0 i_1} R^1\Gamma(U_{i_0 i_1}, F) & \longrightarrow & \prod_{i_0 i_1 i_2} R^1\Gamma(U_{i_0 i_1 i_2}, F) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\prod_{i_0} F(U_{i_0}) & \longrightarrow & \prod_{i_0 i_1} F(U_{i_0 i_1}) & \longrightarrow & \prod_{i_0 i_1 i_2} F(U_{i_0 i_1 i_2}) & \longrightarrow & \cdots
\end{array}
$$

Higher Sheaves

Let us try to interpret the above equation as a descent condition for $R\Gamma(-, F)$:

Definition

An ∞-*sheaf* with values in $D(R)$ is a functor $\mathcal{F} : \mathrm{Open}(X)^{op} \to D(R)$ such that, for any open cover $(U_i \subseteq U)$,

$$\mathcal{F}(U) \overset{\text{qis}}{\simeq} \mathrm{tot}\left(\prod_i \mathcal{F}(U_i) \to \prod_{ij} \mathcal{F}(U_{ij}) \to \dots\right) \simeq$$

$$\simeq \operatorname*{holim}_{[n]\in\Delta^{op}} \left(\prod_{i_1,\dots,i_n} \mathcal{F}(U_{i_1,\dots,i_n})\right)$$

Denote their ∞-category by $\mathcal{S}\mathrm{h}(X; D(R))$.

∞-categories

To be precise, F should not be required to be a functor; for $U \subseteq V \subseteq W$ we should fix a chain homotopy from the composition $\rho_W^V \circ \rho_V^U : F(W) \to F(V) \to F(U)$ to ρ_W^U, as well as higher homotopies for more compositions, in a coherent way.

Definition

An ∞-*category* consists of:

- objects, morphisms, 2-morphisms between morphisms, 3-morphisms between 2-morphisms, ...
- and appropriate compositions, identities, associators, ...
- such that all n-morphisms with $n > 1$ are invertible.

In other words, an ∞-category is a category *up to coherent homotopy*.

∞-categories

Example

- An ordinary category, with only identity *n*-morphisms for $n > 1$.
- For X a topological space, its homotopy type $\mathrm{Sing}(X)$ with objects points of X, morphisms paths, 2-morphisms homotopies, ...
- The *∞-category of spaces* $\mathcal{S}$ with objects CW-complexes, morphisms continuous maps, 2-morphisms homotopies, ...
- The *∞-category of spectra* $\mathcal{S}\mathrm{p}$,
- The *derived ∞-category* $D(R)$, with objects injective chain complexes, morphisms chain maps, 2-morphisms chain homotopies, ...

The last two examples are *stable ∞-categories*.

Further Examples

- Fix $(X_i)_{i \in I}$ a collection of topological spaces, with open subspaces $U_j^{(i)} \subseteq X_i$ for $i, j \in I$, and homeomorphisms $\phi_{ji} : U_j^{(i)} \xrightarrow{\cong} U_i^{(j)}$. Then, we may glue along them to obtain a space
$$X := \bigsqcup_i X_i \big/ x_j \sim \phi_{ji} x_i$$
provided that the cocycle condition $\phi_{kj} \circ \phi_{ji} = \phi_{ki}$ holds on triple intersections.
- Stacks in Algebraic Geometry

Theorem (Lurie)

The functor $\mathrm{Ch}(\mathrm{Sh}(X;R)) \to \mathcal{S}\mathrm{h}(X;D(R))$ *informally given by sending* $F \mapsto R\Gamma(-,F)$ *induces an equivalence*

$$D(\mathrm{Sh}(X;R)) \simeq \mathcal{S}\mathrm{h}^{hyp}(X;D(R))$$

with the ∞*-category of hypercomplete* ∞*-sheaves on* X*.*

Remark

For good X, e.g. CW-complexes, manifolds or Whitney stratified spaces, every sheaf is hypercomplete.

Schwede-Shipley

Theorem (Schwede-Shipley)

For R a ring and HR its associated (associative) ring spectrum, the derived ∞-category

$$D(R) \simeq \mathrm{LMod}_{HR}$$

is equivalent to the ∞-category of module spectra over HR.

Corollary

If we always work with ∞-categories, everything is automatically derived:

$$D(\mathrm{Sh}(X;R)) \simeq \mathcal{S}\mathrm{h}^{hyp}(X;\mathrm{LMod}_{HR}) =: \mathcal{S}\mathrm{h}^{hyp}(X;HR)$$

Verdier Duality

Perfect Complexes

Definition

Given a chain complex C over a ring R, its *dual complex* is

$$C^\vee := \mathrm{RHom}(C, R) .$$

Definition

The ∞-category of *finitely presented chain complexes* $D^{\mathrm{fp}}(R)$ is the smallest full subcategory of $D(R)$ containing R, which is closed under shifts, direct sums and mapping cones.

The ∞-category of *perfect chain complexes* $D^{\mathrm{perf}}(R)$ is the smallest full subcategory which is also closed under direct summands.

Perfect Complexes

Definition

For R a ring spectrum, we can similarly define $\mathrm{LMod}_R^{\mathrm{fp}}$ and $\mathrm{LMod}_R^{\mathrm{perf}} \subseteq \mathrm{LMod}_R$. Obviously, these notions agree for Eilenberg-MacLane spectra.

Proposition

For P a perfect module spectrum, $P^\vee$ is also perfect and $P^{\vee\vee} \cong P$. Similarly in the finitely presented case.

From now on, R is always a ring spectrum.

Compactly supported sections

Let X be a locally compact Hausdorff space, and $\mathcal{V}$ a stable ∞-category.

Definition

Given an ∞-sheaf $\mathcal{F}$, its *compactly supported sections* on an open set U are defined as

$$\mathcal{F}_c(U) := \operatorname*{colim}_{K \subseteq U \text{ cpt}} \operatorname{fib}\left(\mathcal{F}(U) \to \mathcal{F}(U - K)\right)$$

where fib is the (functorial) mapping cocone. The functor $U \mapsto \mathcal{F}_c(U)$ is an ∞-*cosheaf*, i.e. a sheaf valued in $\mathcal{V}^{op}$.

Sheaf-Cosheaf Duality

Theorem (Lurie)

For X a locally compact Hausdorff space and $\mathcal{V}$ a stable ∞-category with limits and colimits, the functor $(-)_c$ induces a contravariant equivalence

$$\mathcal{S}\mathrm{h}(X;\mathcal{V})^{op} \simeq \mathcal{S}\mathrm{h}(X;\mathcal{V}^{op})$$

with inverse $(-)_c$ defined in $\mathcal{V}^{op}$.

Six Functors

We apply this to $\mathcal{V} = \mathrm{LMod}_R$, for R a ring spectrum.

Definition

The *Verdier duality functor* $\mathbb{D} : \mathcal{S}\mathrm{h}(X; R)^{op} \to \mathcal{S}\mathrm{h}(X; R)$ is given by

$$\mathbb{D}\mathcal{F}(U) := (\mathcal{F}_c(U))^\vee$$

so automatically $\mathbb{D}^{op} \dashv \mathbb{D}$.

Warning

This is not an equivalence anymore!

Six Functors

Definition

For $f : X \to Y$ a map of locally compact Hausdorff spaces, *exceptional direct and inverse image* along f are defined as

$$f_! := (-)_c \circ f_* \circ (-)_c \cong \mathbb{D} \circ f_* \circ \mathbb{D}$$
$$f^! := (-)_c \circ f^* \circ (-)_c \cong \mathbb{D} \circ f^* \circ \mathbb{D}$$

By construction, $f_! \dashv f^!$.

Remark

For $t : X \to *$ define the *dualizing complex* $\omega_X = t^! R$, then we can rewrite

$$\mathbb{D}\mathcal{F} = \underline{\mathrm{Hom}}(\mathcal{F}, \omega_X) .$$

Verdier Biduality

Theorem

For X locally compact Hausdorff, the Verdier duality functor restricts to a contravariant autoequivalence

$$\mathbb{D} : \mathcal{S}\mathrm{h}^{hyp}_{\mathrm{perf}}(X; R)^{op} \to \mathcal{S}\mathrm{h}^{hyp}_{\mathrm{perf}}(X; R)$$

of hypersheaves with perfect (or alternatively finitely presented) stalks $x^*\mathcal{F}$ and costalks $x^!F$, with $\mathbb{D}^2 = \mathrm{Id}$.

Proof.

Well-defined, since $x^*\mathbb{D}\mathcal{F} = (\mathbb{D}x^*\mathbb{D}\mathcal{F})^\vee = (x^!\mathcal{F})^\vee$ and similarly for $x^!$; also as a right adjoint $\mathbb{D}$ preserves hypercompleteness.
Equivalence, since $x^*\mathbb{D}\mathbb{D}\mathcal{F} = (x^!\mathbb{D}\mathcal{F})^\vee = (x^*\mathcal{F})^{\vee\vee} = x^*\mathcal{F}$, so by hypercompleteness the canonical map is an isomorphism. □

L-Groups of Sheaves

Motivation

Poincaré Duality can be rephrased as $\mathbb{D}\underline{\mathbb{R}} \cong \underline{\mathbb{R}}[n]$, i.e. the constant sheaf $\underline{\mathbb{R}}$ on an oriented manifold M^n is Verdier self-dual:

$$C_c^{-*}(M)^\vee = \underline{\mathbb{R}}_c(M)^\vee = \mathbb{D}\underline{\mathbb{R}}(M) = \underline{\mathbb{R}}[n](M) = C^{n+*}(M)$$

This is no longer true for stratified manifolds. Are there other interesting Verdier self-dual sheaves?

L-groups of Sheaves

Definition

Given $\mathcal{F} \in \mathcal{S}\mathrm{h}^{hyp}_{\mathrm{perf}}(X; R)$, we define its spectrum of

- *n-dimensional symmetric forms* as
$$\mathcal{Q}^s(\mathcal{F}) := \mathrm{map}(\mathcal{F} \wedge \mathcal{F}, \omega_X[-n])^{hS_2}$$
- *n-dimensional quadratic forms* as
$$\mathcal{Q}^q(\mathcal{F}) := \mathrm{map}(\mathcal{F} \wedge \mathcal{F}, \omega_X[-n])_{hS_2}$$

Both induce a morphism $F \to \mathbb{D}F[-n]$, and are said to exhibit F as *Verdier self-dual* if this is an isomorphism.

L-groups of Sheaves

Both $\mathcal{Q}^s, \mathcal{Q}^q$ make $\mathcal{S}\mathrm{h}^{hyp}_{\mathrm{perf}}(X;R)$ into a Poincaré ∞-category [1], allowing us to define:

Definition

The n-th *quadratic L-group* of $\mathcal{S}\mathrm{h}^{hyp}_{\mathrm{perf}}(X;R)$ is the quotient

$$L^q_n(\mathcal{S}\mathrm{h}^{hyp}_{\mathrm{fp}}(X;R)) := \frac{\left\{\begin{matrix}\text{pairs } (\mathcal{F},q) \text{ of an } \infty\text{-sheaf and an } n\text{-dim.}\\ \text{quadr. form exhibiting it as Verdier self-dual}\end{matrix}\right\}}{\{\text{pairs that admit a } \textit{Lagrangian } \mathcal{L}\to\mathcal{F}\}}$$

Similarly in the perfect case, and for symmetric forms. These are the homotopy groups of the symmetric and quadratic L-spectra:

$$\mathbb{L}^q(\mathcal{S}\mathrm{h}^{hyp}_{\mathrm{fp}}(X;R))$$

$$\mathbb{L}^s(\mathcal{S}\mathrm{h}^{hyp}_{\mathrm{fp}}(X;R))$$

[1] [Calmès, Dotto, Harpaz, Hebestreit, Land, Moi, Nardin, Nikolaus, Steimle]

Constructible Sheaves

Definition

Let $(X \to P)$ be a stratified space. An ∞-sheaf $\mathcal{F} \in \mathcal{S}h(X; R)$ is called *constructible* if its restrictions to all strata $\mathcal{F}|_{X_p}$ are locally constant.

For this to be a good definition, we need further conditions on X. For simplicity, assume it is Whitney-stratified.

Proposition (Volpe)

If $(X \to P)$ is Whitney-stratified, an ∞-sheaf has perfect (f.p.) stalks iff it has perfect (f.p.) costalks.

L-Groups of constructible sheaves

Proposition

For $(X \to P)$ Whitney-stratified, constructible sheaves with a complicated finiteness condition form a Poincaré ∞-category as well, so we may define $\mathbb{L}^q(\mathcal{S}\mathrm{h}^{cbl}(X; R)^{(fp)})$ and $\mathbb{L}^s(\mathcal{S}\mathrm{h}^{cbl}(X; R)^{(fp)})$

Proposition (Lurie)

If M is a connected smooth manifold with the trivial stratification, then $\mathcal{S}\mathrm{h}^{lc}(X; R) \simeq \mathrm{LMod}_{\Omega X \wedge R}$ so by the *algebraic* π*-*π*-theorem*,

$$\mathbb{L}^q(\mathcal{S}\mathrm{h}^{cbl}(X; R)^{(fp)}) \cong \mathbb{L}^q((\pi_0 R)[\pi_1 M])$$

which is the obstruction appearing in the surgery exact sequence.

Surgery-type exact sequence

There is an exact sequence of spectra

$$\mathbb{L}^q(\mathcal{S}\mathrm{h}_{fp}^{\perp cbl}(X;R)) \to \mathbb{L}^q(\mathcal{S}\mathrm{h}_{fp}(X;R)) \to \mathbb{L}^q(\mathcal{S}\mathrm{h}^{cbl}(X;R)^{(fp)}) \ .$$

If X is a PL space, the middle spectrum is equivalent to $\mathbb{L}^q(R) \wedge X$, so the second is an *assembly map* and the sequence is closely related to the *surgery exact sequence*. This is probably wrong in the topological case.

Decomposition into Strata

For X a 2-strata space, we have a fiber sequence

$$\mathbb{L}^q(\mathcal{S}\mathrm{h}^{cbl}(X_-;R)^{(fp)}) \to \mathbb{L}^q(\mathcal{S}\mathrm{h}^{cbl}(X;R)^{(fp)}) \to \mathbb{L}^q(\mathcal{S}\mathrm{h}^{cbl}(X_+;R)^{(fp)})$$

inducing a long exact sequence of L-groups. In the case of more strata, we can successively apply this decomposition. Similarly in the PL case.

Relations to other work

- For $R = Hk$ with k a field of $\mathrm{char}(k) \neq 2$, this agrees with a construction by [Schürmann, Woolf].
- Suppose that X only contains strata of even codimension (or more generally, is a Witt space) and $R = H\mathbb{Q}$. Then the intersection homology sheaf $\mathrm{IC}_{X;\mathbb{Q}}$ is Verdier self-dual, as exhibited by the intersection paring, and hence defines a class in $L_n(\mathcal{S}\mathrm{h}^{cbl}(X;\mathbb{Q})^{(fp)})$.
- In this situation, by the results in their paper our groups agree with the *Witt groups of perverse sheaves*. The decomposition sequence splits in this case.
- Our decomposition and surgery sequence are reminiscent of Browder-Quinn L-groups.

Outlook

L-Groups of classical field theories

- Any perturbative Lagrangian classical field theory on a (pseudo-)manifold M has an associated *BV-complex*. It admits a (-1)-shifted symplectic structure, and thus defines a class

$$[\mathcal{E}] \in L_3^s(\mathcal{S}\mathrm{h}(M; \mathrm{Mod}_{C^\infty})) \; .$$

- For topological field theories, this sheaf is locally constant/ constructible
- If M has a boundary/ corners, the BV-complex of a TFT is a constructible ∞-sheaf, but not longer Verdier self-dual. A mixture of Verdier- and Poincaré-Lefschetz-duality has to be introduced.

L-Groups of classical field theories

Example

For classical Chern-Simons theory, ignoring functional analysis, we obtain the visible symmetric signature. This works in any dimension d, but $[\mathcal{E}]$ will be in L^s_d. For BF-theories, the L-class is trivial.

Informally, $[\mathcal{E}] \in L^s_3$ measures how for our theory is from being a boundary theory. A different story with similar result is WIP by [Reutter, Johnson-Freyd]; they develop a surgery sequence for extended TFTs, involving a Quantum Witt group.