Constructible Factorization Algebras for Field Theories

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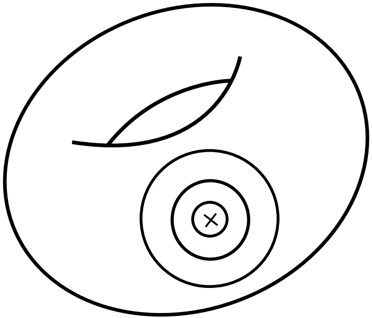
Content

What are Constructible Factorization Algebras?

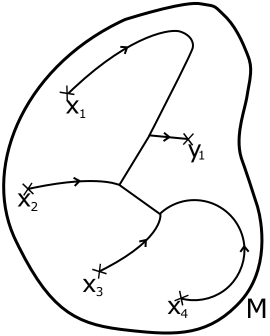
Local Operators

Given a topological/ conformal field theory on an n-manifold M, and a point $x \in M$, write $A(x)$ for the space of local (perturbative, polynomial) operators at x.



Monodromy along Multipaths

For a set of finitely many paths that start at points (x_i), end at (y_j) and are allowed to join together, we can parallel transport operators along such a *multipath*:



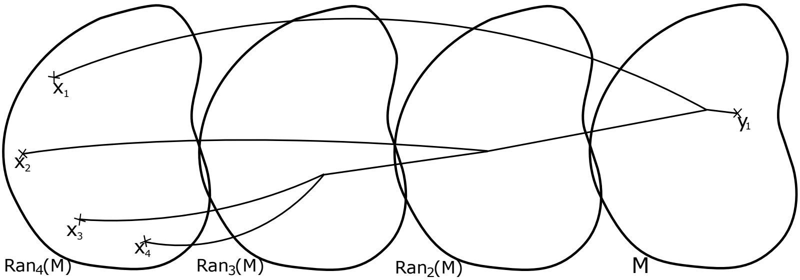
Ran Space

A mathematically precise formulation uses the *Ran space* $\mathrm{Ran}(M)$, a stratified space with

- one point for every non-empty finite subset of M,
- stratification $\mathrm{Ran}(M) \to \mathbb{N}_{>0}$ via cardinality,
- topology on $\mathrm{Ran}(M)_{\leq m}$ the final topology of the map $M^{\times m} \to \mathrm{Ran}(M)_{\leq m}$,
- final topology as a union of these subspaces (non-standard).

Multipaths in M are paths in $\mathrm{Ran}(M)$ that monotonically decrease in the stratification $\mathbb{N}_{>0}$, also called *enter-paths*.

Ran Space



Category of Multipaths

We introduce an ∞-category $\mathrm{Exit}\,\mathrm{Ran}(M)^{op}$ with

- Objects the non-empty finite subsets of M, i.e. points of the Ran space,
- Morphisms from (x_i) to (y_j) given by enter-paths between the respective points in the Ran space,
- 2-morphisms given by homotopies that similarly decrease in the stratification,
- Higher morphisms given by higher homotopies.

Equip it with ∞-operadic structure $\sqcup$.

Factorization Algebras

Let $\mathcal{V}$ be a (good) symmetric monoidal ∞-category.

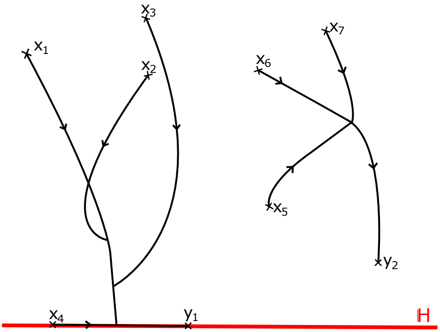
Definition

A *locally constant factorization algebra* on M is a symmetric monoidal functor $A : (\mathrm{Exit}\,\mathrm{Ran}(M)^{op})^{\sqcup} \to \mathcal{V}^{\otimes}$, associating

- to any finite set $S = \{x_1, \ldots, x_m\}$ in M an object $A(S) \in \mathcal{V}$,
- to any multipath from S to T a morphism $A(S) \to A(T)$ in $\mathcal{V}$,
- in a way compatible with concatenation of multipaths, up to coherent homotopy
- such that $A(S) \cong A(x_1) \otimes A(x_2) \otimes \cdots \otimes A(x_m)$, together with higher coherences.

Manifolds with Boundary

In a boundary CFT on the upper half-plane $M = \{\mathrm{Im}(z) \geq 0\}$, we can transport operators from the interior arbitrarily close to the boundary, but not from the boundary into the interior:



Constructible Factorization Algebras

Stratify $M \to \{1 < 2\}$, sending the interior to 2 and the boundary to 1. Then, A has monodromy along *enter-multipaths*.
Define $\mathrm{Ran}(M)$ with a stratification respecting this.

Definition

A *constructible factorization algebra* on a stratified space $M \to P$ is a symmetric monoidal functor $A : (\mathrm{Exit}\,\mathrm{Ran}(M)^{op})^{\sqcup} \to \mathcal{V}^{\otimes}$.

We discuss two ways to make this more explicit.

Exodromy

Definition

A *constructible sheaf* on a stratified space $(M \to P)$ is a sheaf F such that the restriction $F|_{X_p}$ to each stratum is locally constant.

The term "sheaf" will always refer to ∞-sheaves.

Theorem (Exodromy Correspondence, [Lurie, Porta-Teyssier])

For any good stratified space $(M \to P)$ and good ∞-category $\mathcal{V}$,

$$\mathcal{S}h^{cbl}(M, \mathcal{V}) \cong \mathrm{Fun}(\mathrm{Exit}(M), \mathcal{V})$$

where $\mathrm{Exit}(M)$ *is the category of exit-paths.*

Thus, constructible factorization algebras are constructible sheaves on $\mathrm{Ran}(M)$ satisfying a factorization condition.

Basic Open Subsets

Definition

We restrict $M \to P$ to be a *conically smooth stratified space*. By this, we mean that it has an atlas of basics

$$\mathbb{R}^i \times C(L) \to Q^{\triangleleft} ,$$

where $L \to Q$ is a compact conically smooth stratified space, glued together along smooth maps.

Definition

Let $\mathrm{Disk}_{/M}$ be the partially ordered set of open inclusions $j : D \to M$, with D a disjoint union of basic open sets. Equip $\mathrm{Disk}_{/M}$ with the operadic structure $\sqcup$.

Factorization Algebras and Disks

Definition

A *factorization algebra* on $M \to P$ is a symmetric monoidal functor

$$A : \mathrm{Disk}^{\sqcup}_{/M} \to \mathcal{V}^{\otimes} .$$

It is called *constructible* if every inclusion of isotopic basics is sent to an isomorphism by A.

Remark

This is equivalent to the definition above, since [Ayala-Francis-Tanaka]

$$\mathrm{Disk}^{\mathrm{surj}}_{/M}[(\mathcal{I}_M)^{-1}] \simeq \mathrm{Exit}\,\mathrm{Ran}(M)^{op} .$$

Caveat: This only yields non-unital factorization algebras.

Classical Field Theories and the BV complex

Classical Field Theory

A classical field theory on a smooth manifold M consists of

- a space of (off-shell) field configurations $\mathcal{F}$,
- an action $S : \mathcal{F} \to \mathbb{R}$.

Variation of the action $dS = 0$ yields the *covariant phase space*

$$X = \mathrm{dCrit}(S) = \mathrm{Graph}(dS) \times^R_{T^*\mathcal{F}} \mathcal{F}$$

which admits a (-1)-shifted symplectic structure. The tangent complex

$$\mathcal{E} := \mathbb{T}_\phi X$$

is called *BV complex* and describes perturbation theory around ϕ. The shift $\mathcal{E}[-1]$ is an L_∞ algebra.

Linear Operators

- Observables are functions on the covariant phase space X
- Polynomial local observables are $\mathcal{O}bs^{cl} := \mathrm{Sym}(\mathcal{E}^\vee)$, equipped with the Chevalley-Eilenberg differential
- Linear local observables are in $\mathcal{E}^\vee$ and should satisfy monodromy along enter-paths, by exodromy they form a constructible cosheaf

Upshot: The BV-BRST complex is a constructible sheaf! This automatically implies that $\mathcal{O}bs^{cl}$ is a constructible factorization algebra.

Applications for Simplicial BV theories

Simplicial Complexes

Definition

A *simplicial complex* K consists of a set of vertices K_0 and a poset $\mathcal{I}_K$ of faces consisting of nonempty finite subsets of K_0, such that

- for every $v \in K_0$, the set $\{v\}$ is in $\mathcal{I}_K$,
- if $\sigma \in \mathcal{I}_K$ and $\tau \subseteq \sigma$, then $\tau \in \mathcal{I}_K$,
- the partial order relation is given by inclusion.

Fix a finite simplicial complex K, and a stable ∞-category $\mathcal{V}$ with duality functor $(-)^\vee$, for example $D^{\text{perf}}(\mathbb{R})$.

Gluing Data

Definition

If we regard $\mathcal{I}_K$ as an ∞-category,

- Functors in $\mathrm{Fun}(\mathcal{I}_K, \mathcal{V})$ will be called *$\mathcal{V}$-valued constructible sheaves on K*,
- Functors in $\mathrm{Fun}(\mathcal{I}_K^{op}, \mathcal{V})$ will be called *gluing data* on K.

Proposition

This coincides with the usual definition of a constructible ∞-sheaf on the stratified space $|K| \to \mathcal{I}_K$:

$$\mathcal{S}h^{cbl}(|K|, \mathcal{V}) \simeq \mathrm{Fun}(\mathcal{I}_K, \mathcal{V})$$

Duality

Definition

To a constructible sheaf $S : \mathcal{I}_K \to \mathcal{V}$, we can associate a gluing datum $\mathbb{G}S : \mathcal{I}_K^{op} \to \mathcal{V}$ defined by

$$\mathbb{G}S(\sigma) := \lim_{(\tau \subseteq \sigma) \in (\mathcal{I}_K)_{/\sigma}} S(\tau) \,.$$

Definition

To a gluing datum $F : \mathcal{I}_K^{op} \to \mathcal{V}$, we can associate a *dual* gluing datum $DF : \mathcal{I}_K^{op} \to \mathcal{V}$ defined by

$$DF(\sigma) := \lim_{(\tau \subseteq \sigma) \in (\mathcal{I}_K)_{/\sigma}} F(\tau)^{\vee} \,.$$

Duality

Example

If $K = \Delta^1$, then $DF(\{0\}) = F(0)^\vee$, $DF(\{1\}) = F(1)^\vee$ and

$$F(\{0\})^\vee \times_{F(\{0,1\})^\vee} F(\{1\})^\vee \ .$$

Proposition (from Algebraic L-Theory)

For every gluing datum $F \in \mathrm{Fun}(\mathcal{I}_K^{op}; \mathcal{V})$, there is a canonical biduality isomorphism $F \cong DDF$.

Corollary

We can recover S from $\mathbb{G}S$ since $S \cong (D\mathbb{G}S)^\vee \cong (DDS^\vee)^\vee \cong S$.

Definition

For $S \in \mathrm{Fun}(\mathcal{I}_K, \mathcal{V})$ a constructible sheaf, its *global sections* are defined as

$$C^* S := \lim_{\sigma \in \mathcal{I}_K} S(\sigma) \in \mathcal{V} .$$

Similarly $C_* S := \operatorname{colim}_{\sigma \in \mathcal{I}_K} S(\sigma)$. They agree with the simplicial (co-)chain complexes with values in the local system S.

Definition

Similarly, for $F : \mathcal{I}_K^{op} \to \mathcal{V}$ a gluing datum, we define $C^* F$ and $C_* F$ by taking a limit or colimit over $\mathcal{I}_K^{op}$.

Proposition

For S a constructible sheaf, the global sections $C^* S \cong C^* \mathbb{G} S$ agree. This is generally not true for C_*.

Poincaré-Duality

From now on, let K be a (Whitehead) triangulation of a smooth oriented closed n-manifold.

Theorem (Poincaré Duality)

For any gluing datum $F \in \mathrm{Fun}(\mathcal{I}_K^{op}, \mathcal{V})$,

$$C^*(DF) \cong C^*(F)^\vee[-n]$$

Proof.

$$C^*(DF) = \lim_{\sigma \in \mathcal{I}_K^{op}} \lim_{\tau \subseteq \sigma} F(\tau)^\vee \cong \lim_{\tau \in \mathcal{I}_K} F(\tau)^\vee \overset{!}{\cong}$$

$$\cong \operatorname*{colim}_{\sigma \in \mathcal{I}_K} F(\sigma)^\vee[-n] = C^*(F)^\vee[-n] \,.$$

Rewrite $F^\vee =: S$ as a constructible sheaf, then this becomes simplicial Poincaré Duality $C^*S \overset{!}{\cong} C_*S[-n]$. □

Simplicial BV-theories

Definition

An *m-dimensional Poincaré object* (F, ω) in the stable ∞-category of gluing data $\mathrm{Fun}(\mathcal{I}_K, \mathcal{V})$ is an object F equipped with an isomorphism $\omega : F \overset{\cong}{\to} DF[-m]$ induced by a symmetric pairing.

Proposition

For K a triangulation of a compact oriented smooth n-manifold and (F, ω) an m-dimensional Poincaré object in $\mathrm{Fun}(\mathcal{I}_K, \mathcal{V})$, the pair $(C^*F, C^*\omega)$ is an $(n+m)$-dimensional Poincaré object in $\mathcal{V}$.

Proof.

$$C^*F \cong C^*(DF[-m]) \cong (C^*F)^\vee[-n][-m] \cong (C^*F)^\vee[-n-m] \quad \square$$

Definition

A (free, topological) *simplicial BV theory* on a finite simplicial complex K of dimension n is a $(3-n)$-dimensional Poincaré object in $\mathrm{Fun}(\mathcal{I}_K^{op}, \mathcal{V})$.

Remark

Equivalently, a simplicial BV theory is a constructible sheaf S equipped with an isomorphism $\mathbb{G}S \to D\mathbb{G}S[3-n] \cong S^\vee[3-n]$. This is different from Verdier self-duality.

Corollary

Given a simplicial BV theory $F : \mathcal{I}_K^{op} \to \mathcal{V}$ on a triangulation of a smooth oriented manifold, its global sections admit an isomorphism $C^*F \cong (C^*F)^\vee[-3]$, induced by a (-1)-shifted symplectic pairing.

Generally, our definition of a simplicial BV-theory is chosen in a way that

- On every individual top-dimensional simplex, we obtain a datum resembling an extended BV-BFV theory, and
- For a PL triangulation of a manifold with corners, we should obtain an extended BV-BFV theory on it by taking global sections of the gluing datum restricted to the closed strata.

One can view it as a middle ground between Lagrangian extended topological field theories and extended BV-BFV theories.

Theorem

Any simplicial BV-theory $F : \mathcal{I}_K^{op} \to D^{perf}(\mathbb{R})$ *on an n-dimensional finite simplicial complex* K *defines a constructible factorization algebra* $\mathcal{O}bs^{cl} := \mathrm{Sym}\, \mathcal{E}^{\vee}$ *of classical observables on* $|K| \to \mathcal{I}_K$*, where* $\mathcal{E}$ *is the constructible sheaf on* $|K|$ *associated to* F*.*

Proof.

We know that $\mathcal{E}^{\vee}$ is a constructible cosheaf, which is the same thing as a constructible factorization algebra $\mathcal{D}isk^{\sqcup}_{/|K|} \to D^{\mathrm{perf}}(\mathbb{R})^{\oplus}$. Now, use the fact that the functor $\mathrm{Sym} : D(\mathbb{R})^{\oplus} \to D(\mathbb{R})^{\otimes}$ is symmetric monoidal and preserves sifted colimits. □

Definition

For any gluing datum $F : \mathcal{I}_K^{op} \to \mathcal{V}$, the datum $\mathrm{hyp}(F) := F \oplus DF$ is a Poincaré-object in a canonical way, since

$$D(F \oplus DF) \cong DF \oplus D^2 F \cong F \oplus DF \ .$$

We call it the *hyperbolic Poincaré-object* associated to F. Similarly, we define the *n-dimensional Poincaré-object* associated to F as

$$\mathrm{hyp}^{[n]}(F) := F \oplus DF[-n] \ .$$

Construction

For $A : \mathcal{I}_K \to \mathcal{V}$, let *abelian BF-theory* on K with values in A be the simplicial BV theory defined by

$$F_{BF} := \mathrm{hyp}^{[3-n]}(\mathbb{G}A) = \mathbb{G}A \oplus D\mathbb{G}A[-3+n] \; .$$

The global section BV-complex is given by

$$C^* F_{BF} = C^*\mathbb{G}A \; \oplus \; C^* D\mathbb{G}A[-3+n] \cong C^*A \; \oplus \; C^*A^\vee[-3+n] \cong$$
$$\cong C^*_{\mathrm{simp}}(K, A) \; \oplus \; C^{\mathrm{simp}}_{*+n-3}(K, A)^\vee$$

because $C^*A^\vee = \lim A^\vee = (\mathrm{colim}\, A)^\vee = (C_*A)^\vee$.

Example

Concretely, A can arise as composition of a G-local system $\omega : \mathcal{I}_K \to BG$ on K and a representation $\rho : BG \to \mathcal{V}$.

Construction

For K of dimension 3 and $\mathfrak{g}$ an $\mathbb{R}$-vector space with non-degenerate inner product exhibiting $\mathfrak{g} \cong \mathfrak{g}^\vee$, define *abelian Chern-Simons theory* with values in $\mathfrak{g}$ as the simplicial BV theory

$$S = \underline{\mathfrak{g}[0]} : \mathcal{I}_K \to D^{\mathrm{perf}}(\mathbb{R}) \,, \; \sigma \mapsto \mathfrak{g}[0] \,.$$

To see that this is self-dual of dimension $0 = 3 - 3$, evaluate

$$\mathbb{G}\underline{\mathfrak{g}[0]}(\sigma) = \lim_{\tau \subseteq \sigma} \mathfrak{g}[0] \cong C^*_{\mathrm{simp}}(\sigma, \mathfrak{g}) \cong \mathfrak{g}[0] \cong \mathfrak{g}[0]^\vee \,.$$

The global section BV complex is $C^*\underline{\mathfrak{g}[0]} = C^*_{\mathrm{simp}}(K, \mathfrak{g})$.

Similarly, one can write down (classical) simplicial BV theories for

- Higher-dimensional abelian Chern-Simons theory
- Topological Quantum Mechanics
- Non-abelian Chern-Simons theory

Generally, the hope is that any AKSZ theory or Lagrangian extended field theory

$$Z : \mathrm{Bord}_d \to \mathrm{Lagr}_d$$

has an associated simplicial BV theory for each triangulation of a manifold with corners, constructed by

- Evaluating Z on the associated chain of composable d-morphisms in Bord_d, obtaining a system of Lagrangian correspondences of derived Artin stacks
- Taking the tangent complex at a common geometric point ϕ

Simplicial Field Theories connect the cutting-and-gluing locality of extended field theories and the locality of sheaves.

Outlook

Above methods also allow the definition of BV data on

- Regular cell complexes
- PL spaces
- Manifolds with corners
- Topological pseudomanifolds with corners

The last two cases are harder since we should not use exodromy.
It is also possible to introduce boundary conditions or polarizations.