Cauchy-Completions and Higher Idempotents

jt. WIP with David Reutter

Motivation I: Higher Fusion Categories

Cobordism Hypothesis:

$$\left\{\begin{matrix}\text{framed fully extended} \\ n\text{-dim. TFTs in } \mathcal{C}\end{matrix}\right\} \simeq \left\{\begin{matrix}\text{fully dualizable objects} \\ \text{in } \mathcal{C}\end{matrix}\right\}$$

(∞,n)-category

Q: What $\mathcal{C}$ are of (physical) interest? How do we find the f.d. objects?

"Simplest" example: $\mathcal{C} = \{n\text{-vector spaces}\}$ over a field k

$\underline{n=1}$: $\mathrm{Vec}_k^{\text{dual}} = \{\text{fin.-dim. } k\text{-vector spaces}\}$

$\underline{n=2}$ $(2\text{-}\mathrm{Vec}_k)^{2\text{-dual}} \underset{[G\text{-}JF]}{=} \mathrm{Morita}(\text{sep. algebras}/_k)$

$\simeq \mathrm{Morita}(\text{non(co)unital special Frobenius algebras})$

$\underline{n=3}$ $(3\text{-}\mathrm{Vec})^{3\text{-dual}} \overset{[JF]}{\underset{[D\text{-}SP\text{-}S]}{=}} \mathrm{Morita}(\text{sep. multifusion cats})$

$\underline{n=4}$ separable multifusion 2-cats? (are f.d. [D])

$\underline{\text{general } n}$ "multifusion n-cats" [JF]

Our goal:
- Formalize this, using the language of enriched ∞-cats where
$$\underset{\cup\!\!\!\mid}{\mathrm{Cat}_{(\infty,n)}} = \mathrm{Cat}[\underset{\cup\!\!\!\mid}{\mathrm{Cat}_{(\infty,n-1)}}],$$
$$\mathrm{Cat}_{(n,n)} = \mathrm{Cat}[\mathrm{Cat}_{(n-1,n-1)}] \quad \underline{\text{weak}} \ n\text{-categories!}$$
- Allow k to be an algebra object in any* pres. monoidal ∞-cat.

  eg. $R \in D_{\geq 0}(R) \rightsquigarrow$ "derived" fusion (∞,n)-categories
- Construct examples, retain 1-categorical results.

Motivation II: Absolute Colimits

Def A category $\mathcal{C}$ is called $\underline{\text{additive}}$ if
- It has a zero object
- It admits finite products & coproducts
- $\forall c, c' \in \mathcal{C}$ the map $c \sqcup c' \xrightarrow{\begin{pmatrix} id_c & 0 \\ 0 & id_{c'}\end{pmatrix}} c \times c'$ is an isom.
- The addition on $\mathrm{Hom}_{\mathcal{C}}(c,c') \ni f,g$ given by
$$c \xrightarrow{\Delta} c \times c \cong c \sqcup c \xrightarrow{f \sqcup g} c' \sqcup c' \xrightarrow{\nabla} c'$$
admits inverses, ie. makes $\mathrm{Hom}_{\mathcal{C}}(c,c')$ into an abelian group.

Def An additive category is an Ab-enriched category with finite (incl. empty) coproducts.

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$\Rightarrow$ Those automatically agree with products / the terminal object.

Reason: Let $\emptyset$ be the initial object, then $\mathrm{Hom}_{\mathcal{A}}(\emptyset, \emptyset) = \{id_\emptyset\} \in Ab$ so $id_\emptyset = 0$ is the zero object. Therefore, for all $X \in \mathcal{A}$ and $f \in \mathrm{Hom}_{\mathcal{A}}(X, \emptyset)$, we have

$f = id_\emptyset \circ f = 0 \circ f = 0$ by Ab-enrichment. $\longrightarrow$ This proof up to now only uses enrichment in pointed sets

The argument for direct sums is similar:

$$\begin{tikzcd} X \arrow[r, "i_1"] & X \sqcup Y \arrow[r, "id_X \sqcup 0"] \arrow[dr, "0 \sqcup id_Y"'] & X \sqcup 0 \cong X \\ Y \arrow[ur, "i_2"'] & & 0 \sqcup Y \cong Y \end{tikzcd}$$

$\sum p_k i_k = id_{X \sqcup Y}$, $i_k p_{k'} = \{id \}$ "$\square$"

Conceptually: Coproducts are absolute colimits for enrichment over Ab, ie.

- every Ab-enriched functor preserves coproducts
- Coproducts can be written as limits over a dual diagram (products)
- Coproducts are characterized by a diagrammatic property.

$\Rightarrow$ What about other enrichment categories?

Motivation III: Tiny Presheaves

Q: Can we recover:
- A ring R from its module category $\mathrm{Mod}_R(Ab)$?
- —— " —— derived cat. $D(R) = \mathrm{Mod}_{HR}(Sp)$?
- A category $\mathcal{C}$ from its presheaf category $\mathcal{P}(\mathcal{C})$?
- An enriched —— " —— $\mathcal{P}_{\mathcal{V}}(\mathcal{C})$?

$\mathcal{V} = Ab$, $\otimes R$; $\mathcal{V} = Sp$, $\otimes HR$; $\mathcal{V} = \mathcal{S}$; general $\mathcal{V}$ closed mon. pres.

$\longrightarrow$ Yes if we know which presheaves are representable / module is trivial.

Otherwise: R is a dualizable R-module. Similarly,

Def $F \in \mathcal{P}_{\mathcal{V}}(\mathcal{C})$ is tiny $:\Leftrightarrow$ $\mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(F, -) : \mathcal{P}_{\mathcal{V}}(\mathcal{C}) \to \mathcal{V}$ preserves colimits & $\mathcal{V}$-tensoring

$\Rightarrow$ Representable presheaves are tiny. In fact, for $\mathcal{V} = Set$

$F \in \mathcal{P}(\mathcal{C})$ is tiny $\Leftrightarrow$ F is retract of a repres. presheaf.

Enriched ∞-categories

Let $\mathcal{V}$ be a presentably monoidal ∞-category (closed monoidal, has colimits + set-theoretic cond.)

Def A $\mathcal{V}$-enriched ∞-category consists of
- A space of objects $X \in \mathcal{S} = \{\text{CW-complexes}\}$
- An enriched presheaf category $\mathcal{P}_{\mathcal{V}}(\mathcal{C})$ which is pres. $\mathcal{V}$-tensored
- A Yoneda functor $\text{\textsterling}^{\mathcal{V}}_{-} : X \longrightarrow \mathcal{P}_{\mathcal{V}}(\mathcal{C})$

such that:

(i) $\mathrm{Im}(\mathcal{L}^{\mathcal{V}}_-) \subseteq$ tiny objects of $P_{\mathcal{V}}(\mathcal{C})$

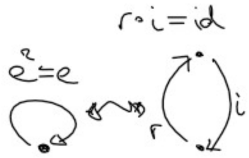
(ii) $\mathrm{Im}(\mathcal{L}^{\mathcal{V}}_-)$ generates $P_{\mathcal{V}}(\mathcal{C})$ under colimits & $\mathcal{V}$-tensoring $\Big\}$ "valent" $\mathcal{V}$-category $\Big\}$ univalent $\mathcal{V}$-category

(iii) $\mathcal{L}^{\mathcal{V}}_-$ exhibits X as the maximal subspace of $\mathrm{Im}(\mathcal{L}^{\mathcal{V}}_-)$

(iv) If $\mathcal{L}^{\mathcal{V}}_-$ hits <u>all</u> tiny objects we call $\mathcal{C}$ <u>Cauchy-complete</u>.]

Def $\mathcal{C}$ is called Cauchy-complete if any tiny presheaf is representable.
Otherwise, $\mathcal{C} \subseteq \{\text{tiny presheaves in } P_{\mathcal{V}}(\mathcal{C})\} =: \hat{\mathcal{C}}^{\mathcal{V}}$ <u>Cauchy-completion</u>.

[<u>Caveat</u>: We work with enriched ∞-categories.]



Ex/
- $\mathrm{Cat}_+(\mathrm{Set})$ = idempotent complete categories
  (every retract of a representable presheaf is representable)
- $\mathrm{Cat}_+(\mathrm{Cat}_{(n-1,m-1)})$ = i.c. (n,m)-categories
- $\mathrm{Cat}_+([0,\infty), \geq, +)$ = Cauchy-complete generalized metric spaces
- $\mathrm{Cat}_+(\mathrm{Set}_*)$ = i.c. categories with zero-object
- $\mathrm{Cat}_+(\mathrm{Ab})$ = i.c. additive categories
- $\mathrm{Cat}_+(\mathrm{Vect}_k)$ = i.c. k-linear categories
- $\mathrm{Cat}_+(\mathrm{Sp}_{cn})$ = i.c. additive ∞-cats
- $\mathrm{Cat}_+(\mathrm{Sp})$ = i.c. stable ∞-cats ($\approx$ i.c. triangulated cats)

<u>Cauchy-Complete (∞, n)-categories</u>

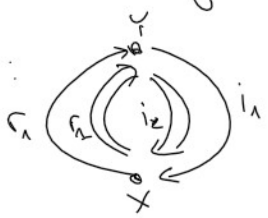
Thm [WIP] Cat_+ : $\left\{\begin{array}{l}\text{symm. monoidal closed}\\ \text{presentable cats}\end{array}\right\} \circlearrowleft$

$\mathrm{Cat}(\mathcal{V})$ is s.m.c. pres. via BM, $\otimes$

$\mathrm{Cat}_+(\mathcal{V})$ too [WIP] via $\widehat{\mathrm{BM}}$, $\hat{\otimes}$

$\Rightarrow$ Can iterate $\mathrm{Cat}^n_+ = \mathrm{Cat}_+[\mathrm{Cat}_+[\cdots]]$ $\rightarrow$ even infinitely
to obtain Cauchy-complete $\mathcal{V}$-enriched (∞, n)-categories.

Ex/ <u>Higher Retracts</u>:



$i_2: r_1 i_1 \Rightarrow \mathrm{id}_X$
$r_2: \mathrm{id}_X \Rightarrow r_1 i_1$ $\Big\}$ 2-section-retraction pair
$r_2 i_2 = \mathrm{id}_{\mathrm{id}_X}$

<u>Generally</u>: $r_1, i_1, r_2, i_2, \ldots, i_n$ with $r_n i_n = \mathrm{id}$
forming the CW-structure of an n-sphere $\Rightarrow$ weak n-category

($\rightarrow$ Our definition: $\{X\} \cup \{Y\} \cup \{r_1\} \cup \ldots =: \mathrm{Retr}_\infty$ (no computad!)

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($\to$ Our definition: $\{X\} \cup \{Y\} \cup \{r_n\} \cup \ldots =: \mathrm{Retr}_\infty$ (no computad!)

$\mathrm{Retr}_n := \mathrm{Retr}_\infty[>n\text{-morphisms}]$)

$\boxed{\text{Thm}}$ $\mathrm{Cat}^n_+(\mathrm{Set}) =$ n-idemp. c. (n,n)-categories

$\mathrm{Cat}^n_+(\mathcal{S}) =$ — " — (∞,n)-categories

$\mathrm{Cat}^n_+(\mathrm{Ab}) =$ n-i.c. additive (n,n)-cats

$\mathrm{Cat}^n_+(\mathrm{Vec}_k)$ [Sp_{cn}] $=$ n-i.c. k-lin. (n,n)-cats [(∞,n)]

[$\mathrm{Cat}^{n-1}_+(\mathrm{Add}_\pi)$ [$D(R)$] $=$ n-i.c. π-semiadd. (∞,n)-cats $\rightsquigarrow$ finite path integrals [WIP Scheimbauer, Walde]

$\mathrm{Cat}^2_+(\mathrm{Sp}) =$ "finitely lax additive $(\infty,2)$-cats" $\rightsquigarrow$ higher K-theory, perverse schobers [CDW]

$\mathrm{Cat}^{n>2}_+(\mathrm{Sp})$?

About our main goal:

$\mathrm{Cat}^n_+(\mathcal{V})$ is symmetric monoidal, with unit $\widehat{B \cdots \widehat{B \widehat{B 1_{\mathcal{V}}}}} =: \Sigma^n 1_{\mathcal{V}}$ "framed orbifold"

$\boxed{\text{Thm}}$ (WIP) Let $\mathcal{V} = \mathrm{Ab}, \mathrm{Vec}_k, \mathrm{Sp}_{cn}, \mathrm{Sp}, D(R), \ldots$

$$\Sigma^n 1_{\mathcal{V}} \simeq \mathrm{Cat}^{n-1}_+(\mathcal{V})^{1\text{-dual}} \simeq \mathrm{Morita}(\mathrm{Cat}^{n-2}_+(\mathcal{V}))^{2\text{-dual}} \overset{\text{Conj.}}{\simeq} \text{"}\mathrm{Morita}^2(\mathrm{Cat}^{n-3}_+(\mathcal{V}))\text{"} \ldots$$

is fully dualizable, even "f.d. objects in $1_{\mathcal{V}}$-vector spaces" [s.p.] [s.p.] $\cup\!\!\!\mid$ $\mathrm{Morita}_{E_2}(\mathrm{Cat}^{n-2}_+(\mathcal{V}))$

Ex/ $\Sigma 1_k \simeq \mathrm{Vec}_k^{f.d.}$

$\Sigma^2 1_k \simeq$ semisimple cats $\simeq$ Morita(sep. algebras)

$\Sigma^3 1_k \simeq$ s.s. 2-cats $\simeq$ Morita(sep. MFCs) ($\simeq$ ss. quasi-Hopf algebras)

" $\Sigma^4 1_k \simeq$ s.s. 3-cats $\simeq$ Morita(sep MF2Cs) $\simeq$ ss. quasi-Hopf cats $\simeq$ trialgebras "
$\simeq$ [is] braided MFCs [D]

$\Sigma 1_{D(R)} = D(R)^{perf}$

k field of char 0

$\Sigma^2 1_{D(R)} =$ s.p. R-linear stable ∞-cats $\simeq$ Morita(smooth proper algs k)

$\Sigma^3 1_{D(R)}$ studied for "derived Turaev-Viro" [G]

$\Rightarrow$ $\mathrm{Morita}^{s.p.}(\mathrm{Cat}^n_+(\mathcal{V})) =:$ "$\mathcal{V}$-fusion n-categories" $\rightsquigarrow$ Much more to do! $\rightsquigarrow$ $\exists$ examples!