Master Thesis in Physics

Constructible Factorization Algebras for Field Theories

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Abstract

Factorization algebras were developed to describe the algebraic structure on the space of local operators in perturbative field theories, inherited from homotopy-theoretic information of the underlying manifold and its tangent bundle. We give a relatively self-contained introduction to their definition, their construction from physical theories, and the necessary mathematical preliminaries involving higher category theory. Thereafter, we generalize our background spaces from manifolds to a large class of stratified spaces, including for example manifolds with corners, conifolds and complex varieties. Topological field theories are in this case described by constructible factorization algebras, which we extract in several exemplary cases by considering their BV-BRST complex as a constructible sheaf. Finally, we generalize our approach by defining BV data endowed with local (-1)-shifted symplectic structures on simplicial complexes, regular CW complexes, PL spaces and pseudomanifolds to which we can associate such algebras.

Zusammenfassung

Faktorisierungsalgebren wurden mit dem Zweck entwickelt, die algebraische Struktur auf dem Raum der lokalen Operatoren in perturbativen Feldtheorien zu beschreiben, welche von den homotopietheoretischen Eigenschaften der unterliegenden Mannigfaltigkeit und ihres Tangentialbündels abhängt. Wir geben eine Einführung in die Definition dieser Strukturen, ihre Konstruktion aus physikalischen Feldtheorien, sowie die notwendigen mathematischen Vorkenntnisse insbesondere in höherer Kategorientheorie. Damit ausgestattet gehen wir von Mannigfaltigkeiten als Hintergründen über zu einer großen Klasse an stratifizierten Räumen, welche insbesondere Mannigfaltigkeiten mit Ecken, Kegelpunkten und komplexe Varietäten enthält. Topologische Feldtheorien werden in diesem Fall von konstruierbaren Faktorisierungsalgebren beschrieben, welche wir an einer Reihe von Beispielen bestimmen indem wir den BV-BRST Komplex als konstruierbare Garbe beschreiben. Schließlich verallgemeinern wir unsere Vorgehensweise und führen BV-Theorien, ausgestattet mit lokalen (-1)-geshifteten Strukturen, auf Simplizialkomplexen, regulären CW-Komplexen, PL-Räumen und Pseudomannigfaltigkeiten ein aus denen sich solche Algebren konstruieren lassen.

Contents

Introduction

'Die Geisterwelt ist nicht verschlossen;
Dein Sinn ist zu, dein Herz ist tot!
Auf, bade, Schüler, unverdrossen
Die ird'sche Brust im Morgenrot!'

Wie alles sich zum Ganzen webt,
Eins in dem andern wirkt und lebt!
Wie Himmelskräfte auf und nieder steigen
Und sich die goldnen Eimer reichen!

Johann Wolfgang von Goethe, Faust I

The concept of *factorization algebras* has its origins in the study of Conformal Field Theories. In order to improve the unsatisfactory definition of a vertex operator algebra, Beilinson and Drinfel'd in [BDD04] introduced *chiral algebras* to axiomatically describe the space of chiral operators on an algebraic curve. These should be regarded as precursors of (locally constant) factorization algebras, the main difference being that they are defined as objects of algebraic geometry. In his work on extended topological field theories [Lur09], Lurie sketched an analogous definition on topological manifolds of *arbitrary* dimension instead of algebraic curves. We will work with this topological flavor exclusively, but should not forget to mention that factorization algebras have had a tremendous impact in algebra as well, in particular in Gaitsgory and Lurie's proof of a version of the Weil Conjectures for function fields [GL18].

A proper mathematical description of factorization algebras in the topological setting was finally worked out by Lurie in [HA], see Sections 5.4 and 5.5 and Appendix B. His methods have laid the foundation for a large body of subsequent work in this direction, and essentially established the subject of stratified homotopy theory which we present in Appendix B. Finally, the usefulness of factorization algebras in physics was pointed out in [CG16] not only for conformal, but even for arbitrary perturbative field theories, opening up an entirely new world of applications.

The goals of this thesis are the following:

- Above, we have pointed out three works that have had tremendous impact on the theory of factorization algebras. Surprisingly, their very definition is completely different in each of them: Beilinson-Drinfel'd view them as cosheaves on the Ran space, Lurie uses ∞-operads and the book of Costello and Gwilliam works with Weiss cosheaves. We try to clarify how these and other definitions are related, allowing us to switch around between them if this simplifies an argument.

- Our broad mathematical discussion of factorization algebras allows for a very streamlined transition to the physical point of view; we can optimize parts of the discussion in [CG16] by giving shorter and more conceptual proofs.
- We explain the definition of (constructible) factorization algebras on stratified spaces in [AFT14a], and deduce a few structure results on them that are helpful for subsequent applications in physics.
- Using examples of field theories on manifolds with boundary or corners, we point out a connection between constructible factorization algebras, constructible sheaves and the (extended) BV-BFV formalism [CMR14].
- We introduce a formalism for BV theories on very general classes of spaces, including simplicial complexes and regular cell complexes where it generalizes cellular BV-theory as in [CMR20], but also PL spaces and topological pseudomanifolds.

Since our methods involve a large volume of mathematical subjects, we have included an Appendix A on Higher Category Theory and Appendix B on stratified spaces and stratified homotopy theory, which we make use of extensively. We have tried to strike a balance between giving intuitive motivations and only insightful and short proofs, but including enough technicalities to be able to work make statements on a mathematically precise level. In particular, all notions we use will be properly defined, except for a few cases where this would be impractical (e.g. presentable ∞-categories); but some proofs will ignore technical details like higher coherences. The main text also involves some digressions into the realms of derived algebraic, differential and symplectic geometry as well as hermitian K-theory, but those are kept at a more basic level.

In Chapter 1, we motivate the need to work in the realm of ∞-categories for the remainder of the text. Using the example of Chern-Simons theory that we explicitly develop, we explain a derived geometric point of view on the BV-BRST formalism and how it is used to describe perturbative aspects of classical field theories. Many other examples are included as well.

Chapter 2 begins by motivating the mathematical definition of factorization algebras as a method of describing physical operators. Afterwards, we introduce the large amount of different definitions we have teased, and sketch the reasons why they are equivalent. Of course, we equip the reader with many examples and helpful statements to build a working intuition.

We combine this mathematical theory with physics in 3 by explaining how the factorization algebra of classical (and, in the free case, of quantum) operators in a given field theory can be deduced from the BV-BRST complex, and how it can be used to extract information about the field theory.

Finally, chapter 4 defines what a (constructible) factorization algebra on a conically smooth stratified space is, developing basic properties and examples; and Chapter 5 motivates their application in physics in many specific situations. Our main claim is that the BV-datum of a free field theories on a large class of spaces should consist of a constructible ∞-sheaf with

- A stalk-wise finiteness condition,
- A local (-1)-shifted symplectic structure on the interior of strata,
- Satisfying a Poincaré-Lefschetz style duality between multiple strata.

We make this explicit for simple situations, in particular simplicial complexes, in 5.7 and 5.8.

How to read

The following diagram captures (up to a few exceptions) the dependencies between the respective chapters. The left column contains most of the physical applications, while the middle column is about factorization algebras as mathematical object and the right column contains the mathematical background.

$$\begin{tikzcd}
\text{Physics} & \text{Mathematics} & A \arrow[dl] \arrow[d] \\
1 \arrow[d] & 2 \arrow[dl] \arrow[d] & B \arrow[dl] \\
3 \arrow[d] & 4 \arrow[dl] & \\
5 & &
\end{tikzcd}$$

To understand the central concepts without most of the technical subtleties, or to obtain a quick overview over the covered material, we recommend the following accelerated reading path: In Chapters 1 and 2, we have used the first subsection to motivate, and the last section to conclude the chapter, so reading them clarifies the main ideas. The examples in 2.4 are also indispensable for further understanding.

In Chapter 3, the first two sections contain generalities on the BV formalism and free field theories, and the examples in the remainder of the chapter may at first be skipped. Conversely, in Chapter 4, the definitions are very technical while the examples in 4.4 and 4.5 are helpful to go through. Finally, after 5.1, 5.2 and the example of Topological Quantum Mechanics in 5.3, the remaining sections in Chapter 5 are mostly independent (although ordered by increasing complexity) and may be read in order of interest.

What is new?

To our knowledge, the following short results can not be found in the previous literature, although many of them might be considered folklore:

- The characterization of absolute factorization algebras as relative factorization algebras compatibly on all manifolds 2.3.6
- An intuitive multipath picture for constructible sheaves on the Ran space in 2.5
- The construction of constructible factorization algebras from constructible sheaves in 4.3.8
- Our reasoning that a BV datum on a stratified spaces should consist of a constructible sheaf (with extra structures) in 5.1.4
- Our self-gluing proposition for Poincaré objects in 5.2.17
- The statements 5.3.1, 5.3.4 and 5.3.5 on Topological Quantum Mechanics
- Our discussion of the scalar field 5.5, although as mentioned there it is motivated by similar discussions in slightly different settings
- The speculation on Hamiltonian Field Theory and boundaries of Verdier self-dual sheaves in 5.5.7
- The calculation of the exit-path category in the case of regular CW complexes in B.2.18

More substantially, the applications of our reasoning about BV-data and constructible sheaves to simplicial field theories in 5.7 as well as the further ideas in 5.8 are completely new (motivated by similar constructions in surgery theory and algebraic L-theory).

Finally, we have tried to streamline and improve on the discussion of many covered subjects in the current literature; in particular many proofs were substantially changed or completely rewritten. Since a large part of this text is about examples, some of them are not covered anywhere else, e.g. the calculation of Hochschild Homology with monodromy in 3.3.9 and the definition of Witt spaces via tangential structures in 4.2.7.

Notation and Conventions

- We denote the natural numbers including zero by $\mathbb{N}_0$, and excluding zero by $\mathbb{N}^+$ to avoid confusion.
- CW complexes are always locally finite.
- Unless stated otherwise, we always use cohomological grading for chain complexes. The grading increases from left to right, and the shift acts as $C[1]_{-1} = C_0$.

- We work with Grothendieck pretopologies instead of Grothendieck topologies.
- In an adjunction, the upper arrow is always the left adjoint.
- We ignore size arrows of ordinary and ∞-categories; however they all disappear if we fix a small and a large Grothendieck universe.
- The term "∞-categories" always refers to $(\infty, 1)$-categories; and "ordinary categories" refers to 1-categories. If they have a name that does not just consist of a single letter, we try to distinguish them by starting with a calligraphic letter in the first, but not the second case, as in $\mathcal{D}isk$ and Disk. We do not attempt this for single letters as it would lead to confusions with objects in the respective categories. We similarly write 2-categories in boldface and $(\infty, 2)$-categories boldface with a calligraphic first letter.
- Our model of choice for ∞-categories are quasi-categories, as developed in [HTT].
- Higher categories are always weak, not strict.
- We try to always clarify whether we are working with ordinary categories or ∞-categories, either explicitly or by context. When in doubt, we are usually working in the ∞-setting.
- Unless it is clear and clutters the notation, we denote symmetric monoidal or ∞-operadic structures on ∞-categories by a superscript, as in $\mathcal{V}^{\otimes}$ if $\otimes$ is a symmetric monoidal product on $\mathcal{V}$.
- Unlike [CG16], we only work with factorization algebras valued in sifted complete symmetric monoidal ∞-categories instead of more general ∞-operads, and our factorization algebras are always multiplicative (and not lax) in the sense that $A(U) \otimes A(V) \cong A(U \sqcup V)$ instead of there just being a canonical morphism in one direction.

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1. BV-BRST and Classical Chern-Simons Theory

In this chapter, we give an introduction to the perturbative treatment of classical field theories via the BV formalism, using Chern-Simons Theory as an example. The underlying derived geometry is introduced in an intuitive way, following [Yoob], [Cal14] and omitting most technicalities. Finally, we foreshadow how factorization algebras enter the picture.

1.1. Chern-Simons Theory

Let M be a connected and oriented smooth 3-manifold without boundary, and G a compact Lie group. In what follows, we will define a classical field theory on M with gauge symmetry, called (classical) Chern-Simons theory with gauge group G, and find its equations of motion as well as their solutions (while taking into account the gauge symmetry). Our discussion roughly follows [Fre92] and [Kel19].

Proposition 1.1.1. If G is simply connected, then every principal G-bundle $P \to M$ is trivializable.

Proof. Isomorphism classes of principal G-bundles are in bijective correspondence with homotopy classes of maps into the classifying space of G,

$$\{\text{Principal } G\text{-bundles on } M\}\big/_{\cong} \cong [M, BG]\ . \tag{1.1}$$

But since BG is a delooping, $\pi_n BG \cong \pi_{n-1} G$ and $\pi_0 G = \pi_1 G = \pi_2 G = 0$ for a compact simply connected Lie group. If we model M by a 3-skeletal CW-complex, this means that there can be no obstructions to retracting a continuous map $M \to BG$ to a point, meaning that all principal G-bundles on M must be isomorphic to the trivial bundle. $\square$

Corollary 1.1.2. Any orientable 3-manifold is parallelizable.

Proof. We need to show that the tangent classifier $\tau_M : M \to \mathrm{BO}(3)$, i.e. the map into the classifying space $\mathrm{BO}(3)$ of smooth vector bundles of rank 3 classifying the tangent bundle, can be lifted to $\mathrm{BSpin}(3)$:

$$\begin{tikzcd} & \mathrm{BSpin}(n) \arrow[d] \\ & \mathrm{BSO}(n) \arrow[d] \\ M \arrow[r, "\tau_M"] \arrow[ur, dashed] \arrow[uur, dashed] & \mathrm{BO}(n) \end{tikzcd}$$

Then, we can apply above proposition since Spin(3) is simply connected. By obstruction theory, we need to make sure that the first and second Stiefel-Whitney classes vanish, and this follows since M is an orientable 3-manifold and therefore by a general fact (a calculation with Wu classes) admits a spin structure. □

However, there is a very important gauge group we would like to consider that is not simply connected – namely, we will later take a close look at abelian Chern-Simons-Theory, and the only abelian Lie groups are products of $\mathbb{R}$ and U(1). The latter factors are a problem, and solving it is a very subtle matter, see [Kel19] 6.1, 6.2 and the pointers therein (however, the fact that $\mathbb{R}$ is not compact will be no problem for us). In this work, we will mostly talk about *perturbative* aspects of Chern-Simons theory, and it will turn out that those do only depend on the Lie algebra $\mathfrak{g}$ of G.

Disclaimer 1.1.3. In the following, by *abelian Chern-Simons theory* we always mean $\mathbb{R}$-Chern-Simons theory, since it is perturbatively indistinguishable from the U(1)-theory.

Let G from now on be a simply connected Lie group and $P \to M$ therefore necessarily trivializable. Fix a trivialization, given by a global section $\sigma : M \to P$. We start by introducing the dynamical field of our theory:

Definition 1.1.4. A *connection* on a principal G-bundle $P \to M$ is a Lie-algebra-valued 1-form $A \in \Omega^1(P, \mathfrak{g})$ such that:

- For any $g \in G$, the right translation $R_g^*\omega = \mathrm{ad}_{g^{-1}}\,\omega$
- For all $X \in \mathfrak{g}$ and $p \in P$, the differential of the map $L_p : G \to P,\ g \mapsto p \cdot g$ yields a corresponding *vertical vector* $\tilde{X}_p := dL_p(X) \in T_pP$. We require $\omega(\tilde{X}_p) = X$.

Definition 1.1.5. The *adjoint vector bundle* $P \times_G \mathfrak{g} \to M$ associated to P is the vector bundle obtained by taking the quotient of $P \times \mathfrak{g}$ by the group action of G defined by

$$(p, X) \cdot g := (p \cdot g, \mathrm{ad}_g X)\ . \tag{1.2}$$

Similarly, the *adjoint principal bundle* $P \times_G G \to M$ is the quotient of $P \times G$ by the action

$$(p, h) \cdot g := (p \cdot g, ghg^{-1})\ . \tag{1.3}$$

Proposition 1.1.6 ([Kel19, Lemma 2.6 and Corollary 2.1]). A $\mathfrak{g}$-valued n-form on a G-principal bundle P that is invariant under right translation via G, and vanishes if one of its entries is vertical, is the same thing as an n-form on M with values in $P \times_G \mathfrak{g}$:

$$\Omega^n(P, \mathfrak{g})^G_{hor} \cong \Omega^n(M, P \times_G \mathfrak{g}) \tag{1.4}$$

In particular, this means that the space of connections $\mathrm{Conn}(M)$ is an affine space built from the vector space $\Omega^1(M, P \times_G \mathfrak{g})$. Since P is always trivial for us, we obtain an identification $\Omega^n(P, \mathfrak{g})^G_{hor} \cong \Omega^n(M, \mathfrak{g})$ so that a connection is completely determined by its restriction $A := \sigma^*\omega \in \Omega^1(M, \mathfrak{g})$ for $\sigma : M \to P$ a fixed section (= trivialization). We will freely switch between those two points of view.

For $X, Y, Z \in \mathfrak{g}$ with coefficients $X = X^a T_a$ in a basis $(T_a)_{a=1,\dots,r}$ of $\mathfrak{g}$, we denote by $\langle X, Y\rangle = k^{ab} X_a Y_b$ the (symmetric) Killing form on $\mathfrak{g}$. Also, we remind of the total antisymmetry of the cocycle $\langle X, [Y, Z]\rangle = f_{abc} X^a Y^b Z^c$ where f^a_{bc} are the structure constants. When tensoring with the exterior product, we obtain on $\mathfrak{g}$-valued 1-forms an antisymmetric expression $\langle - \wedge -\rangle$ and a symmetric expression $\langle - \wedge [- \wedge -]\rangle$.

Definition 1.1.7. A connection ω on $P \to G$ induces a *covariant exterior derivative*

$$D_\omega = d + [\omega \wedge -] : \Omega^n(P, \mathfrak{g}) \to \Omega^{n+1}(P, \mathfrak{g}) \ . \tag{1.5}$$

We identify it with the corresponding expression in a trivialization, $D_A = d + [A \wedge -] : \Omega^n(M, P \times_G \mathfrak{g}) \to \Omega^{n+1}(M, P \times_G \mathfrak{g})$.

Definition 1.1.8. The *curvature* $F_A = \sigma^*\Omega_A$ of the connection A is given by

$$\Omega_A = d\omega + \frac{1}{2}[\omega \wedge \omega] \in \Omega^2(P, \mathfrak{g})^G \ ; \quad F_A = dA + \frac{1}{2}[A \wedge A] \in \Omega^2(M, P \times_G \mathfrak{g}) \ . \tag{1.6}$$

It satisfies the *Bianchi identity*

$$D_A F_A = d^2 A + [dA \wedge A] + [A \wedge dA] + \frac{1}{2}[A \wedge [A \wedge A]] = 0 \, , \ D_\omega \Omega = 0 \ . \tag{1.7}$$

Definition 1.1.9. A *gauge transformation* of $P \to M$ is an automorphism $\phi : P \xrightarrow{\cong} P$ as a principal G-bundle over M. The group $\mathcal{G}(P)$ of gauge transformations acts on the space of connections $\mathrm{Conn}(P)$ via $A \mapsto \phi^* A$.

Remark. One can identify $\mathcal{G}(P) \cong C^\infty(P, G)^G \cong \Omega^0(M, P \times_G G)$ by sending, for $\sigma_i : M \supseteq U_i \to P|_{U_i}$ a local trivialization of P,

$$(p \mapsto p \cdot \hat{\chi}(g)) \leftarrow\!\shortmid (\hat{\chi} : P \to G) \mapsto \text{gluing of } (\sigma_i^* \hat{\chi})_i \quad . \tag{1.8}$$

Again, since P is trivial in our case, $\mathcal{G}(P) \cong C^\infty(M, G)$.

Definition 1.1.10. Classical Chern-Simons theory of level $k \in \mathbb{Z}$ on M is a classical field theory with field content a connection $\omega \in \mathrm{Conn}(P)$, $A = \sigma^*\omega \in \Omega^1(M, \mathfrak{g})$, and action

$$\begin{aligned} S_{CS} :& \mathrm{Conn}(P) \to \mathbb{R}/\mathbb{Z} \\ A \mapsto & S_{CS}[A] := \frac{k}{4\pi} \int_M \sigma^* \left(\langle \omega \wedge d\omega \rangle + \frac{1}{3} \langle \omega \wedge [\omega \wedge \omega] \rangle \right) = \\ &= \frac{k}{4\pi} \int_M \langle A \wedge dA \rangle + \frac{1}{3} \langle A \wedge [A \wedge A] \rangle =: \frac{k}{4\pi} \int_M \mathrm{cs}(A) \quad . \end{aligned} \tag{1.9}$$

Gauge transformations are given by smooth G-valued functions $\chi \in \mathcal{G} \cong C^\infty(M, G)$, acting on the connection via

$$A \mapsto \mathrm{ad}_{\chi^{-1}} A + \chi^*\theta = \chi^{-1} A \chi + \chi^{-1} d\chi \, . \tag{1.10}$$

Here, θ is the Maurer-Cartan-form on G, and the last expression is a simpler description in matrix Lie groups.

Proof of gauge-invariance. One obtains this expression for the gauge action by going through the isomorphisms in 1.8. It is important to show that S_{CS} does not depend on the trivialization σ, and is indeed gauge invariant. The former follows from the latter, since for a different section $\sigma' : M \to P$ the unique map $d : M \to G$ that translates $\sigma(m) \cdot d(m) = \sigma'(m)$ for all $m \in M$ is a gauge transformation.

For gauge invariance, we give a quick and dirty argument and refer to [FSS15], sections 2.1 and 2.2 for a broader discussion.

We assume that M is compact and use a theorem of Rohlin which states that every compact 3-manifold is nullbordant, so we can find a compact oriented 4-manifold 4 such that $\partial W = M$. If A, A' are two gauge equivalent connections on M, one can show that they can be extended to connections $\hat{A}', \hat{A}''$ on two copies W^+, W^- of W. Gluing those together along a collar, we obtain a closed oriented 4-manifold $W^+ \cup_{M \times \mathbb{R}} W^-$ with connection A'' (the connections can be glued as well because they are gauge equivalent). At this point, we notice that $cs(A'')$ is an antiderivative of the first Pontrjagin (= second Chern) class, if we define it via Chern-Weil theory:

$$d\,\mathrm{cs}(A'') = \langle dA'' \wedge dA'' \rangle + \langle dA'' \wedge [A'' \wedge A''] \rangle = \langle F_{A''} \wedge F_{A''} \rangle = p_1 \tag{1.11}$$

Notice that the term $\langle [A'' \wedge A''] \wedge [A'' \wedge A''] \rangle$ vanishes because of the Jacobi identity in $\mathfrak{g}$. This does not mean that p_1 is exact, since the expression for A is only locally defined. Now, we use Stokes:

$$\begin{aligned} S_{CS}[A] - S_{CS}[A'] &= \frac{k}{4\pi} \int_{\partial W} (cs(A) - cs(A')) = \\ = \frac{k}{4\pi} \left(\int_{W^+} p_1 - \int_{W_-} p_1 \right) &= \frac{k}{4\pi} \int_{W^+ \cup_{M \times \mathbb{R}} W^-} p_1 = k \cdot p_1[W] = 0 \in \mathbb{R}/\mathbb{Z} \end{aligned} \tag{1.12}$$

$\square$

Next, we take a look at the equations of motion. Variation with respect to A yields:

$$\begin{aligned}0 \overset{!}{=} \frac{4\pi}{k}\delta S = &\int_M \langle \delta A \wedge dA\rangle + \langle A \wedge d\delta A\rangle + \langle \delta A \wedge [A \wedge A]\rangle = \\ =&2\int_M \langle \delta A \wedge F_A\rangle + \int_{\partial M} \langle \delta A \wedge A\rangle\end{aligned} \tag{1.13}$$

We use the total symmetry of the expression $\langle . \wedge [. \wedge .]\rangle$ in the first step, and partial integration in the second step ($d\delta A = \delta d A$ by definition of a variation via jet prolongation). Since the Killing form is non-degenerate and $\partial M = \emptyset$, solutions of the Euler-Lagrange equation are precisely the flat connections on our principal bundle P.

Perturbative methods rely on deforming a given background solution A_0 inside the space of solutions. Therefore, let us write an arbitrary connection as $A = A_0 + \alpha$, where $F_{A_0} = 0$, and regard $\alpha \in \Omega^1(M, \mathfrak{g})$ as the dynamical field when varying the action:

$$\begin{aligned}0 \overset{!}{=} &\frac{4\pi}{k}\delta S = \int_M \langle \delta\alpha \wedge dA\rangle + \langle A \wedge d\delta\alpha\rangle + \langle \delta\alpha \wedge [A \wedge A]\rangle = \\ =&\int_M 2\langle \delta\alpha \wedge F_{A_0}\rangle + \delta\alpha \wedge d\alpha + \delta\alpha \wedge ([\alpha \wedge \alpha] + 2[\alpha \wedge A_0]) + \int_{\partial M} \langle \delta\alpha \wedge A\rangle = \\ =&\int_M 2\langle \delta\alpha \wedge \left(D_{A_0}\alpha + \frac{1}{2}[\alpha \wedge \alpha]\right)\rangle\end{aligned} \tag{1.14}$$

If we assume that α is only an infinitesimal variation (which is the realm of perturbation theory), the Euler-Lagrange equations are $D_{A_0}\alpha = 0$. The perturbative picture also only cares about infinitesimal gauge transformations, parametrized by $\gamma \in \mathrm{Lie}(\mathcal{G}) = \mathrm{Lie}(\Omega^0(M, \mathfrak{g})) = \Omega^0(M, \mathfrak{g})$.

For simplicity, let us assume G is a matrix group and write $\chi = 1 - \gamma$. Then,

$$A \mapsto (1+\gamma)A(1-\gamma) + (1+\gamma)d\gamma = A + d\gamma + [A, \gamma] \tag{1.15}$$

This holds generally by a similar calculation, and for $A = A_0 + \alpha$ with α, γ infinitesimal we see that the Lie algebra action $\Omega^0(M, \mathfrak{g}) \to \Omega^1(M, \mathfrak{g})$ is given by $\gamma \mapsto D_{A_0}\gamma$.

Definition 1.1.11. Perturbative Chern-Simons theory on a principal G-bundle $P \to M$ around the fixed background solution A_0 has a dynamical field $\alpha \in \Omega^1(M, P \times \mathfrak{g})$ with equations of motion

$$D_{A_0}\alpha = 0\ , \tag{1.16}$$

and gauge transformations parametrized by $\gamma \in \Omega^0(M, P \times \mathfrak{g})$ given by

$$\alpha' = \alpha + D_{A_0}\gamma\ . \tag{1.17}$$

It seems like perturbative Chern-Simons theory is intimately related to the chain complex $(\Omega^\bullet(M, \mathfrak{g}), D_{A_0})$, where $D_{A_0}^2 = 0$ because A_0 is flat. In fact, if we introduce a source $J \in \Omega^2(P, \mathfrak{g})$ that enters the equations of motion $D_A A = J$, we need to impose the Bianchi identity $D_A J = 0$. The above chain complex hence contains gauge transformations in degree 0, fields in degree 1, and sources in degree 2. Let us investigate this further.

1.2. Derived Geometry and the Lagrangian Formalism

1.2.1. Constraints and Gauge Symmetries

To deepen our understanding of Chern-Simons theory, we need to take a detour and look at general Classical Field Theories. Many subtleties arising in their Lagrangian description are often completely overlooked in the usual treatment, and our goal will be to give a fairly general formulation of it that includes gauge symmetries, as well as a field theory analogue of *Dirac's theory of constraints* from classical mechanics. Our main tool for this will be *derived geometry*, more precisely derived algebraic geometric since we do not want to think about functional analytic subtleties.

A short reminder on what the issues with a more naive description even are. In classical mechanics, given a Lagrange function $L(q, \dot{q})$, we define the canonical momentum $p_i := \frac{\partial L(q,\dot{q})}{\partial \dot{q}_i} = p_i(q, \dot{q})$. The main problem when trying to pass to a Hamiltonian description is to invert this into $\dot{q}_i(q, p)$, so that the Hamiltonian

$$H(q,p) = \sum_i p_i \cdot \dot{q}_i(q,p) - L(q, \dot{q}(q,p)) \tag{1.18}$$

can be defined. By the inverse function theorem, this can only be done when the Hessian $\frac{\partial^2 L(q,\dot{q})}{\partial \dot{q}_i \dot{q}_j}$ is non-degenerate. If it is not, one can remedy this by adding extra variables that act as Lagrange multipliers, implementing the non-invertible equations as constraints.

A similar issue arises in field theory. Since this is our main case of interest, let us fix some notation: On a (smooth) spacetime manifold by M, let $\mathcal{F}$ denote the off-shell space of fields, which is for all intents and purposes given as the space of global sections of (or connections on) some fiber bundle on M. For Klein-Gordon theory, as an example, $\mathcal{F} = \Gamma(M, M \times \mathbb{C}) = C^\infty(M, \mathbb{C})$. An action is a smooth function $S : \mathcal{F} \to \mathbb{R}$, where the smooth structure on $\mathcal{F}$ can be constructed as in [CG16] via the language of differentiable vector spaces - let us not think too much about technicalities right now.

The usual description of quantum field theory, via the path integral formalism, uses the propagator of our theory. This is the inverse of the (functional) Hessian $\frac{\delta^2 S[\phi]}{\delta\phi(x)\delta\phi(y)}$, where $\phi(x)$ should be some coordinates on $\mathcal{F}$ – our notation should hopefully be clear when comparing to the Klein-Gordon case. Issues arise for example in Maxwell theory, where the non-invertibility along the transverse direction must be accounted for in some way (e.g. using the Gupta-Bleuler formalism). Generally, the Hessian should be non-degenerate!

Similar issues also arise in general Yang-Mills-theory, where they are usually resolved by the introduction of Fadeev-Popov-ghosts, or in the so called BRST-formalism. However, this formalism can not deal with situations where gauge symmetry gives rise to a non-integral distribution on the space of fields because it only closes off-shell; this happens for the Poisson sigma model and nonabelian BF-theory in dimensions > 3 [CMR14]. This also involves the distinction between primary and secondary constraints in Dirac's theory, we refer to [HT92] for an extensive treatment. To take care of these cases we will introduce the BV-BRST formalism – but let us first summarize our problems:

- Given a classical field theory, the equations of motion are given by the Hamiltonian principle $dS = 0$, i.e. by extremizing the action. We note that, since $S \in C^\infty(\mathcal{F})$, we have $dS \in \Omega^1(\mathcal{F})$, where d is the exterior differential.
- The *covariant phase space* X of our physical theory is the space of solutions of this set of partial differential equations. In other words, it is the intersection of the section dS of the cotangent bundle with the zero section inside $T^*\mathcal{F}$:

$$X = \mathrm{Graph}(dS) \times_{T^*\mathcal{F}} \mathcal{F} \tag{1.19}$$

- Problems arise if the Hessian of S is degenerate on this intersection, in other words, if this intersection is non-transverse.

There seems to be an obvious first step to improve our situation: Since a group of gauge symmetries $\mathcal{G}$ acting on $\mathcal{F}$ introduces a large amount of redundancy in our description, it would be wise to descend to the (set-theoretic) quotient $\mathcal{F}/\mathcal{G}$. But $\mathcal{G}$ generally can not be expected to act freely and transitively, so this procedure destroys the smooth structure that we need to even write down a differential equation. To proceed further, we will define a nicer quotient; this procedure turns out to be equivalent to the BRST formalism. Mathematically realizing this, and the full BV-BRST formalism, requires a small detour.

1.2.2. Derived Stacks

In fact, we will need an entirely new type of geometry. To simplify the mathematics, we retreat to the algebraic world, looking at schemes and varieties instead of manifolds (a differential geometric framework is currently work in progress, see [CS19] and [Ste]). We follow the notes [Cal14] and [Yooa].

Non-transverse intersections are problematic in the algebraic world, as well. We calculate the following intersection inside $\mathbb{A}^2_{\mathbb{C}}$:

$$\begin{aligned} \{x=0\} \cap \{x=a\} &= \mathrm{Spec}\left(\frac{\mathbb{C}[x,y]}{(x)}\right) \times_{\mathrm{Spec}(\mathbb{C}[x,y])} \mathrm{Spec}\left(\frac{\mathbb{C}[x,y]}{(x-a)}\right) \cong \\ &\cong \mathrm{Spec}\left(\frac{\mathbb{C}[x,y]}{(x)} \otimes_{\mathbb{C}[x,y]} \frac{\mathbb{C}[x,y]}{(x-a)}\right) \cong \mathrm{Spec}\,\frac{\mathbb{C}[y]}{(0-a)} \cong \begin{cases} \emptyset, & \text{for } a \neq 0 \\ \{x=0\}, & \text{for } a = 0 \end{cases} \end{aligned} \tag{1.20}$$

It is very puzzling how the dimension of this intersection jumps between the transverse and non-transverse case. Similar issues arise all over intersection theory. They can be remedied by a simple, yet extremely surprising step that involves replacing $\otimes$ by the derived functor $\otimes^L$:

$$\begin{aligned}\{x=0\}\cap^R\{x=a\} &= \operatorname{Spec}\left(\frac{\mathbb{C}[x,y]}{(x)}\otimes^L_{\mathbb{C}[x,y]}\frac{\mathbb{C}[x,y]}{(x-a)}\right)\cong\\ &\cong \operatorname{Spec}\left(\left(\mathbb{C}[x,y]\xrightarrow{\cdot x}\mathbb{C}[x,y]\right)\otimes_{\mathbb{C}[x,y]}\frac{\mathbb{C}[x,y]}{(x-a)}\right)\cong\\ &\cong \operatorname{Spec}\left(\mathbb{C}[y]\xrightarrow{\cdot a}\mathbb{C}[y]\right)\overset{qis}{\simeq}\begin{cases}\operatorname{Spec}0=\emptyset, & \text{for } a\neq 0\\ \operatorname{Spec}\left(\mathbb{C}[y]\xrightarrow{0}\mathbb{C}[y]\right)=\mathbb{A}_{\mathbb{C}}[1]\times\mathbb{A}_{\mathbb{C}}, & \text{for } a=0\end{cases}\end{aligned}\tag{1.21}$$

We have resolved $\mathbb{C}[x,y]/(x)$ as a $\mathbb{C}[x,y]$-algebra via a commutative differential graded algebra, its Koszul resolution. Let us for later reference also write this resolution as $\mathbb{C}[x,y,\xi]$, where we introduce a variable ξ of (homological) degree -1 satisfying $d\xi=x$. Note that by graded symmetry, $\xi^2=0$. Then, the result for $a=0$ is $\operatorname{Spec}\mathbb{C}[y,\xi]$, resembling super-affine space $\mathbb{A}^{1|1}_{\mathbb{C}}$.

We notice that the derived intersection is equipped, in this formalism, with a non-zero entry in degree -1 that, in spirit (e.g. calculating an Euler characteristic), cancels out the one-dimensional scheme in degree 0. To put it in pictorial terms, there is an invisible anti-line lying over the $\{x=0\}$ line. Similar considerations lead to extremely elegant results in intersection theory, like a refinement of Bézout's theorem.

But what even is the spectrum of a chain complex? It can only depend on the complex up to quasi-isomorphism, since $\otimes^L$ can not be defined more precisely. Also, evidently, we shouldn't think about chain complexes, but about commutative differential graded algebras in non-positive degrees (and resolve by those). Let us denote the category of those by $\mathrm{cdga}_{\leq 0}$. In fact, we have to Dwyer-Kan-localize (in the sense of A.2.3) this category at the quasi-isomorphisms, yielding an ∞-category that enhances the corresponding derived category, but again – let us not get too technical here.

Now, remember that the category of affine schemes Aff is just the opposite of the category of commutative rings cRing. Similarly, let us set the category of affine derived schemes $\mathrm{dAff} := \mathrm{cdga}^{op}_{\leq 0}$. More general schemes are defined as abstract gluings of these affine schemes, via their functor of points: A scheme is just a functor $\mathrm{Aff}^{op}=\mathrm{cRing}\to\mathrm{Set}$ that satisfies some additional conditions, in particular Zariski descent. By the universal property of the presheaf category, such functors can be understood as abstract gluings of affine parts along open inclusions, just as we wanted.

Definition 1.2.1. A *derived scheme* is a functor $X:\mathrm{dAff}^{op}=\mathrm{cdga}_{\leq 0}\to\mathrm{Set}$ that satisfies a version of Zariski (or étale) descent.

While this fixes our problems with intersections, we still need to find a good notion of quotient in the category of schemes. Again, the answer to this is as abstract as it is surprising: Instead of losing information by identifying field configurations that are gauge-equivalent, we should just keep them distinct, but somehow mark the fact that there is a gauge transformation identifying them in the back of our minds. In other words, we should not only think about individual points of phase space, but also about (invertible) arrows that join gauge equivalent points.

Mathematically, the functor of points of phase space should not take values in sets, but in groupoids, where the arrows in the groupoids are gauge transformations sending the source to the target (family of) field configurations. Since we already work with ∞-categories anyway, we might as well add 2-morphisms as gauge-of-gauge transformations, and so on:

Definition 1.2.2. A *derived stack* is a functor $X : \mathrm{dAff}^{op} = \mathrm{cdga}_{\leq 0} \to \mathcal{S}$, where $\mathcal{S}$ is the ∞-category of spaces (i.e. ∞-groupoids). It should also satisfy Zariski/ étale descent, but we will not need this in the following.

1.2.3. The Derived Covariant Phase Space

Equipped with this formalism, let us solve our transversality problem.

- In the situation above, assume that $\mathcal{F}$ and $\mathcal{G}$ are derived stacks instead of smooth manifolds (e.g. they could be varieties) and let us write $\mathcal{F}' := [\mathcal{F}/\mathcal{G}]$ for the *stacky quotient* of the space of (off-shell) field histories by the gauge symmetries – we won't explain what that means, but maybe the discussion of stacks above gives a hint of an idea. Since the action is gauge invariant, we should be able to identify S with a function on $\mathcal{F}'$.
- We replace the usual critical locus of the action by the *derived critical locus*, calculated as a derived intersection of the graph of the section dS in the cotangent bundle with the graph of the zero section:

$$X = \mathrm{dCrit}(S) = \mathrm{Graph}(dS) \times^R_{T^*\mathcal{F}'} \mathcal{F}' \ . \tag{1.22}$$

We call the resulting derived stack X the *derived covariant phase space* of our classical field theory; we will often simply call it the phase space (it should not be confused with the Hamiltonian phase space from 5.5.7).

Up to this point, we have always worked on the whole spacetime manifold M. To better understand locality and local-to-global properties, we could do the same discussion on an open subset $U \subseteq M$, provided we know how to restrict $\mathcal{F}$ to a space of field histories in U, and similarly S and $\mathcal{G}$. In particular, this works if $\mathcal{F}$ is given by the space of global sections of a vector bundle on M and $\mathcal{G}$ and S are sufficiently local – as almost always in physics.

The same discussion as above then yields a derived covariant phase space $X(U)$ for the restricted classical field theory on U, and in good cases the association $U \mapsto X(U)$ will be a (higher) sheaf of derived stacks.

1.3. The BV-BRST complex

The goal of perturbation theory is to study field histories that are only small perturbations of a fixed background field configuration $\phi \in X$. In fact, we want to make these perturbations infinitesimal so that we can perform a power series expansion in them – an algebraic geometer would say we want to look at the *formal neighborhood* of ϕ in X.

Technical Remark. To put this yet in other words, we want to make the global moduli problem of solving the Euler-Lagrange equations into a formal moduli problem, studying deformations of a fixed geometric point. This means that the functor of points $X : \mathrm{cdga}_{\leq 0} \to \mathcal{S}$ should be restricted to the subcategory of differential graded Artin rings, that restrict to the geometric point ϕ on the residue field.

This seems very difficult to describe, but luckily, a deep result in deformation theory comes to our rescue. While the idea can be traced back to Quillen, its modern formulation stems from Lurie's [SAG]. It roughly states: *Formal moduli problems are the same thing as L_∞-algebras!*

Because of this statement, it is enough for perturbation theory to look at the so-called *tangent complex* $\mathcal{E} := \mathbb{T}_\phi X$ of X at the background solution ϕ, which generalizes the tangent space of a manifold and captures information about deformation theory to all orders (not just linear deformations!). It is also called the *BV-BRST complex*, or simply BV-complex. We try to give a rough explanation on what these terms actually mean, and refer to the excellent introduction [Ane18] for more information.

Technically, one defines this tangent complex as the dual of the cotangent complex, which itself is defined as the non-abelian derived functor of taking Kähler differentials (see A.8). In particular for an ordinary scheme, it agrees with the cotangent complex from deformation theory. In the case of smooth varieties, the tangent complex is concentrated in degree 0 where it agrees with the usual tangent space; meaning that it is not necessary to keep track of higher order deformations in the smooth case so the tangent space is really a linear approximation (as it is for manifolds).

Definition 1.3.1. An *L_∞-algebra* is a $\mathbb{Z}$-graded vector space L equipped with multilinear operations, called higher Lie brackets

$$\ell_n = [-, \ldots, -] : L^{\otimes n} \to L[2-n],\ n \in \mathbb{N}_{\geq 1}\ . \tag{1.23}$$

These should preserve the grading (in other words, ℓ_n is a map of degree $2-n$ to L) and satisfy, for $x_1, \ldots, x_n \in L$ with $x_i \in L_{|x_i|}$ homogeneous:

- *Graded Antisymmetry:* For all $n \geq 2$,

$$\ell_n(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = -(-1)^{|x_i||x_{i+1}|}\ell_n(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n) \tag{1.24}$$

- *Graded Jacobi Identity:* For all $n \geq 1$,

$$0 = \sum_{k=1}^{n} (-1)^k \sum_{\epsilon \in \mathrm{UnShuff}(k,n-k)} \mathrm{sign}^{(|x_1|,\dots,|x_n|)}(\epsilon)\, \ell_{n-k+1}(\ell_k(x_{i_1}, \dots x_{i_k}), x_{j_1}, \dots, x_{j_{n-k}}) \tag{1.25}$$

Here, ϵ goes through all unshuffles, which are permutations of the form

$$\begin{pmatrix} 1 & 2 & \dots & k & k+1 & \dots & n \\ i_1 & i_2 & \dots & i_k & j_1 & \dots & j_{n-k} \end{pmatrix} , \tag{1.26}$$

with $i_1 < \dots i_k$ and $j_1 < \dots < j_{n-k}$; and $\mathrm{sign}^{(|x_1|,\dots,|x_n|)}(\epsilon)$ is the graded anti-symmetric sign of this permutation, in the sense that transposing x_1 and x_2 acquires the sign $-(-1)^{|x_1||x_2|}$.

Example 1.3.2.

- An L_∞-algebra concentrated in degree 0 is an ordinary Lie algebra.
- An L_∞-algebra where all brackets $\ell_n = 0$ vanish for $n > 1$ is a chain complex, where ℓ_1 is the differential and $\ell_1^2 = 0$ because of the Jacobi Identity with $n = 1$.
- An L_∞-algebra with $\ell_n = 0$ for $n > 2$ is a *differential graded Lie algebra*, and if $\ell_1 = 0$ a *graded Lie algebra.* The Leibniz rule follows from the Jacobi Identity.
- For $n = 1, 2, 3$, the higher Jacobi Identities can be written down as

$$\begin{aligned} & d^2 = 0 , \quad [dx_1, x_2] - (-1)^{|x_1||x_2|}[dx_2, x_1] = d[x_1, x_2] , \\ & [[x_1, x_2], x_3] \pm [[x_2, x_3], x_1] \pm [[x_1, x_3], x_2] = \\ & \quad = [dx_1, x_2, x_3] \pm [dx_3, x_1, x_2] \pm [dx_2, x_1, x_3] + d[x_1, x_2, x_3] , \end{aligned}$$

where the $\pm$-signs in the last expression are the graded anti-symmetric signs of the underlying permutations (be aware that Appendix A of [CG21], as well as many other references, contain sign errors). In particular, the usual Jacobi Identity is satisfied up to a cocycle (the *Jacobiator*), and the higher Jacobi Identities are coherence conditions on this cocycle (the Jacobiatorator, and so on). L_∞-algebras are therefore also called homotopy Lie algebras.

Definition 1.3.3. Given a L_∞-algebra L, its *Chevalley-Eilenberg algebra* is the commutative differential graded algebra with underlying graded algebra

$$\mathrm{CE}^*(L) = \mathrm{Sym}_\Pi(L[1]^*) = \prod_{n=0}^{\infty} ((\mathfrak{g}[1]^*)^{\otimes n})_{S_n} \tag{1.27}$$

and differential obtained by extending the duals $\frac{\ell_n^*}{n!} : L^* \to ((L^*)^{\otimes n})_{S_n}$ to graded derivations, and summing over those. Dually, we can define the *homological Chevalley-Eilenberg complex*

$$\mathrm{CE}_*(L) = \mathrm{Sym}(L[1]) = \bigoplus_{n=0}^{\infty} ((\mathfrak{g}[1])^{\otimes n})_{S_n} \tag{1.28}$$

which can be made into a dg cocommutative coalgebra, with differential induced by the sum of the $\frac{\ell_n}{n!}$ extended to a coderivation.

Remark. The factors $n!$ simplify the description of interactions in field theories, they should be ignored for (dg) Lie algebras. The functor CE_* is fully faithful, meaning that all the higher Lie brackets can be recovered from the differential in this dg coalgebra. Explicitly, if we filter by the power of $\mathfrak{g}[1]$ in the symmetric algebra, ℓ_n features as the differential in the nth page of the associated spectral sequence to this filtered chain complex.

In the above discussion, we hinted at the fact that the tangent complex $\mathbb{T}_\phi X$ captures the formal moduli problem of deforming ϕ inside X, so that it should correspond to an L_∞-algebra. It turns out that this is the shifted tangent complex $\mathbb{T}_\phi X[-1]$, also denoted $\mathbb{T}_\phi[-1]X$: It carries the structure of an L_∞-algebra describing precisely this problem! For intuition, compare it with the odd tangent space in supergeometry (i.e. the parity shift of the tangent bundle). There is an intuitive argument for why this works out, that however cannot currently be made mathematically precise yet:

We have argued that in derived geometry, compared to ordinary geometry, non-transverse pullbacks are better behaved. In particular, it turns out that taking the tangent complex (as a chain complex) commutes with pullbacks, i.e. $\mathbb{T}(X \times_Z Y) = \mathbb{T}X \times_{\mathbb{T}Z} \mathbb{T}Y$. Of course all our pullbacks are in the sense of ∞-categories, i.e. this homotopy pullback of chain complexes is given by a variant of the mapping cone construction. Now, we can form the loop space $\Omega X := * \times_X *$ of our derived stack at the geometric point $\phi : * \to X$; for an ordinary scheme, this would just be a point, but since we take homotopy pullbacks we should rather compare this with the loop space $\Omega Y = * \times^h_Y *$ of a topological space Y. Just as loops can be concatenated in the topological world, we may imagine that ΩX obtains a sort of Lie-group-like structure. This implies that its tangent space at the constant loop $e = \mathrm{const}_\phi$ should be a homotopy-coherent version of a Lie algebra, i.e. an L_∞-algebra. We finish by calculating

$$\mathbb{T}_e \Omega X = \mathbb{T}_e(* \times_X *) = \mathbb{T}_* * \times_{\mathbb{T}_\phi X} \mathbb{T}_* * = 0 \times_{\mathbb{T}_\phi X} 0 = \Omega \mathbb{T}_\phi X = \mathbb{T}_\phi X[-1] \tag{1.29}$$

using how the shift in a stable ∞-category is defined in A.3.3.

1.3.1. Example Calculations

While our explanation makes it seem like the L_∞-algebra $\mathbb{T}[-1]X$ would be incredibly difficult to calculate, it is actually quite amenable to physical intuition. We give a view examples, letting $\mathcal{F}$ finite-dimensional and S a polynomial function on it so that we can work in the algebraic world and do not need any functional analysis. Again, we loosely follow [Yoob].

Example 1.3.4 (Free Field without gauge symmetries).
Let $\mathcal{G}$ be trivial, $\mathcal{F} = \operatorname{Spec}\mathbb{R}[x_1,\dots,x_n]$, and $S \in \mathbb{R}[x_1,\dots,x_n] =: \mathbb{R}[\underline{x}]$. We want to calculate the derived critical locus of S, given by the derived intersection

$$\mathrm{dCrit}(S) = \left\{ y_i = \frac{\partial S}{\partial x_i}(x_1,\dots,x_n) \right\} \cap^R \{y_i = 0\} \tag{1.30}$$

where we identify the cotangent space $T^*\mathcal{F}$ in which we take the intersection with $\operatorname{Spec}\mathbb{R}[x_1,\dots,x_n,y_1,\dots,y_n]$.

Let us introduce, for any coordinate x_i, a corresponding fermionic coordinate ξ_i of degree -1 with $d\xi_i = y_i$ that we can use to write down a Koszul resolution of the zero section:

$$\mathbb{R}[\underline{x}] = \frac{\mathbb{R}[\underline{x},\underline{y}]}{(y_1,\dots,y_n)} \overset{qis}{\simeq} \mathbb{R}[\underline{x},\underline{y},\underline{\xi}] \tag{1.31}$$

Using this, the derived critical locus is given by the spectrum of

$$\begin{aligned} \mathcal{O}(\mathrm{dCrit}(S)) &= \frac{\mathbb{R}[\underline{x},\underline{y}]}{(y_i - \frac{\partial S}{\partial x_i})} \otimes_{\mathbb{R}[\underline{x},\underline{y}]} \mathbb{R}[\underline{x},\underline{y},\underline{\xi}] = \\ &= \frac{\mathbb{R}[\underline{x},\underline{\xi}]}{(d\xi_i - \frac{\partial S}{\partial x_i})} \ . \end{aligned} \tag{1.32}$$

In particular, $H_0\mathcal{O}(\mathrm{dCrit}(S)) = \mathcal{O}(\mathcal{F})\big/\left(\frac{\partial S}{\partial x_i}\right)$ are functions on the ordinary critical locus, also known as the *Jacobi ring* of S.

Let us give $\mathrm{dCrit}(S)$ a slightly more geometric interpretation. We can imagine $\mathbb{R}[\underline{x},\underline{\xi}]$ as a product $\mathbb{A}^n_{\mathbb{R}} \times \mathbb{A}^n_{\mathbb{R}}[1]$, and dividing by the above ideal means that odd and even coordinates are related by $d\xi_i = \partial_i S$. Since this derived stack is affine, we may identify it with its tangent complex $\mathbb{T}_0 X =: \mathcal{E}$, and write (viewing the ξ_i as a basis dual to x_i):

$$\mathcal{E} = (0 \to \mathcal{F} \overset{dS}{\to} \mathcal{F}^*[-1] \to 0) \tag{1.33}$$

We see that if this intersection is smooth, meaning that the partial derivatives $\frac{\partial S}{\partial x_i}$ form a regular sequence so that the homology of the Koszul complex is concentrated in degree 0, then $\mathcal{E}$ is also concentrated in degree 0 so there is a tangent space instead of a tangent complex. Also, one can show that even if it is not, the higher Lie brackets ℓ_n for $n \leq 2$ all vanish.

Example 1.3.5. We can translate above calculation for non-affine $\mathcal{F}$. The Koszul resolution we used can be written as

$$\begin{aligned} \mathcal{O}(\mathcal{F}) &\overset{qis}{\simeq} \left(\mathrm{Sym}_{\mathcal{O}(\mathcal{F})}(T_{\mathcal{F}}[1] \oplus T_{\mathcal{F}}), d \hat{=} \mathrm{id}_{T_{\mathcal{F}}}\right) = \left(\mathrm{Sym}_{\mathcal{O}(\mathcal{F})}(T_{\mathcal{F}}[1]) \otimes_{\mathcal{O}(\mathcal{F})} \mathcal{O}(T^*_{\mathcal{F}}), d\right) \\ &= \left(\cdots \to \mathrm{PV}^2_{\mathcal{F}} \otimes_{\mathcal{O}}(\mathcal{F})\mathcal{O}(T^*\mathcal{F})[2] \to \mathrm{PV}^1_{\mathcal{F}} \otimes_{\mathcal{O}}(\mathcal{F})\mathcal{O}(T^*\mathcal{F})[1] \to \mathcal{O}(T^*\mathcal{F})\right) \end{aligned} \tag{1.34}$$

The symmetric powers of the shifted tangent bundle yield *polyvector fields* $\mathrm{PV}^n_{\mathcal{F}} := \bigwedge^n T^*_{\mathcal{O}_{\mathcal{F}}}$. Continuing the calculation, one can show that tensoring with $\mathcal{O}\mathrm{Graph}(dS)$ collapses the factor $\mathcal{O}(T^*_{\mathcal{F}})$, thereby changing the differential to the *Schouten-Nijenhuis bracket* with S acting on polyvector fields:

$$\mathcal{O}(\mathrm{dCrit}(S)) = \big(\mathrm{Sym}_{\mathcal{O}(\mathcal{F})}(T_{\mathcal{F}}[1]), \{-, S\}\big) = (\mathrm{PV}^*_{\mathcal{F}}, \{-, S\}) \tag{1.35}$$

When calculating the tangent complex, only the $\mathcal{O}(\mathcal{F})$-linear part of $\{-, S\}$, that stems from the quadratic part of S at its critical points, survives – the higher-order terms turn into the higher Lie brackets of the L_∞-structure. See 3.4.1 for more.

Remark. In the differential geometric case, we analogously would like to define

$$C^\infty(\mathrm{dCrit}(S)) = C^\infty(\mathrm{Graph}(S)) \otimes^L_{C^\infty(T^*\mathcal{F})} C^\infty(\mathcal{F}) = (\mathrm{PV}^*(\mathcal{F}), \{-, S\}) \tag{1.36}$$

While this is difficult to understand rigorously, a tentative definition of the polyvector fields involved in this wish will be given in 3.1.

Example 1.3.6 (Free field theory with gauge symmetry)**.** Now, let us assume we again have $\mathcal{F} = \mathrm{Spec}\,\mathbb{R}[\underline{x}]$ but there is a gauge group G acting on it. Let us assume we are interested in perturbation theory around the origin of $\mathcal{F}$, then our result should not change if we restrict our fields to lie in a small neighborhood of $0 \in \mathcal{F}$, which again looks like the linear space $\mathcal{F}$, and replace our group action by its differential $\mathfrak{g} \to \mathcal{F}$. In other words, we have linearized the group action.

As we have discussed above, we don't want to form a set theoretic quotient with respect to this action, but a stacky quotient. In our ∞-categorical setting this is not very difficult; we simply take the derived functor of the quotient. Explicitly,

$$\mathcal{O}\left([\mathcal{F}/\mathfrak{g}]\right) = \mathcal{O}(\mathcal{F})^{h\mathfrak{g}} = \mathrm{RHom}_{U\mathfrak{g}}(\mathbb{C}, \mathcal{O}(\mathcal{F})) = \mathrm{CE}^*(\mathcal{O}(\mathcal{F}), \mathfrak{g})\ . \tag{1.37}$$

In shorter words, the Chevalley-Eilenberg cochains are the derived functor of taking the $\mathfrak{g}$-invariants (i.e. the homotopy invariants, which we denote using $h\mathfrak{g}$). Write $\mathcal{F} = \mathrm{Sym}(\mathbb{R}\langle\underline{x}\rangle)$ and remember that

$$\mathrm{CE}^*(M, \mathfrak{g}) := \mathrm{Sym}(\mathfrak{g}^*[-1]) \otimes M \tag{1.38}$$

with differential induced by the Lie bracket and Lie algebra action. Therefore,

$$\mathcal{O}([\mathcal{F}, \mathfrak{g}]) = \mathrm{Sym}(\mathfrak{g}^*[-1] \oplus \mathbb{R}\langle\underline{x}\rangle) = \mathrm{Sym}(\mathfrak{g}[1] \oplus \mathcal{F})^* \tag{1.39}$$

We obtain $[\mathcal{F}/\mathfrak{g}] = \mathfrak{g}[1] \oplus \mathcal{F}$ with differential induced by the Lie-algebra action. An analogous calculation to 1.3.4 yields the following BV-BRST complex:

$$\mathcal{E} = (\mathfrak{g}[1] \to \mathcal{F} \stackrel{dS}{\to} \mathcal{F}^*[-1] \to \mathfrak{g}^*[-2]) \tag{1.40}$$

Note the symmetry of this expression, there is a canonical pairing between degrees 0 and 1, -1 and 2 that we later call the (-1)-shifted symplectic structure on the BV-BRST complex. Of course, we have not explained yet how to understand the L_∞-structure on $\mathcal{E}$, as this is only the underlying chain complex (as in the example above). We return this issue from a physical point of view in 3.4.1, a mathematical explanations via the homological Chevalley-Eilenberg complex can again be found in [Yoob].

Remark. Studying the result 1.40 from a physical point of view, we see that the BV-BRST complex $\mathcal{E}$ of a field theory contains:

- In negative degrees, gauge symmetries (and if there were any, gauge-of-gauge symmetries),
- The differential d_{-1} is the action of the gauge symmetries,
- In degree 0, the (off-shell) fields,
- The equations of motion are expressed by d_0,
- In degree 1, the target of the equations of motion, i.e. source fields,
- In higher degrees, constraints on those source fields stemming from gauge symmetry (i.e. conservation laws).

1.4. Digression: The Moduli Space of Flat Connections

We have seen in 1.13 that the Euler-Lagrange equations of Chern-Simons theory single out the flat connections. However, our discussion of derived geometry teaches us that we are not only interested in the set of solutions, but in their moduli space. We will not give a precise definition of it to avoid developing derived differential geometry, but we give a rough discussion how spaces from geometric representations theory like the Betti, deRham and Dolbeault space fit into the context of derived stacks. But let us start by classifying flat connections as a set:

Proposition 1.4.1. Let M be a smooth connected manifold with $x \in M$ and G a connected Lie group, then there is a one-to-one correspondence between the following sets:

$$\left\{ \begin{matrix} \text{Principal } G\text{-bundles on } M \\ \text{with flat connection } A \end{matrix} \right\} \Big/ \cong \;\cong \mathrm{Hom}_{\mathrm{Grp}}(\pi_1(M,x), G) \Big/ \text{conjugation} \tag{1.41}$$

More precisely (and even if M and G are not connected), there is an equivalence of categories that reduces to above statement by taking connected components. Let BG be the delooping groupoid of G, and $\mathrm{Flat}_G(M)$ be the groupoid with objects principal

G-bundles on M together with a flat connection, and morphisms of principal bundles (are always isomorphisms) that preserve the connection; then

$$\mathrm{Flat}_G(M) \simeq \mathrm{Fun}(\pi_{\leq 1}(M), \mathrm{BG}) \; . \tag{1.42}$$

This already knows abut gauge symmetries, since those are the morphisms on both sides. It could even be generalized to arbitrary (reasonably good) topological spaces, if we replace principal G-bundles with flat connections by G-local systems. However, like in B.5.6, we would like to follow the spirit of higher category theory and equip this space with homotopy coherent data; we are trying to replace $\pi_{\leq 1}M$ by the fundamental ∞-groupoid $\mathrm{Sing}(M)$ from A.1.11. There is a roundabout way to explain this; we assume the reader has read Appendix A (the rest of this chapter might also be skipped on a first reading). In order to categorify our statement, let us first decategorify it:

Proposition 1.4.2. For X a topological space and S a set, there is a natural bijection

$$\mathrm{Hom}_{\mathrm{Top}}(X, \mathrm{Disc}(S)) \cong \mathrm{Hom}_{\mathrm{Set}}(\pi_0 X, S) \tag{1.43}$$

where $\mathrm{Disc}(S)$ is S equipped with the discrete topology. This fits into an adjoint quadruple with the underlying-set functor U and the indiscrete topology coDisc:

$$\begin{tikzcd}
\mathrm{Top} \arrow[r, shift left=4.5, "\pi_0" description] \arrow[r, shift right=1.5, "U" description] & \mathrm{Set} \arrow[l, shift right=1.5, "\mathrm{Disc}" description] \arrow[l, shift left=4.5, "\mathrm{coDisc}" description]
\end{tikzcd}$$

Since general topological spaces are too ill-behaved for our considerations, let us restrict to the nicer model of simplicial sets:

Theorem 1.4.3. The functor const : Set $\to$ sSet fits into an adjoint quadruple:

$$\begin{tikzcd}
\mathrm{sSet} \arrow[r, shift left=4.5, "\pi_0" description] \arrow[r, shift right=1.5, "(-)_0" description] & \mathrm{Set} \arrow[l, hook, shift right=1.5, "\mathrm{const}" description] \arrow[l, hook, shift left=4.5, "\mathrm{cosk}_0" description]
\end{tikzcd}$$

Here, $(-)_0$ is the set of vertices, and cosk_0 is the 0-coskeleton. Also, π_0 commutes with finite products and $\mathrm{Disc}, \mathrm{cosk}_0$ are fully faithful.

Proof. The terminal functor $\Delta \to *$ has a left adjoint $* \to \Delta$ sending $*$ to $[0]$, since this is an initial object. Above quadruple is obtained from this adjunction by left and right Kan extending. The statement about π_0 is standard. $\square$

Definition 1.4.4. As a presheaf category, sSet is a topos (see A.4.7) with terminal geometric morphism given by const $\dashv (-)_0$. We call a topos with the above properties (a quadruple of adjunctions involving the terminal geometric morphism, such that π_0 preserves finite products) *cohesive*, and similarly for ∞-topoi.

Note how the uppermost adjunction is again a correspondence between maps into a discrete simplicial set (something like local systems) and maps from the fundamental 0-groupoid to this discrete set. If we generalize to ∞-topoi, it makes sense to assume that we obtain representations of the fundamental ∞-groupoid instead, just as we hope.

Definition 1.4.5. Let Man be the site of smooth manifolds, with morphisms given by smooth maps and coverings given by collections of open embeddings whose images form a covering. A *smooth ∞-groupoid* is an ∞-sheaf on this site.

Example 1.4.6. Since a smooth map $f \in C^\infty(M, N)$ is determined by its restrictions to an open cover of M, this site is subcanonical. Hence, the Yoneda-embedding yields a fully faithful functor from manifolds into smooth ∞-groupoids. In fact, this functor can be extended to both Fréchet manifolds and orbifolds.

Theorem 1.4.7. The ∞-topos of smooth ∞-groupoids is cohesive and hypercomplete. Explicitly, the global sections functor and its adjoints can be written as

$$\mathrm{Disc} : \mathcal{S} \to \mathcal{Sh}(\mathrm{Man}),\ K \mapsto (M \mapsto \mathrm{Map}(\mathrm{Sing}(M), K)) \tag{1.44}$$

$$\Gamma : \mathcal{Sh}(\mathrm{Man}) \to \mathcal{S},\ X \mapsto \mathrm{Nat}(\mathrm{const}_*, X) \cong X(*) \tag{1.45}$$

$$\mathrm{coDisc} : \mathcal{S} \to \mathcal{Sh}(\mathrm{Man}),\ K \mapsto \left(M \mapsto \prod_{m \in M} K\right) \tag{1.46}$$

$$\begin{tikzcd}[column sep=huge] \mathcal{Sh}(\mathrm{Man}) \arrow[r, hook, shift left=4.5, "\Pi"] \arrow[r, shift right=1.5, "\Gamma"] & \mathcal{S} \arrow[l, shift right=1.5, "\mathrm{Disc}"'] \arrow[l, shift left=4.5, "\mathrm{coDisc}"] \end{tikzcd}$$

Proof. Combine [ADH21], 4.1.2, 4.3.9, 5.1.8, A.5.3; see 5.1.6 for a description of Π. $\square$

Corollary 1.4.8. If we view a manifold M as a smooth ∞-groupoid via the Yoneda embedding, then $\Pi(h_M)$ agrees with the fundamental ∞-groupoid $\mathrm{Sing}(M)$, since:

$$\mathrm{Map}(\Pi(h_M), K) = \mathrm{Map}(h_M, (N \mapsto \mathrm{Map}(\mathrm{Sing(N)}, \mathrm{K}))) \overset{\textbf{Yoneda}}{=} \mathrm{Map}(\mathrm{Sing}(M), K)$$

Definition 1.4.9. We call the composition $\flat := \mathrm{Disc} \circ \Gamma$ the *flat modality*, or *Betti space*; the composition $\int := \mathrm{Disc} \circ \Pi$ the *shape modality* and $\sharp := \mathrm{coDisc} \circ \Gamma$ the *sharp modality*. Since adjunctions compose,

$$\int \dashv \flat \dashv \sharp\,. \tag{1.47}$$

Definition 1.4.10. Let G be a group object in smooth ∞-groupoids (for example, a Lie algebra) and $\mathbb{B}G$ its delooping in the topos. Then, we define the space of *flat ∞-connections* on a smooth ∞-groupoid X as one of the homotopy equivalent mapping spaces

$$\mathrm{Map}(X, \flat\mathbb{B}G) \cong \mathrm{Map}(\Pi X, \Gamma\mathbb{B}G) \cong \mathrm{Map}(\textstyle\int X, \mathbb{B}G) \ . \tag{1.48}$$

In particular, for M a smooth manifold, flat ∞-connections are classified by the space of simplicial maps from $\mathrm{Sing}(M)$ to $\Gamma\mathbb{B}G$. Compare [BNV13, Lemma 5.2].

This is already very good, but unfortunately, it does not fully resolve our problem. While smooth ∞-groupoids are a nice setting to do higher stacky geometry in, it is *not* a setting for derived geometry since the problem of forming well-behaved intersections is not resolved. One would have to replace the site of manifolds by a site of either derived manifolds or derived cartesian spaces, ie. the opposite category of cdgas of the form

$$\cdots \to A_{-3}[3] \to A_{-2}[2] \to A_{-1}[1] \to C^\infty(\mathbb{R}^n) \to 0 \to \dots \ . \tag{1.49}$$

In fact, a nicer (but, by a version of Dold-Kan, equivalent) candidate site are the finitely presented objects in the non-abelian derived category (as defined in A.8) $\mathrm{Fun}^\pi(\mathrm{Cart}, \mathcal{S})^{fp}$. See [CS19] for more.

Let us, for simplicity, focus on the algebraic setting of derived stacks. Remember that they too were defined as higher sheaves on the Zariski or étale site of cdgas, so they form an ∞-topos as well.

Definition 1.4.11. The left adjoint in the terminal geometric morphism of the topos of derived stacks is called the *Betti stack* functor $\flat : \mathcal{S} \to \mathrm{dSt}$. Explicitly, $\flat K$ is given by sheafifying the constant functor $\mathrm{const}_K : \mathrm{cdga}_{\leq 0} \to \mathcal{S}$.

Definition 1.4.12. Let G be a group object in dSt, e.g. an algebraic group, and $\mathbb{B}G$ its delooping (can be defined in any ∞-topos). Alternatively, it is the stacky quotient $\mathbb{B}G := [*/G]$ and classifies G-principal bundles over stacks. Then, we define for $M \in \mathrm{sSet}$ (in particular smooth manifolds) the moduli space of flat connections

$$\mathbb{L}\mathrm{oc}_G(M) := \underline{\mathrm{Hom}}(\flat M, \mathbb{B}G) \ . \tag{1.50}$$

Here, $\underline{\mathrm{Hom}}$ is the *mapping stack* or internal Hom, given by its functor of points

$$\underline{\mathrm{Hom}}(X, Y) : (R \in \mathrm{cdga}_{\leq 0}) \mapsto \mathrm{Hom}_{\mathrm{dSt}}(X \times \mathrm{Spec}\, R, Y) \ . \tag{1.51}$$

Observation 1.4.13. Following [Cal21, Example 2.19], one can calculate the tangent complex of $\mathbb{L}oc_G(M)$ using the formula $\mathbb{T}_\phi[-1]\underline{\mathrm{Hom}}(X, Y) \cong \Gamma(X, \phi^*\mathbb{T}_Y)$ with $\phi : X \to Y$ a geometric point:

$$\mathbb{T}_\phi[-1]\mathbb{L}oc_G(M) \cong \Gamma(\flat M, \phi^*\mathbb{T}_{\mathbb{B}G}) \tag{1.52}$$

Since we work with derived categories, Γ automatically calculates derived sections, i.e. sheaf cohomology. Given a suitable setting for derived differential geometry, this complex calculates $H^*(M, P \times_G \mathfrak{g})$ where P is the principal bundle classified by ϕ.

Definition 1.4.14. Given a cdga $A \in \mathrm{cdga}_{\leq 0}$, its 0th homology group H^0A is a commutative ring; let H^0A^{red} be its reduction. One can show that this construction is compatible with étale covers, so precomposing a derived stack $X \in \mathrm{dSt}$ with it yields a new derived stack, its *deRham shape*

$$X_{dR} : \mathrm{cdga}_{\leq 0} \to \mathcal{S}\,, \quad R \mapsto X(H_0 R^{red})\;. \tag{1.53}$$

We can define a *moduli space of flat connections on* X as the mapping stack

$$\mathbb{F}\mathrm{lat}_G(X) := \underline{\mathrm{Hom}}(X_{dR}, \mathbb{B}G)\;. \tag{1.54}$$

Remark. The reason for this nomenclature lies in the technology of *crystals*, we will not get into that. Be aware that unlike in differential geometry, flat connections and local systems are generally different things in the algebraic world.

Along these lines, we can construct many moduli spaces from geometric representation theory via derived stacks, for example the *Dolbeault shape* – see [PS] for more.

1.5. Conclusion and Operator Algebras

Since this chapter has involved a lot of mathematical terminology and definitions we could not introduce, let us recapitulate the most important ideas from a conceptual point of view.

- A classical field theory consists of the following data: A spacetime manifold M, a space of off-shell field histories $\mathcal{F}$ that is, for our intents and purposes, given as the space of sections of a smooth fiber bundle on M, an action of the group of gauge symmetries $\mathcal{G}$ on $\mathcal{F}$, and a gauge-invariant action functional $S : [\mathcal{F}/\mathcal{G}] \to \mathbb{R}$.
- The derived covariant phase space X is defined as the moduli space of solutions to the Euler-Lagrange equations $dS = 0$ modulo gauge symmetry.
- The BV-BRST complex is defined as the tangent complex $\mathcal{E} = \mathbb{T}_\phi X$ of the phase space at a fixed background solution ϕ. It parametrizes perturbations, or deformations, of ϕ inside the moduli space of solutions; and $\mathcal{E}[-1]$ is an L_∞-algebra. Using 1.3.6, we see how this complex contains gauge symmetries, fields and sources.
- The algebra of (perturbative polynomial local) observables is defined as the space of polynomial functions on this tangent space, ie. the symmetric algebra on its dual space. Taking the higher brackets into account, this is exactly what the Chevalley-Eilenberg algebra 1.3.3 incorporates:

$$\mathcal{O}bs^{cl}(M) = \mathrm{CE}^*(\mathcal{E}[-1]) = (\mathrm{Sym}\,\mathcal{E}^\vee, d_{\mathrm{CE}}) \tag{1.55}$$

Let us work out the example of Chern-Simons theory:

- For M a 3-manifold, $\mathcal{F}$ is the space of connections on the trivial principal bundle on M. Gauge symmetries are then given by smooth functions from M to the gauge group G, so $\mathcal{G} = C^\infty(M, G)$. Finally, the gauge-invariant action $S : \mathcal{F} \to \mathbb{R}$ is given by 1.9, where we ignore issues concerning integrability.
- Since the equations of motions merely state that $F = 0$, we know that X must be the derived moduli space of flat connections on M. As is discussed in 1.4.12, this is in the algebraic world given by the mapping stack

$$X = \mathbb{L}\mathrm{oc}_G(M) = \mathrm{Map}(\flat M, \mathbb{B}G) \ . \tag{1.56}$$

- The BV-BRST complex, following our observations after 1.1.11, should be given by the chain complex $(\Omega^\bullet(M, \mathfrak{g})[1], D_{A_0})$ where A_0 is the fixed background solution. This agrees with the expression derived in 1.4.13, and the one in 1.3.6 by Poincare duality (and ignoring functional analytic issues for now); and we will see another way to derive it in 3.1.
- The classical observables are given by

$$\mathcal{O}bs^{cl}(M) = \mathrm{CE}^*\left((\Omega^\bullet(M, \mathfrak{g}), D_{A_0}, [-,-])\right) = \left(\mathrm{Sym}\left(\bar{\Omega}^*_c(M, \mathfrak{g})[2], D_{A_0}, [-,-]\right), d_{\mathrm{CE}}\right)$$

where we have used that the integration pairing $\int_M (- \wedge -) : \bar{\Omega}^{n-i}_c(M) \otimes \Omega^i(M)$ is perfect (the bar indicates distributional sections, we will talk about this later).

Our general calculation of 1.40 also allows us to guess the BV-BRST complexes of several other interesting physical (and mathematical) theories. Note that, since the fields are usually given as the space of sections of a vector bundle, we could always replace M by an open subset U, making $\mathcal{E}$ into a sheaf.

- *Scalar Field:* If (M, g) is a Riemannian manifold, a real scalar field on M has configuration space $\mathcal{F} = C^\infty(M, \mathbb{R})$, no (local) gauge symmetries and action

$$S_{\mathrm{scalar}}[\phi] = \frac{1}{2} \int_M \phi(\Delta_g + m^2)\phi \ . \tag{1.57}$$

Its BV-BRST complex is given by

$$\mathcal{E} = \left(0 \to C^\infty(M, \mathbb{R}) \overset{\Delta_g + m^2}{\longrightarrow} C^\infty(M, \mathbb{R})[-1] \to 0\right) \ . \tag{1.58}$$

We interpret the degree 0 elements as fields, and the degree 1 elements as antifields or possible source terms. All Lie brackets vanish.

- *Abelian Chern-Simons theory:* If we set $\mathfrak{g} = \mathbb{R}$ and take $A_0 = 0$ as background connection, the BV-BRST complex for Chern-Simons theory is the deRham complex

$$\mathcal{E} = (0 \to \Omega^0(M)[1] \overset{d}{\to} \Omega^1(M) \overset{d}{\to} \Omega^2(M)[-1] \overset{d}{\to} \Omega^3(M)[-2] \to 0) \ . \tag{1.59}$$

- *Yang-Mills-Theory* with gauge group G on an n-manifold M: Fields and gauge transformations are the same as for Chern-Simons theory, but the action is

$$S_{YM} = \frac{1}{2}\int_M \langle F_A \wedge \star F_A \rangle \ . \tag{1.60}$$

In particular, it depends on the metric through the Hodge star operator. We can again expand around a background solution A_0, yielding

$$\begin{aligned} 0 \overset{!}{=} D_A \star F_A =& D_{A_0} \star F_{A_0} + D_{A_0} \star (d\alpha + [A_0 \wedge \alpha]) + [\alpha \wedge \star F_{A_0}] = \\ =& (D_{A_0} \star D_{A_0} + [- \wedge \star F_{A_0}])\alpha \end{aligned} \tag{1.61}$$

where we again write $A = A_0 + \alpha$ with α infinitesimal. The BV-BRST complex hence looks like this:

$$\mathcal{E} = \left(\begin{array}{l} 0 \longrightarrow \Omega^0(M,\mathfrak{g})[1] \xrightarrow{D_{A_0}} \Omega^1(M,\mathfrak{g}) \\ \xrightarrow{D_{A_0}\star D_{A_0} + [-\wedge\star F_{D_{A_0}}]} \\ \Omega^{n-1}(M,\mathfrak{g})[-1] \xrightarrow{D_{A_0}} \Omega^n(M,\mathfrak{g})[-2] \longrightarrow 0 \end{array} \right) \tag{1.62}$$

This is indeed a chain complex, i.e. the equations of motion are gauge invariant and they can be modified by a current $J \in \Omega^{n-1}(M,\mathfrak{g})$ iff it satisfies $D_{A_0} J = 0$:

$$\begin{aligned} &(D_{A_0} \star D_{A_0} + [- \wedge \star F_{A_0}])(D_{A_0}\gamma) = D_{A_0} \star [F_{A_0} \wedge \gamma] + [D_{A_0}\gamma \wedge \star F_{A_0}] = \\ &\qquad = -D_{A_0}[\gamma \wedge \star F_{A_0}] + [D_{A_0}\gamma \wedge \star F_{A_0}] = -[\gamma \wedge D_{A_0} \star F_{A_0}] = 0 \\ &D_{A_0}(D_{A_0} \star D_{A_0}\alpha + [\alpha \wedge \star F_{A_0}]) = [F_{A_0} \wedge \star D_{A_0}\alpha] + [D_{A_0}\alpha \wedge \star F_{A_0}] - [\alpha \wedge D_{A_0} \star F_{A_0}] = \\ &\qquad = -[\star F_{A_0} \wedge D_{A_0}\alpha] + [D_{A_0}\alpha \wedge \star F_{A_0}] = 0 \end{aligned}$$

- *Abelian B-Field:* Similarly to Electrodynamics, where the dynamical field A is a 1-form (or connection on a trivial principal U(1)-bundle), one can postulate a theory where it is a 2-form B (or connection on a trivial bundle gerbe). Using the field strength $H = dB$, one can also mimic its action:

$$S_B[B] = \int_M H \wedge \star H \tag{1.63}$$

This yields the Euler-Lagrange equations $dH = 0$, the theory is obviously invariant under transformations $B \mapsto B + d\chi$, with $\chi \in \Omega^1(M)$. However, the gauge group is *not* $\Omega^1(M)$, since the 1-forms χ and $\chi + d\zeta$ for $\zeta \in \Omega^0(M)$ yield the same gauge transformation – we call ζ a *gauge-of-gauge transformation.*

We can see that gauge transformations do not form a group, but a groupoid; meaning that the stacky quotient $[\mathcal{F}/\mathcal{G}]$ will have a functor of points taking values in 2-groupoids: The points of $[\mathcal{F}/\mathcal{G}]$ are field histories, morphisms between two of them are gauge transformations that send one to the other, and 2-morphisms are gauge-of-gauge transformations between the respective morphisms. In our calculation 1.3.4, this means that we have to add another component to the complex representing (the tangent space of) the stacky quotient, yielding the BV-BRST complex:

$$\mathcal{E} = \left(\begin{tikzcd} 0 \arrow[r] & \Omega^0(M)[2] \arrow[r, "d"] & \Omega^1(M)[1] \arrow[r, "d"] & \Omega^2(M) \arrow[dll, rounded corners, "d\star d" description] & \\ & \Omega^{n-2}(M)[-1] \arrow[r, "d"] & \Omega^{n-1}(M)[-2] \arrow[r, "d"] & \Omega^n(M)[-3] \arrow[r] & 0 \end{tikzcd} \right)$$

- *Associative Algebras:* Not only phase spaces of physical theories have an associated tangent complex; every moduli problem can be restricted to its formal moduli problem around a fixed solution. For example, deformations of k-algebra structures around a fixed algebra A are parametrized by the Hochschild complex

$$\mathcal{E} = (\mathrm{HC}^\bullet(A)[1], d_{HH}, \text{Gerstenhaber bracket}, 0, \dots) \tag{1.64}$$

  We have already seen this in 1.3.4; the complex $A \otimes A \otimes A \to A \otimes A \to A \to 0$ obtains an L_∞ algebra structure with the natural differential induced by the algebra multiplication, the *Gerstenhaber bracket* and vanishing higher Lie brackets.

- *Poisson Structures:* As studied by Kontsevich in [Kon03], deformations of Poisson structures on a manifold M correspond to the L_∞-algebra

$$\mathcal{E} = (\mathrm{PV}^*(M)[1], d = 0, \text{Schouten-Nijenhuis bracket}, 0, \dots) \,. \tag{1.65}$$

- *Flat Holomorphic Connections* with gauge group G on a complex manifold N have a similar deformation L_∞-algebra as Chern-Simons theory (i.e. flat smooth connections):

$$\mathcal{E} = \big(\Omega^{0,\bullet}(N, \mathfrak{g}), \bar{\partial}, [-,-], 0, \dots\big) \tag{1.66}$$

- *Complex Structures on a complex manifold N:*

$$\mathcal{E} = \big(\Omega^{0,\bullet}(N, T_N^{1,0}), \bar{\partial}, \text{vector field commutator}, 0, \dots\big) \tag{1.67}$$

2. Factorization Algebras on Manifolds

2.1. Motivation: Local Operators in Perturbative QFTs

In this chapter, we pose ourselves the problem of understanding the algebraic structure that local (polynomial) operators in Lagrangian field theories posses. This structure, as we will see, crucially depends on the topology of the underlying spacetime manifold M, while further retaining the homotopy-theoretic subtleties we had encountered with the sheaf of L_∞-algebras locally describing the derived covariant phase space – the BV-BRST complex. We fix a small number of fundamental insights, or axioms, we naturally have about perturbative field theories to narrow down (in a slightly ad hoc way) the kind of objects we want to obtain.

Goal: Find a functor $A : \mathrm{Open}(M) \to \mathcal{V}$ that associates to each open subset of the spacetime manifold a space of local operators on it. Here, $\mathcal{V}$ should be the symmetric monoidal ∞-category of chain complexes (with the usual tensor product) or some functional analytic refinement of it, like chain complexes of Costello's *convenient/ differentiable vector spaces* in [CG16] (see 3.1 for more on those).

Observation 2.1.1. This functor (of ∞-categories) must be covariant, since an operator localized inside an open set U can always be regarded as an operator localized inside a larger open set $V \supseteq U$; we say that A is a precosheaf on M with values in $\mathcal{V}$. Compare to the space of fields $\mathcal{F}$, the covariant phase space X or the BV-complex $\mathcal{E}$, which are sheaves on M – since operators are something we can *apply* to states/ field histories, i.e. related to distributions, such a change of variance makes sense.

Observation 2.1.2. As known from usual quantum mechanics, we would expect the operators on disjoint open sets to be independent of each other (just like for non-interacting systems), so for $U \cap V = \emptyset$ we expect the *factorization axiom* $A(U \sqcup V) \cong A(U) \otimes A(V)$ to hold.

One might object that this independence of observables at first glance seems to only be a reasonable assumption for causally independent open sets, however our statement is more low-level than causal independence since we are only talking about which observables can be measured at all in some spacetime region, not how these measurements commute. The relationship between algebras of observables and causality is explored in perturbative Algebraic Quantum Field Theory (pAQFT), but the connection to the factorization algebra machinery is subtle and has only began to be understood, see [GR17].

Observation 2.1.3. Since we only want to describe perturbative QFT, what we are finally interested in is calculating many-point-functions, i.e. the expectation value of a finite product of operators, each localized at a single point. We do not expect to see anything more, for example (non-perturbative) line or area operators (but we can talk about their perturbative expressions via path-ordered exponentials). Therefore one might assume that, if we specify, for any finite set of points $x_0, \ldots, x_r \in M$, the space $A(U)$ for some open neighborhood $x_0, \ldots, x_r \in U \subset M$ of all the points (and, additionally, assume that the set of such U is closed under intersections), then we can glue these to find $A(M)$. This looks very much like a cosheaf condition – but unlike for example the BV-BRST complex $\mathcal{E}$, the space of operators is not a (co-)sheaf or homotopy (co-)sheaf!

Warning. Let $\mathcal{F} : \mathrm{Open}(M) \to \mathrm{Vect}_{\mathbb{R}}$ be an ordinary sheaf of vector spaces over M. Then, for $U, V \subseteq M$ disjoint open subsets,

$$\mathcal{F}(U \sqcup V) = \{\text{compatible sections over } U \text{ and } V\} = \lim_{* \sqcup *} F|_{\{U,V\}} = F(U) \oplus F(V)\ . \quad (2.1)$$

However, if we pointwise form the symmetric algebra, this doesn't hold any more:

$$\mathrm{Sym}(\mathcal{F}(U \sqcup V)) = \mathrm{Sym}(\mathcal{F}(U) \oplus \mathcal{F}(V)) = \mathrm{Sym}(\mathcal{F}(U)) \otimes \mathrm{Sym}(\mathcal{F}(V)) \quad (2.2)$$

Similarly for Chevalley-Eilenberg algebras, that we use to construct the observables $\mathcal{O}bs^{cl}$ as explained in 1.5, since their underlying vector space is given by the symmetric algebra. In fact, the factorization axiom tells us that the value of A on a disjoint union must always be given by a tensor product, and not a direct sum. Therefore, we need a modified cosheaf condition.

Definition 2.1.4. A collection of inclusions (or open embeddings) $\{U_i \hookrightarrow M\}_{i \in I}$ is called a *Weiss cover* of M if, for any $S \subseteq M$ finite, there is an $i_S \in I$ such that $S \subset U_{i_S}$.

Proposition 2.1.5. Weiss covers form a Grothendieck pretopology (see A.4.2) on the category $\mathrm{Open}(M)$, making it into a subcanonical site. We call ∞-(co-)sheaves on this site *Weiss (co-)sheaves.*

Remark. In the following, when we talk about (co-)sheaves we always mean homotopy or ∞-(co-)sheaves as defined in A.4. As a reminder, a $\mathcal{V}$-valued cosheaf on a site $\mathcal{C}$ is a $\mathcal{V}^{op}$-valued sheaf on $\mathcal{C}$.

Proof. Clearly, $\{U \subseteq U\}$ is a Weiss covering for all $U \subseteq X$ open, and transitivity of Weiss covers as well as stability under intersections is also trivial to check. Subcanonicity is slightly more subtle; for $\{U_i \subseteq U\}$ a Weiss cover and $W \in \mathrm{Open}(M)$ we want

$$\operatorname*{colim}_i \mathrm{Hom}_{\mathrm{Open}(X)}(U_i, W) \cong \mathrm{Hom}_{\mathrm{Open}(X)}(U, W) \tag{2.3}$$

Note that this is a 1-categorical colimit, and it is empty iff any of the morphism spaces inside it is empty, since the colimit cone then would include a morphism into the empty set. Since both sides can have cardinality 0 or 1 only, we are finished. □

Remark. Using A.5, we can spell out what it means for $A : \mathrm{Open}(M) \to \mathcal{V}$ to be a Weiss cosheaf: For any Weiss cover $\{U_i \to U\}$, we want

$$A(U) \cong \operatorname*{colim}_\Delta \left(\coprod_i A(U_i) \longleftarrow \coprod_{i,j} A(U_i \cap U_j) \Lleftarrow \cdots \right). \tag{2.4}$$

where the colimit cone is given by the natural maps induced by functoriality of A. Note that this should be an ∞-colimit, for example in $\mathcal{V} = \mathrm{Ch}(\mathbb{C})$ it can be written as the totalization of a Čech complex as carried out later in 3.2.1.

Observation 2.1.6. Finally, for a topological field theory, we expect no dependence on the metric or length scales, so for $U \subseteq V$ two *disks* in M, i.e. both homeomorphic to $\mathbb{R}^n$, we want the natural map $A(U) \xrightarrow{\cong} A(V)$ to be an isomorphism. We say that A is *locally constant.*

Definition 2.1.7. A *factorization algebra* on a topological manifold M^n is a *factorizable Weiss cosheaf* on M, this means it is a functor $A : \mathrm{Open}(M) \to \mathcal{V}$ satisfying above factorization property and descent for Weiss covers.

Similarly, a *locally constant factorization algebra* is a *locally constant factorizable Weiss cosheaf* on M. Denote the respective categories by $co\mathcal{S}h^{\otimes}(M_{Weiss}, \mathcal{V})$ and $co\mathcal{S}h^{\otimes,lc}(M_{Weiss}, \mathcal{V})$.

Remark. This construction will later describe operators in arbitrary classical or quantum field theories, specializing to the locally constant case for topological, but also free or conformal field theories.

2.2. Absolute and Relative Factorization Algebras

From this point on, we assume that the reader is familiar with Appendix A on higher category theory since we will use it to define factorization algebras following [AF20], [Mat13] and [HA]. We will see that they come in two flavors: Relative and absolute.

This reflects the two perspectives on field theories used by physicists – either a field theory can be defined on a fixed spacetime manifold, or it can be applied analogously to a whole class of manifolds, potentially of a fixed dimension or equipped with some extra structure like an orientation.

Definition 2.2.1. Let $\mathrm{Mfd_n}$ be the category of topological manifolds with open embeddings as morphisms. We denote by $\mathrm{Charts}(n)$ the full subcategory spanned by the single object $\mathbb{R}^n$, and by $\mathrm{Disk}(n)$ the full subcategory spanned by finite disjoint unions of $\mathbb{R}^n$.

Definition 2.2.2. Let $\mathcal{M}\mathit{fd}_n$ be the ∞-category of topological manifolds with morphism spaces given by

$$\mathrm{Map}_{\mathcal{M}\mathit{fd}_n}(M,N) := \mathrm{Sing}\,\mathrm{Emb}(M,N)\ . \tag{2.5}$$

Here, we equip the set of open embeddings $\mathrm{Emb}(M,N)$ with the compact-open topology. This determines an ∞-category since we can apply the homotopy coherent nerve $A.2.7$ to this Kan-enriched category.

We again fix two full subcategories of it: $\mathcal{C}\mathit{harts}(n)$ is spanned by the single object $\mathbb{R}^n \in \mathcal{M}\mathit{fd}_n$, and $\mathcal{D}\mathit{isk}(n)$ is spanned by finite disjoint unions of $\mathbb{R}^n$ (including $\emptyset$).

Warning. Unfortunately, the terminology for these categories is not consistent in the literature; for example our Disk is sometimes denoted Disj. Also, the notation $\mathcal{D}\mathit{isk}(n)$ is aimed to connect with $\mathcal{D}\mathit{isk}(\mathcal{B})$ which we later define on stratified spaces.

Remark. The category $\mathcal{C}\mathit{harts}(n)$ is also called $\mathrm{BTop}(n)$, since the morphism space of this one-object ∞-category is homotopy equivalent to a topological group:

$$\mathrm{Map}_{\mathcal{C}\mathit{harts}(n)}(\mathbb{R}^n,\mathbb{R}^n) = \mathrm{Emb}(\mathbb{R}^n,\mathbb{R}^n) \simeq \mathrm{Homeo}(\mathbb{R}^n) =: \mathrm{Top}(n) \tag{2.6}$$

The middle equality is a result by Kister and Mazur, see [HA, 5.4.1.5] for a proof. It roughly states that every embedding of $\mathbb{R}^n$ into itself can be deformed into a homeomorphism. This space is fairly important in the theory of topological manifolds, since the geometric realization $|\,\mathrm{BTop}(n)|$ is a classifying space for *topological microbundles*. The unfamiliar reader can take this as the definition of a microbundle, these serve as a refinement of vector bundles to the context of topological (instead of smooth) manifolds. For example, we will see in 2.2.7 that every topological manifold has a *tangent microbundle* (but no tangent vector bundle, unless it can be smoothened).

Definition 2.2.3. We can equip the ordinary categories Mfd_n and $\mathrm{Disk}(n)$ with symmetric monoidal structures, where the product is given by disjoint union. Similarly for the ∞-categories $\mathcal{M}\mathit{fd}_n$ and $\mathcal{D}\mathit{isk}(n)$. Let us denote the corresponding ∞-operads by $\mathrm{Mfd}_n^{\sqcup}, \mathrm{Disk}(n)^{\sqcup}, \mathcal{M}\mathit{fd}_n^{\sqcup}, \mathcal{D}\mathit{isk}(n)^{\sqcup}$.

This does not work for $\mathrm{Charts}(n)$ and $\mathcal{C}harts(n)$, since their objects are not closed under forming disjoint unions. However, $\sqcup$ still induces operadic structures on them: $\mathrm{Charts}(n)^{\sqcup}$ is an ordinary operad with a single object $\mathbb{R}^n$ and k-ary multimorphisms

$$\mathrm{Mul}_{\mathrm{Charts}(n)}(\mathbb{R}^n, \dots, \mathbb{R}^n; \mathbb{R}^n) = \mathrm{Hom}_{\mathrm{Mfd}_n}(\mathbb{R}^n \times \{1, \dots, k\}, \mathbb{R}^n) \in \mathrm{Set} \ . \tag{2.7}$$

Analogously, $\mathcal{C}harts(n)$ is the underlying ∞-category of an ∞-operad $\mathcal{C}harts(n)^{\sqcup}$, with a single object $\mathbb{R}^n$ and multimorphisms with k sources given by

$$\mathrm{Mul}_{\mathcal{C}harts(n)}(\mathbb{R}^n, \dots, \mathbb{R}^n; \mathbb{R}^n) = \mathrm{Sing}\,\mathrm{Emb}(\mathbb{R}^n \times \{1, \dots, k\}, \mathbb{R}^n) \in \mathcal{S} \ . \tag{2.8}$$

Warning. Beware that $\sqcup$ is not the coproduct in these categories; in fact, not a single coproduct exists in any of them (except for coproducts with $\emptyset$ since this is the initial object).

These operads will be useful for defining absolute factorization algebras; in the relative case over a fixed topological n-manifold M we need the following variants:

Definition 2.2.4. Denote by $\mathrm{Open}(M)$ the poset of open subsets of M, regarded as a thin category; by $\mathrm{Charts}(M)$ the full subcategory spanned by those that are homeomorphic to $\mathbb{R}^n$ (let us call those *disks* or balls); and by $\mathrm{Disk}(M)$ the full subcategory on finite (possibly empty) disjoint unions of disks.

Remark ([HA, 5.4.5.8]). $\mathrm{Charts}(M)$ is equivalent to the slice category $\mathrm{Charts}(n)_{/M}$, via the functor that sends an embedding $j : \mathbb{R}^n \hookrightarrow M$ in the slice category to the disk $j(\mathbb{R}^n)$. This is clearly essentially surjective, and since both categories are posets ordered by inclusion, it is also fully faithful: Note for $j' : \mathbb{R}^n \hookrightarrow M$ another embedding, morphisms from j to j' in $\mathrm{Charts}(n)_{/M}$ are commutative diagrams

$$\begin{tikzcd} \mathbb{R}^n \arrow[rr, hook, "k"] \arrow[dr, hook, "j"'] & & \mathbb{R}^n \arrow[dl, hook', "j'"] \\ & M & \end{tikzcd}$$

where k is uniquely determined as $(j')^{-1} \circ j$, as j' is a homeomorphism onto its image, which contains the image of j. Similarly, $\mathrm{Disk}(M) \simeq \mathrm{Disk}(n)_{/M}$, and $\mathrm{Open}(M) \simeq (\mathrm{Mfd}_n)_{/M}$ so we don't need a new name for this.

Definition 2.2.5. Again, these can be made into ∞-categories. The easiest way to do this is to follow the last remark and use the slice categories $\mathcal{M}fd_{/M} := (\mathcal{M}fd_n)_{/M}$, $\mathcal{C}harts_{/M} := \mathcal{C}harts(n) \times_{\mathcal{M}fd_n} \mathcal{M}fd_{/M}$ and $\mathcal{D}isk_{/M} := \mathcal{D}isk(n) \times_{\mathcal{M}fd_n} \mathcal{M}fd_{/M}$.

Objects of these slice categories are clearly open embeddings of n-manifolds, $\mathbb{R}^n$ and disjoint unions of $\mathbb{R}^n$ into M, respectively. Also, for embeddings $j : N \hookrightarrow M$ and $j' : N' \hookrightarrow M$, morphisms between them in $\mathcal{M}fd_{/M}$ are open embeddings $k : N \hookrightarrow N'$ making the triangle

$$\begin{tikzcd} N \arrow[rr, hook, "k"] \arrow[dr, hook, "j"'] & & N' \arrow[dl, hook', "j'"] \\ & M & \end{tikzcd}$$

commute up to a specified isotopy from j to $j' \circ k$.

Remark. $\mathcal{C}harts_{/M}$ is equivalent to the full subcategory $\mathcal{C}harts(M)$ of $\mathcal{M}fd_{/M}$ spanned by the subsets of M that are homeomorphic to $\mathbb{R}^n$, via the functor sending an open embedding $j : \mathbb{R}^n \hookrightarrow M$ to its image $j(\mathbb{R}^n) \subseteq M$. This is clearly essentially surjective, and it is fully faithful since we can compute the mapping spaces in the category $\mathcal{M}fd_{/M}$ containing both as full subcategories, whose objects are related via homeomorphisms. Now, note that pre- and postcomposing with isomorphisms in an ∞-category induces homotopy equivalences on mapping spaces. Alternatively, one could show that our map $\mathcal{C}harts_{/M} \to \mathcal{C}harts(M)$ is a coCartesian fibration with contractible fibers, thus a trivial fibration.

A similar result $\mathcal{D}isk_{/M} \simeq \mathcal{D}isk(M)$ also holds, so we can identify disks over M and disks in M without problems. This is very helpful for our intuition as $\mathcal{C}harts(M)$ is much smaller than $\mathcal{C}harts_{/M}$ since we only look at inclusions, not all embeddings. However, in the stratified case, it turns out to be the wrong definition as explained in 4.1.

Theorem 2.2.6 ([HA, 5.4.5.2])**.** The ∞-category $\mathcal{C}harts_{/M}$ of charts in M is homotopy equivalent to $\mathrm{Sing}(M)$.

Remark. Intuitively, every chart $j : \mathbb{R}^n \hookrightarrow M$ can be contracted to the point $j(0)$, which makes isotopies between charts into paths, and so on. We will further discuss this striking result and its applications in 4.2, but let us already tease an important one:

Definition 2.2.7. The projection out of the slice theory

$$\mathrm{Sing}(M) \simeq \mathcal{C}harts(n)_{/M} \to \mathcal{C}harts(n) \simeq \mathrm{BTop}(n) \tag{2.9}$$

is called the *tangent classifier* τ_M of M, and the microbundle classified by this map the *tangent microbundle*. It turns out that if we regard the orthogonal group $\mathrm{O}(n)$ as a subgroup of the topological group $\mathrm{Homeo}(\mathbb{R}^n)$, then factoring τ_M through $\mathrm{BO}(n) \subseteq \mathrm{BTop}(n)$ is equivalent to choosing a smooth structure on M by smoothing theory, and the resulting map $\mathrm{Sing}(M) \to \mathrm{BO}(n)$, or rather the adjoint map $M \to |\,\mathrm{BO}(n)|$, classifies the tangent vector bundle of M as in B.5.7. See 4.2 for more.

Definition 2.2.8. Since a disjoint union can only be formed for disjoint subsets, this operation induces a *partial* symmetric monoidal structure in the relative case. As discussed in A.7.6, we can express this via operadic structures on the above categories:

- $\mathrm{Open}(M)^{\sqcup}$ with multimorphism spaces either empty or single points:

$$\mathrm{Mul}_{\mathrm{Open}(M)}(U_1, U_2, \ldots, U_m; U) = \begin{cases} \{\iota : \coprod_i U_i \hookrightarrow U\}, & \text{if } (U_i) \text{ disjoint subsets of } U \\ \emptyset, & \text{otherwise} \end{cases}$$

- $\mathrm{Disk}(M)^{\sqcup}, \mathrm{Charts}(M)^{\sqcup}$ can be defined analogously.
- $\mathcal{M}fd_{/M}, \mathcal{C}harts_{/M}$ and $\mathcal{D}isk_{/M}$ are also the underlying categories of ∞-operads, however those are more tricky to define. The multimorphism spaces in $\mathcal{C}harts^{\sqcup}_{/M}$ and $\mathcal{D}isk^{\sqcup}_{/M}$ will agree with those we construct for $\mathcal{M}fd^{\sqcup}_{/M}$, if we embed them as full subcategories. It therefore suffices to define, for open embeddings $j_i : N_i \hookrightarrow M$ and $j : N \hookrightarrow M$, the multimorphism space $\mathrm{Mul}_{\mathcal{M}fd_{/M}}(j_1, \ldots, j_m; j)$ as

$$\mathrm{Sing}\,\mathrm{Emb}(N_1 \sqcup \cdots \sqcup N_m; N) \times_{\prod_i \mathrm{Sing}\,\mathrm{Emb}(N_i, N)} \prod_i \mathrm{Map}_{\mathcal{M}fd_{/M}}(j_i, j) \,.$$

  Here, the disjoint union $N_1 \sqcup \ldots N_m$ is taken as topological manifolds independent of M; the images of the j_i do not have to be disjoint. This looks very intimidating, but is actually fairly easy to understand: A multimorphisms from the embeddings j_i to j consists of, for any i, an embedding $k_i : N_i \hookrightarrow N$ and an isotopy from j_i to $j \circ k_i$, such that the images of all the k_i in N are disjoint. We will discuss in 4.1.8 why this is actually an ∞-operad, using the abstract construction A.7.14.

In the following, let $\mathcal{V}^{\otimes}$ be a sifted complete (see A.8.9) symmetric monoidal ∞-category. This means that its underlying ∞-category has all sifted colimits, and $\otimes$ preserves them (in each argument, or equivalently, in $\mathcal{V} \otimes \mathcal{V}$).

Definition 2.2.9. An *absolute factorization algebra* with values in $\mathcal{V}^{\otimes}$ is a symmetric monoidal functor $A : \mathrm{Disk}(n)^{\sqcup} \to \mathcal{V}^{\otimes}$.

Similarly, a *relative factorization algebra* with values in $\mathcal{V}^{\otimes}$ is a symmetric monoidal functor $A : \mathrm{Disk}(M)^{\sqcup} \to \mathcal{V}^{\otimes}$. Denote the respective ∞-categories of symmetric monoidal functors by $\mathrm{FA}(\mathcal{V})$ and $\mathrm{FA}(M; \mathcal{V})$.

Remark. In the rest of this text, we will mostly work with relative factorization algebras, still we feel like the absolute case is fairly underappreciated.

Definition 2.2.10. A relative factorization algebra $A : \mathrm{Disk}(M)^{\sqcup} \to \mathcal{V}^{\otimes}$ is called *locally constant* if for any disk inclusion $D \subseteq D'$ with $D, D' \in \mathrm{Charts}(M)$, its image under A is an isomorphism $A(D) \xrightarrow{\cong} A(D')$.

Remark. In particular, a relative factorization algebra is locally constant iff the underlying functor $\mathrm{Disk}(M) \to \mathcal{V}$ factors through the localization $\mathrm{Disk}(M)[\mathcal{J}_M^{-1}]$, where $\mathcal{J}_M$ is the class of morphisms that are (disjoint unions of) disk inclusions. We denote the full subcategory on them by $\mathrm{FA}^{lc}(M;\mathcal{V})$.

Theorem 2.2.11 ([AFT14a, 2.22])**.** The canonical map $\mathrm{Disk}(M) \simeq \mathrm{Disk}_{/M} \to \mathcal{D}isk_{/M}$ induces an equivalence of categories $\mathrm{Disk}(M)[\mathcal{J}_M^{-1}] \simeq \mathcal{D}isk_{/M}$. Since $\mathcal{J}_M$ is compatible with the operadic structure, this induces an operadic structure on $\mathcal{D}isk_{/M}$, which agrees with the one defined above.

Definition 2.2.12. An absolute factorization algebra $A : \mathrm{Disk}(n)^{\sqcup} \to \mathcal{V}^{\otimes}$ is called *locally constant* if it factors through $\mathrm{Disk}(n)^{\sqcup} \to \mathcal{D}isk(n)^{\sqcup}$. Take care as $\mathcal{D}isk(n)^{\sqcup}$ is not symmetric monoidal, but only a possess a weak symmetric monoidal structure in the sense of A.7.14. As explained there, this allows us to still define symmetric monoidal functors as maps of operads preserving $\sqcup$.

Conjecture 2.2.13. Equivalently, an absolute factorization algebra is locally constant iff it inverts disk inclusions.

Proposition 2.2.14 ([Mat13, after 2.7 and 2.18])**.** The inclusions $\mathrm{Charts}(n)^{\sqcup} \hookrightarrow \mathrm{Disk}(n)^{\sqcup}$ and $\mathcal{C}harts(M)^{\sqcup} \hookrightarrow \mathcal{D}isk(M)^{\sqcup}$ induce equivalent descriptions for factorization algebras:

$$\begin{aligned}
\mathrm{FA}(\mathcal{V}) &:= \mathrm{Fun}^{\otimes}(\mathrm{Disk}(n)^{\sqcup}, \mathcal{V}^{\otimes}) \simeq \mathrm{Alg}_{\mathrm{Charts}(n)^{\sqcup}}(\mathcal{V}^{\otimes})\\
\mathrm{FA}(M;\mathcal{V}) &:= \mathrm{Fun}^{\otimes}(\mathrm{Disk}(M)^{\sqcup}, \mathcal{V}^{\otimes}) \simeq \mathrm{Alg}_{\mathrm{Charts}(M)^{\sqcup}}(\mathcal{V}^{\otimes})\\
\mathrm{FA}^{lc}(\mathcal{V}) &:= \mathrm{Fun}^{\otimes}(\mathcal{D}isk(n)^{\sqcup}, \mathcal{V}^{\otimes}) \simeq \mathrm{Alg}_{\mathcal{C}harts(n)^{\sqcup}}(\mathcal{V}^{\otimes})\\
\mathrm{FA}^{lc}(M;\mathcal{V}) &:= \mathrm{Fun}^{\otimes}(\mathcal{D}isk^{\sqcup}_{/M}, \mathcal{V}^{\otimes}) \simeq \mathrm{Alg}_{\mathcal{C}harts^{\sqcup}_{/M}}(\mathcal{V}^{\otimes})
\end{aligned}$$

Here, $\mathrm{Alg}_{\mathrm{Charts}(n)^{\sqcup}}(\mathcal{V}^{\otimes})$ denotes the ∞-category of maps of ∞-operads.

Proof Sketch. The first two and the second two statements are similar, let us therefore look at the first. We need to show that a symmetric monoidal functor $A : \mathrm{Disk}(n)^{\sqcup} \to \mathcal{V}^{\otimes}$ is determined by its restriction to $\mathrm{Charts}(n)^{\sqcup} \subseteq \mathrm{Disk}(n)^{\sqcup}$, which is a map of operads since this inclusion preserves the operadic structure. However, any object of $\mathrm{Disk}(n)$ is a disjoint union of disks $D_1 \sqcup D_2 \sqcup \cdots \sqcup D_m \subseteq M$, so the value of A on it is the tensor product of the values $A(D_i)$. The values of A on morphisms and higher simplices are determined by the restriction to $\mathrm{Charts}(n)$ as well, and this kind of extension works for any algebra on $\mathrm{Charts}(n)$, but proving this rigorously would require too much operad theory we have not developed. We say that $\mathrm{Disk}(n)^{\sqcup}$ is the *symmetric monoidal envelope* of $\mathrm{Charts}(n)^{\sqcup}$, this is however wrong over M since $\mathrm{Disk}(M)^{\sqcup}$ is not symmetric monoidal.

The second two statements are slightly more difficult, consider e.g. the operad $\mathcal{C}harts^{\sqcup}_{/M}$. Remember that multimorphisms from $(j_i : D_i \hookrightarrow M)$ to $j : D \hookrightarrow M$ are embeddings $k_i : D_i \hookrightarrow D$ with disjoint images and isotopies from j_i to $j \circ k_i$. On the other side, morphisms in $\mathcal{D}isk^{\sqcup}_{/M}$ from $J : \bigsqcup D_i \hookrightarrow M$ to $j : D \to M$, to which those should correspond, are given by an embedding $K : \bigsqcup D_i \hookrightarrow D$ together with a single isotopy from J to $j \circ K$. While giving K and k_i is equivalent, there are many more (families of) isotopies in the first case. Still, it turns out that up to homotopy, this is no problem – see the reference. □

2.3. Weiss Cosheaves

Why is the definition of factorization algebras as disk algebras equivalent to the first definition we gave via Weiss descent?

Definition 2.3.1. Remember that we had equipped the category $\mathrm{Open}(M)$ with a Grothendieck pretopology, where covers were given by Weiss covers, in 2.1.5. Similarly, we can define a Grothendieck pretopology on Mfd_n, where a family $(U_i \subseteq V)$ of open embeddings of V is a covering iff the (U_i) form a Weiss cover of V.

Definition 2.3.2. A cosheaf $F \in co\mathcal{S}h(M, \mathcal{V}) := \mathcal{S}h(M, \mathcal{V}^{op})$ is called *factorizable* iff the underlying functor $\mathrm{Open}(M) \to \mathcal{V}$ is symmetric monoidal with respect to disjoint union of open subsets. Denote the full subcategory of factorizable cosheaves by $co\mathcal{S}h^{\otimes}(M, \mathcal{V})$.

Theorem 2.3.3. The canonical inclusions $\mathrm{Disk}(n)^{\sqcup} \hookrightarrow \mathrm{Mfd}_n^{\sqcup}$ and $\mathrm{Disk}(M)^{\sqcup} \hookrightarrow \mathrm{Open}(M)^{\sqcup}$ induce equivalences of categories

$$\mathrm{FA}(\mathcal{V}) := co\mathcal{S}h^{\otimes}(\mathrm{Mfd}_{n,\mathrm{Weiss}}, \mathcal{V}) \simeq \mathrm{Fun}^{\otimes}(\mathrm{Disk}(n)^{\sqcup}, \mathcal{V}^{\otimes}) \,, \tag{2.10}$$
$$\mathrm{FA}(M; \mathcal{V}) := co\mathcal{S}h^{\otimes}(\mathrm{Open}(M)_{\mathrm{Weiss}}, \mathcal{V}) \simeq \mathrm{Fun}^{\otimes}(\mathrm{Disk}(M)^{\sqcup}, \mathcal{V}^{\otimes}) \,. \tag{2.11}$$

Proof. Precomposing with the canonical inclusions does induce maps from the left to the right, since they are symmetric monoidal. Their inverses are given by left Kan extension along these inclusions. Showing this in the absolute case can be reduced to the relative case using the characterization of absolute factorization algebras we prove below.

The relative case will be discussed in the more general stratified case in 4.3.5. In particular, the relevant Kan extensions exist since, as we will see, all involved colimits can be cofinally replaced by sifted colimits. □

Remark. In the relative case locally constant factorization algebras on M correspond under this equivalence precisely to locally constant Weiss cosheaves (i.e. those that send disk inclusions to isomorphisms). In the absolute case, a similar characterization would require our conjecture 2.2.13 to hold.

Proposition 2.3.4. A functor $A : \mathrm{Disk}(n) \to \mathcal{V}$ is an absolute factorization algebra iff the restrictions $A|_M : \mathrm{Disk}(M) \to \mathcal{V}$ are relative factorization algebras, for any manifold M.

Proof. We need to show that A is symmetric monoidal iff all $A|_M$ are. The *only if* direction is clear, since the slice projection $\mathrm{Disk}(M) \simeq \mathrm{Disk}(n)_{/M} \hookrightarrow \mathrm{Disk}(n)$ is symmetric monoidal. For the *if* direction, we only show that for $D, D' \in \mathrm{Disk}(n)$ we have $A(D \sqcup D') \cong A(D) \otimes A(D')$; while being symmetric monoidal also involves preserving the higher coherence conditions in a symmetric monoidal ∞-category (compare A.7.9), let us avoid these technicalities. Since we know that $A|_{D \sqcup D'}$ is symmetric monoidal, we are finished. □

Proposition 2.3.5. A functor $A : \mathrm{Mfd} \to \mathcal{V}$ is a Weiss cosheaf iff for any manifold M, its restriction $A|_M : \mathrm{Open}(M) \to \mathcal{V}$ is one.

Proof. We can apply A.4.13 since the inclusion $\mathrm{Open}(M) \to \mathrm{Mfd}_n$ preserves pullbacks and covering families; in particular it is a morphism of sites (and even satisfies the covering-lifting property). Therefore, precomposition $A \mapsto A|_M$ preserves Weiss (co-)sheaves and sheafification. For the converse direction, if $A : \mathrm{Mfd} \to \mathcal{V}$ is a functor such that $A|_M$ are cosheaves for all M, then denote by $s : A \to A^{sh}$ the unit from A into its sheafification. Since restriction preserves sheafification, $s|_M$ is the unit $A|_M \to (A|_M)^{sh}$, which is an isomorphism. Hence, s must already be an isomorphism, since if it was not there had to be a manifold M with $s(M) : A(M) \to A^{sh}(M)$ not an isomorphism. □

Remark. In particular, we see from the proof that isomorphisms of Weiss cosheaves on Mfd can be recognized on the family of restrictions to all M; in other words the functors $(\mathrm{FA}(M; \mathcal{V}) \to \mathrm{FA}(\mathcal{V}))_{M \in \mathrm{Mfd}_n}$ are jointly conservative.

Corollary 2.3.6. Let $A : \mathrm{Mfd}_n \to \mathcal{V}$ be a functor. Then it is an absolute factorization algebra, using the equivalence of 2.3.3, iff for any topological manifold M, the restriction $A|_M : \mathrm{Open}(M)^{\sqcup} \to \mathcal{V}$ is a relative factorization algebra on M.

Proof. Follows immediately from the last two propositions. □

Conjecture 2.3.7. A factorization algebra $A : \mathrm{Disk}_n \to \mathcal{V}$ is locally constant, i.e. factors through $\mathcal{D}isk_n$, iff the restrictions $A|_M$ are locally constant for every $M \in \mathrm{Mfd}_n$. Proving this would require a characterization of $\mathcal{D}isk_n$ as a localization of Disk_n (if possible, including local structures as in 4.2).

2.4. Examples

2.4.1. Little Cubes Operads

After going through all the hassle of actually defining factorization algebras, examples are definitely in order. We will start with the simplest ones, letting $\mathcal{V}^{\otimes}$ still be a sifted complete symmetric monoidal ∞-category:

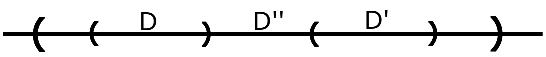
Example 2.4.1. On $M = \emptyset$, there is only one factorization algebra, given by $A : \mathrm{Open}(\emptyset) \to \mathcal{V}$ sending $\emptyset \mapsto 1_{\mathcal{V}}$.

Example 2.4.2. A factorization algebra A on $M = \{*\}$ is determined by its value on the open set $M \subseteq M$, which can be arbitrary, and a morphism $A(\emptyset) = 1_{\mathcal{V}} \to A(M)$. Therefore, $\mathrm{FA}(*, \mathcal{V}) \simeq \mathcal{V}_* = \mathrm{Alg}_{\mathbb{E}_0}(\mathcal{V})$, the category of pointed objects in $\mathcal{V}$. Here, $\mathbb{E}_0$ is the operad we define in A.6.2 and A.6.5.

For non-trivial manifolds, the category of factorization algebras is usually too big to explicitly describe, similar to the category of sheaves. We therefore restrict to locally constant ones.

Example 2.4.3. A locally constant factorization algebra A on $\mathbb{R}$ associates to every open subset $U \subseteq \mathbb{R}$ an element in $\mathcal{V}$. Let us assume that D and D' are disks in $\mathbb{R}$ (in other words, possibly non-bounded open intervals), then we can embed them into the larger disk $D'' = \mathbb{R}$. Since A is locally constant, it sends disk inclusions to isomorphisms, so we can identify $A_0 := A(\mathbb{R}) \cong A(D) \cong A(D')$. Further, if we assume that D and D' are disjoint, the factorization property tells us that $A(D \sqcup D') \cong A(D) \otimes A(D')$.

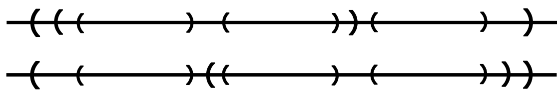
We know that a factorization algebra is determined by its restriction $A : \mathrm{Disk}(M) \to \mathcal{V}$, and the above tells us completely how this restriction acts on objects. However, we are not finished, since as a functor A should also associate a value to any morphism in $\mathrm{Disk}(M)$. These in particular contain inclusions of multiple disks into one, like the following:



An inclusion of three disks into one can be factored as two subsequent inclusions of two disks into one, so we for now disregard inclusions of more than two disks. This leaves the unit $A(\emptyset) \cong 1_{\mathcal{V}} \to A_0 \cong A(D)$ and the diagram above, yielding a morphism

$$A(D \sqcup D') \cong A(D) \otimes A(D') \to A(D'') \tag{2.12}$$

that we identify via the above isomorphisms as a multiplication map $A_0 \otimes A_0 \to A_0$. It is associative because both compositions of disk inclusions below should agree (up to isomorphism) with the inclusion of three disks into one; but it does not have to be commutative, since we can not deform D and D' via a sequence of disk inclusions to exchange their ordering along the real axis, without making them non-disjoint. It is impossible for intervals in $\mathbb{R}$ to continuously be moved past each other.



However, the identification $A(D) \cong A(D')$ above as well as the argument for associativity contain non-canonical isomorphisms. They again satisfy higher coherence conditions, induced by inclusions of multiple disks into one, and trying to understand those by hand can get very unwieldy. This is resolved by the following

Proposition 2.4.4. Factorization algebras on $\mathbb{R}$ with values in $\mathcal{V}^{\otimes}$ are the same thing as associative algebra objects in $\mathcal{V}^{\otimes}$:

$$\mathrm{FA}^{lc}(\mathbb{R}, \mathcal{V}) \simeq \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{V}) \tag{2.13}$$

Proof. It suffices to show that the ∞-operads $\mathbb{E}_1^{\otimes}$ and $\mathit{Charts}_{/\mathbb{R}}^{\sqcup}$ are equivalent. $\mathbb{E}_1^{\otimes}$ only has a single object, and the underlying ∞-category $\mathit{Charts}_{/\mathbb{R}} \simeq \mathrm{Sing}(\mathbb{R})$ by 2.2.6 is contractible, so it suffices to show that there is a functor between them inducing homotopy equivalences on the multimorphism spaces. Those are given by the space of rectilinear embeddings $[-1,1] \times \{1, \dots, k\} \to [-1,1]$ for $\mathbb{E}_1^{\otimes}$; and for $\mathit{Charts}_{/\mathbb{R}}^{\sqcup}$ multimorphisms from $(j_i : \mathbb{R} \hookrightarrow \mathbb{R})$ to $j : \mathbb{R} \to \mathbb{R}$ are embeddings $\{1, \dots, k\} \times \mathbb{R} \to \mathbb{R}$ together with isotopies from their components k_i, postcomposed with j, to j_i. Both spaces can be deformation retracted on the values of the respective embeddings at 0, since $\mathbb{R}$ is contractible. Also, the space of possible choices of isotopies in the second case is contractible for the same reason. So, they are both homotopy equivalent to $\mathrm{Emb}(\{1, \dots, k\}, \mathbb{R}) =: \mathrm{Conf}_k(\mathbb{R})$, in a way that is compatible with composition. The functor $\mathit{Charts}_{/\mathbb{R}}^{\sqcup} \to \mathbb{E}_1^{\otimes}$ that sends all objects to the singe one in $\mathbb{E}_1$, and induces this chain of homotopy equivalences on mapping spaces, is therefore an equivalence. $\square$

Example 2.4.5. If $M = \mathbb{R}^2$, the same argumentation shows that a locally constant factorization algebra is completely determined by its value on a disk $D \subseteq \mathbb{R}^2$, and $A_0 := A(D)$ obtains an algebra structure. Now, however, we *can* move two disks D, D' in $\mathbb{R}^2$ around each other; there is no ordering on $\mathbb{R}^2$ distinguishing $A(D) \otimes A(D') \to A(\mathbb{R}^2)$ and $A(D') \otimes A(D) \to A(\mathbb{R}^2)$ like in $\mathbb{R}$.

This means that A_0 must be a commutative algebra, up to homotopy, since the composition of disk inclusions we use to exchange the positions of two disks again induces an isomorphism that is a priori not controllable. More specifically, moving the disk D completely around the disk D' to its original position via a series of disks inclusions yields a generally non-trivial isomorphism we definitely have to keep track of, a *braiding*.

Abstractly, the same argument as in the above proposition can be used to show that locally constant factorization algebras on $\mathbb{R}^2$ are the same thing as $\mathbb{E}_2$-algebras. We have studied these objects in A.7.7, where we have seen that e.g. for $\mathcal{V}^\otimes$ the (Duskin nerve of the) $(2,1)$-category of ordinary categories, $\mathbb{E}_2$-algebras are braided monoidal categories – their braiding is induced precisely via the above considerations. For $\mathcal{V}^\otimes = \mathcal{C}h(\mathbb{R})$ we obtained dg-algebras with commutative multiplication up to cocycles, where the mentioned braiding is connected to the R-matrix of the Tannaka-dual quasitriangular Hopf algebra. One can show that $\mathbb{E}_2$-algebras in $\mathcal{C}h(\mathbb{R})$ are very close to the definition of vertex operator algebras in conformal field theories, see [CG16, Chapter 5].

Example 2.4.6. For $M = \mathbb{R}^n$, an analogous argument shows that $\mathcal{C}harts^{\sqcup}_{/\mathbb{R}^n} \simeq \mathbb{E}_n^\otimes$. In particular,

$$\mathrm{FA}^{lc}(\mathbb{R}^n, \mathcal{V}) \simeq \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{V}) \ . \tag{2.14}$$

This means that the algebra object $A_0 := A(\mathbb{R}^n)$ becomes more and more commutative as n is increased, as we would expect since it becomes easier to move disks around each other in high dimensions.

Remark. Locally constant factorization algebras with values in an ordinary categories $\mathcal{V}_0$ are less interesting, since for $n \leq 2$ they are all just commutative algebra objects, $\mathrm{Alg}_{\mathbb{E}_2}(\mathcal{V}_0) \simeq \cdots \simeq \mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{V}_0)$ as discussed in A.7.7. In this case, the arguments above are precisely the same reasons for which given a pointed topological space X, the homotopy group $\pi_0(X)$ is a pointed set, $\pi_1(X)$ is a group and $\pi_n(X)$ for $n \geq 2$ is an abelian group. Generally, if $\mathcal{V}$ is an m-category, we reach complete symmetry for $n = m+1$, but this is not the case for $\mathcal{C}h(\mathbb{R})$ or $D(\mathbb{R})$.

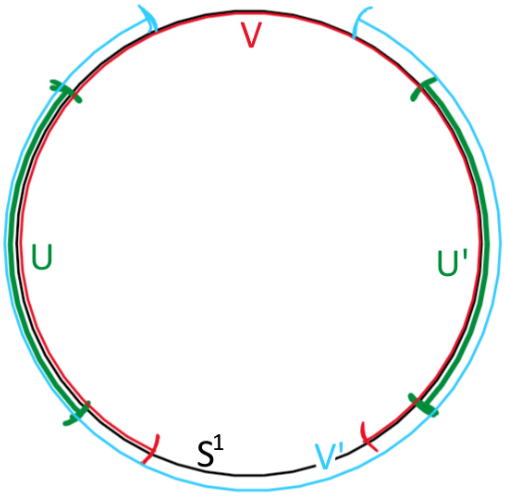
Notation 2.4.7. Because of this result, we will generally also denote $\mathcal{C}harts^{\sqcup}_{/M}$ by $\mathbb{E}_M^\otimes$.

2.4.2. Hochschild Homology and Variants

Next, let us take a look at a locally constant factorization algebra $A : \mathrm{Open}(S^1) \to \mathcal{V}$ over a circle S^1.

For $U, U' \subseteq S^1$ two disjoint disks, we can embed them into a third, bigger disk V, such that the disk inclusions induce isomorphisms $A_0 := A(U) \cong A(V) \cong A(U')$. Therefore, A again takes on the same value on every disk, so that we only need to take a look at the structure maps to obtain a complete understanding of it. As for factorization algebras on $\mathbb{R}$, the inclusion $U \sqcup U' \subseteq V$ induces a multiplication map $A_0 \otimes A_0 \to A_0$, and the inclusion $\emptyset \subseteq U$ induces a unit $1_{\mathcal{V}} = A(\emptyset) \to A_0$. But this is not everything.

Instead of identifying $A(U) \cong A(U')$ via V, we could have used V'. This yield a diagram of isomorphisms:



$$\begin{tikzcd}
 & A(V) \arrow[dr, "\cong"] & \\
A(U) \arrow[ur, "\cong"] \arrow[dr, "\cong"'] & & A(U') \\
 & A(V') \arrow[ur, "\cong"'] &
\end{tikzcd}$$

The composition around this square yields an automorphism $\mu : A(U) \xrightarrow{\cong} A(U)$, and generally this will not just be the identity. In other words, the process of moving a disk around the circle induces a monodromy automorphism on A_0. This is actually the only difference to the case of $\mathbb{R}$:

Proposition 2.4.8. Locally constant factorization algebras on the circle S^1 with values in $\mathcal{V}$ are the same thing as associative algebra objects in $\mathcal{V}$ equipped with an automorphism.

We are, however, not finished. While we now understand the value of A on a disk, and (because of the factorization axiom) on disjoint unions of disks, there is one open subset of S^1 that cannot be written as such a disjoint union: What are the global sections $A(S^1)$?

Definition 2.4.9. For M a topological manifold and $A \in \mathrm{FA}(M; \mathcal{V})$, we call the global sections

$$\int_M A := A(M) = \operatorname*{colim}_{D \in \mathrm{Disk}(M)} A(D) \tag{2.15}$$

the *factorization homology* of A. Similarly, for absolute $A' \in \mathrm{FA}(\mathcal{V})$, its factorization homology $\int_M A$ is the value of the associated Weiss cosheaf $\mathrm{Mfd}_n \to \mathcal{V}$ on M, calculated via the same colimit.

Remark. The formula 2.15 is a special case of Weiss descent; it computes the value of the Left Kan extension that induces the equivalence in 4.3.5 on M.

This should be compared with the definition of sheaf cohomology as a derived functor of taking global sections. Another common name for this construction is *topological chiral homology*, we will see in a moment why. But first, why call it homology at all?

Definition 2.4.10. Given a topological manifold M, a *collar-gluing* of M is a continuous map $f : M \to [0,1]$ such that the restriction $f| : f^{-1}((0,1)) \to (0,1)$ is a fiber bundle with fiber a topological manifold M_0. For $M' := f^{-1}[0,1)$ and $M'' := f^{-1}(0,1]$, which, being open subsets of M, are topological manifolds themselves, we write

$$M = M' \cup_{M_0 \times \mathbb{R}} M'' \ . \tag{2.16}$$

Definition 2.4.11 (Eilenberg-Steenrod Axioms for (factorization) homology theories on manifolds, [AF15, 3.15])**.** A functor $A : \mathcal{M}fd_n \to \mathcal{V}$ is called a *homology theory on n-manifolds* if

- it is symmetric monoidal,
- it preserves sequential colimits, and
- it is excisive: For any collar-gluing $M \cong M' \cup_{M_0 \times \mathbb{R}} M''$, the canonical inclusion maps induce an isomorphism

$$A(M) \cong A(M') \otimes_{A(M_0)} A(M'') \ . \tag{2.17}$$

Denote the full subcategory on homology theories by $\mathcal{H}(\mathcal{M}fd_n, \mathcal{V}) \subseteq \mathrm{Fun}(\mathcal{M}fd_n, \mathcal{V})$. Similarly, for a fixed topological n-manifold M, we can define a category $\mathcal{H}(M, \mathcal{V}) \subseteq \mathrm{Fun}^{\otimes}((\mathcal{M}fd_n)_{/M}, \mathcal{V})$ on those functors that are symmetric monoidal, compatible with collar gluings over M, and preserve sequential colimits (unions) of submanifolds in M

Remark. A colimit is sequential if it is parametrized over $\mathbb{N}$. We can omit the respective axiom if we restrict e.g. to *finitary* manifolds, i.e. manifolds that admit a finite open cover $(U_i)_{i \in I}$ such that for each $S \subseteq I$, the intersection $\bigcap_{i \in S} U_i$ is either empty, or homeomorphic to $\mathbb{R}^n$.

Remark. In the excision axiom, we use the *relative tensor product* of a right- and a left module over the algebra object $A(M_0)$, which is defined via the two-sided bar construction. These right- and left module structures are induced from the characterization of factorization algebras over $[0,1]$ in 4.4.5, since we can pushforward A along the collar gluing $f : M \to [0,1]$ via $f_* A(U) := A(f^{-1}(U))$. This reader not familiar with the bar construction can take the referenced discussion as a definition of the relative tensor product; in the case of S^1 it will be obvious what to do.

Theorem 2.4.12 ([AFT14a, 2.43])**.** Given a locally constant factorization algebra $A : \mathcal{D}isk(n)^{\sqcup} \to \mathcal{V}^{\otimes}$, we can calculate its factorization homology $\int_M A \in \mathcal{V}$ for every manifold M, yielding the associated Weiss cosheaf $\mathcal{M}fd_n^{\sqcup} \to \mathcal{V}^{\otimes}$. This induces equivalences of categories

$$\mathrm{FA}^{lc}(\mathcal{V}) \simeq \mathcal{H}(\mathcal{M}fd_n, \mathcal{V})$$
$$\mathrm{FA}^{lc}(M, \mathcal{V}) \simeq \mathcal{H}(M, \mathcal{V})$$

between factorization algebras and manifold homology theories, both in the absolute and relative case.

Proof Idea. This essentially relies on the fact that any topological manifold can be written as a union of finitary topological manifolds, and the latter always possess a handlebody decomposition (which is a special case of a collar gluing). An exception is dimension 4, where some trickery using a smoothing is necessary. □

Example 2.4.13 ([AF20, 3.31])**.** Write $M = S^1$ as a collar gluing for two copies of $M' = M'' = \mathbb{R}$ along $S^0 \times \mathbb{R}$. Identify $A \in \mathrm{FA}^{lc}(S^1, \mathcal{V})$ with an associative algebra A_0 equipped with a monodromy automorphism $\mu : A \to A$ as above. If $\mu = 0$, its factorization homology can be calculated using the excision axiom as

$$A(S^1) = A(\mathbb{R}) \otimes_{A(S^0 \times \mathbb{R})} A(\mathbb{R}) = A_0 \otimes_{A_0^{op} \otimes A_0} A_0 =: \mathrm{HH}(A_0) \; . \tag{2.18}$$

Since the bar construction used to calculate the relative tensor product is simply the Hochschild complex, for $\mathcal{V}^{\otimes} = \mathcal{C}h(R)^{\otimes}$ this yields usual Hochschild Homology, while for $\mathcal{V}^{\otimes} = D(R)^{\otimes^L}$ one obtains *Shukla Homology.* For non-vanishing μ, we must define A_0 by evaluating A on a disk on either the left or right side of this gluing, so that the module structure of A_0 on the other side must be precomposed with μ. Denote by ${}^{\mu}A_0$ the bimodule over $A_0^{op} \otimes A_0$ with underlying object A_0 and this μ-twisted module action. Then,

$$A(S^1) = A_0 \otimes_{A_0^{op} \otimes A_0} {}^{\mu}A_0 =: \mathrm{HH}(A_0, {}^{\mu}A_0) \; . \tag{2.19}$$

See 3.3.9 for an exemplary calculation from physics.

Next, we state two results that allow for the characterization of locally constant factorization algebras on a large class of manifolds.

Proposition 2.4.14 ([HA, 5.4.5.4])**.** Given topological n-manifolds M, N, restricting along the canonical inclusions $M, N \hookrightarrow M \sqcup N$ induces equivalences

$$\begin{aligned} \mathrm{FA}^{lc}(M \sqcup N; \mathcal{V}) &\simeq \mathrm{FA}^{lc}(M; \mathcal{V}) \times \mathrm{FA}^{lc}(N; \mathcal{V}) \; , \\ \mathrm{FA}(M \sqcup N; \mathcal{V}) &\simeq \mathrm{FA}(M; \mathcal{V}) \times \mathrm{FA}(N; \mathcal{V}) \; . \end{aligned} \tag{2.20}$$

If we use the notation $\mathbb{E}_M := \mathcal{C}harts_{/M}$, one can even write $\mathbb{E}_{M \sqcup N} = \mathbb{E}_M \amalg \mathbb{E}_N$ as a coproduct of ∞-operads.

Proof. Notice that $\mathrm{Disk}(M \sqcup N) \cong \mathrm{Disk}(M) \times \mathrm{Disk}(N)$ and $\mathcal{Disk}_{/M\sqcup N} \cong \mathcal{Disk}_{/M} \times \mathcal{Disk}_{/N}$, since a disjoint union of disks inside $M \sqcup N$ consists by some disks in M, and some in N. Therefore, we can define an inverse to precomposition with $\mathrm{Disk}(M), \mathrm{Disk}(N) \hookrightarrow \mathrm{Disk}(M \sqcup N)$, sending a pair of factorization algebras A, B to the symmetric monoidal functor

$$\mathrm{Disk}(M \sqcup N) \simeq \mathrm{Disk}(M) \times \mathrm{Disk}(N) \to \mathcal{V}\,; \quad (U, V) \mapsto A(U) \otimes B(V) \tag{2.21}$$

and similarly in the locally constant case. □

Theorem 2.4.15 ([HA, 5.4.5.5])**.** Given topological manifolds M^m, N^n, the projections out of the product induce an equivalence of ∞-operads $\mathbb{E}_M \otimes \mathbb{E}_N \simeq \mathbb{E}_{M\times N}$. This means that

$$\mathrm{FA}^{lc}(M \times N; \mathcal{V}) \simeq \mathrm{FA}^{lc}(M; \mathrm{FA}^{lc}(N; \mathcal{V}))\ . \tag{2.22}$$

Remark. This is a generalization of Dunn additivity A.7.8.

Proposition 2.4.16 ([Gin13, Lemma 11])**.** Let M be a topological n-manifold and $\mathcal{V}^{\sqcup}$ be an ∞-category admitting finite coproducts, equipped with the coCartesian symmetric monoidal structure from A.7.10 (i.e. coproduct as multiplication). Then, a factorization algebra in $\mathrm{FA}(M; \mathcal{V})$ is the same thing as a $\mathcal{V}$-valued cosheaf on M. It is locally constant iff the corresponding cosheaf is locally constant. Also, factorization homology agrees with cosheaf cohomology.

Proof. Since every Weiss cover is in particular an open cover, every cosheaf $F : \mathrm{Open}(M) \to \mathcal{V}$ is a Weiss cosheaf. Also, for U, V disjoint,

$$F(U \sqcup V) = \operatorname*{colim}_{\Delta}(\coprod_{U,V} F \to 0) = F(U) \amalg F(V) \tag{2.23}$$

making F symmetric monoidal (let us ignore higher coherences).

Conversely, for $A : \mathrm{Open}(M) \to \mathcal{V}$ a factorizable Weiss cosheaf, let us check that it is also a cosheaf. Using that M is Hausdorff, we can refine any basis of its topology to a *factorizing basis* $\mathfrak{B}$, meaning that for any finite set $S \subseteq M$, any neighborhood of S contains pairwise disjoint open subsets $(U_s)_{s\in S} \in \mathfrak{B}$ such that $s \in U_s$. Now, let $\mathfrak{U} = (U_i \subseteq U)$ be a cover of $U \subseteq X$ open such that without loss of generality, $\mathfrak{U}$ is closed under intersections. We need to show that

$$A(U) \cong \operatorname*{colim}_{U_i \in \mathfrak{U}} A(U_i)\ . \tag{2.24}$$

Denote by $\mathcal{P}\mathfrak{B}_{U_i}$ the set of disjoint unions of elements in $\mathfrak{B}$ that are contained in U_i, and by $\mathcal{P}\mathfrak{B}_{\mathfrak{U}}$ the set of disjoint unions of elements in $\mathcal{P}\mathfrak{B}_{U_i}$ for possibly distinct i. By definition, $\mathcal{P}\mathfrak{B}_{U_i}$ are Weiss covers; and the isomorphism

$$\operatorname*{colim}_{B \in \mathcal{P}\mathfrak{B}_{\mathfrak{U}}} A(B) \cong \operatorname*{colim}_{U_i \in \mathfrak{U}} \operatorname*{colim}_{B_i \in \mathcal{P}\mathfrak{B}_{U_i}} A(B_i) \tag{2.25}$$

follows from the factorization property of A. We are finished when we show that $\mathcal{P}\mathfrak{B}_{\mathfrak{U}}$ is a Weiss cover, but this is clear since $\mathfrak{U}$ is a cover, $\mathfrak{B}$ a factorizing basis and we take disjoint unions.

Now, the fact that factorization homology and cosheaf cohomology agree is automatic, since the latter is given by the global section functor (in the ∞-setting, we do not need to derive) which is calculated by the same Čech nerve as the factorization homology. Also, by B.3.7, a cosheaf is locally constant iff it sends disk inclusions to isomorphisms, which again is the same condition as for factorization algebras. □

Remark. While the proof in the reference is a lot shorter, we feel like it omits some important details.

Example 2.4.17. A locally constant factorization algebra on the torus $S^1 \times S^1$ is, by Dunn additivity, the same thing as an algebra with automorphism in the category of algebras with automorphisms in $\mathcal{V}$, which is just an $\mathbb{E}_2$-algebra with two distinct automorphisms. Similarly, a locally constant factorization algebra on the cylinder $\mathbb{R} \times S^1$ is an $\mathbb{E}_2$-algebra with automorphism.

Example 2.4.18. A locally constant factorization algebra on S^2 is more difficult to describe: If we cut out the north and south pole, we obtain a cylinder, restricting to which we should obtain an $\mathbb{E}_2$-algebra with automorphism. Adding back the pole caps amounts to giving two distinct paths from this automorphism to the identity, in the space of all automorphism (note that adding only one pole would give $\mathbb{R}^2$, just as a single path would allow us to identify our automorphism with the identity). One way to make this precise is the following theorem.

Theorem 2.4.19 ([HA, 5.4.5.2])**.** Given a topological manifold M of dimension n, the ∞-operad $\mathbb{E}_M$ is the colimit over $\mathrm{Sing}(M) \simeq \mathcal{C}harts(M)$ of the diagram sending each point to $\mathbb{E}_n$, with values on paths and higher simplices induced by transport using disk inclusions:

$$\begin{gathered} \mathbb{E}_M \simeq \operatorname*{colim}_{\mathrm{Sing}(M)} \mathbb{E}_n \\ \mathrm{FA}^{lc}(M;\mathcal{V}) \simeq \lim_{\mathrm{Sing}(M)} \mathrm{FA}^{lc}(\mathbb{R}^n;\mathcal{V}) \simeq \lim_{\mathrm{Sing}(M)} \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{V}) \end{gathered} \tag{2.26}$$

Remark. Notice how the latter statement follows immediately from the first, since mapping spaces in the ∞-category $\mathcal{O}p_\infty$ that the colimit is taken in are given by spaces of algebra objects (i.e. maps of ∞-operads).

Finally, let us think about absolute factorization algebras:

- $\mathcal{D}isk(-1)$ only consists of the empty set and empty morphism, so there is only a single (-1)-dimensional absolute factorization algebra, namely $A(\emptyset) = 1_{\mathcal{V}}$. We regard $\emptyset$ as a manifold of any dimension.

- $\mathcal{D}isk(0)$ consists of (possibly empty) disjoint unions of points, so using the factorization axiom, 0-dimensional absolute factorization algebras are just pointed objects in $\mathcal{V}$, i.e. $\mathcal{D}isk(0)^{\sqcup} \simeq \mathbb{E}_0^{\otimes}$.
- $\mathcal{D}isk(1)$ consists of disjoint unions of $\mathbb{R}$, with multimorphisms given by embeddings $\mathbb{R} \times \{1, \dots, k\} \hookrightarrow \mathbb{R}$. As explained before, we can deformation retract the space of such embeddings such that up to homotopy, it only depends on the ordering of the open intervals corresponding to the summands of $\mathbb{R}$, and also their orientations! Therefore, a 1-dimensional absolute factorization algebra is determined by an associative algebra A together with an involution $i : A \cong A^{op}$ (since changing the orientation of an embedding changes the order of multiplications when composing).
- One can define 1-dimensional absolute factorization algebras for only oriented, or framed, manifolds. In this case, $\mathcal{D}isk^{or}(1)^{\sqcup} \simeq \mathcal{D}isk^{fr}(1)^{\sqcup} \simeq \mathbb{E}_1^{\otimes}$ as the involution disappears. See 4.2 for more.
- Generally, by [HA, 5.4.2.9], one can obtain $\mathcal{D}isk(n)^{\sqcup}$ as a colimit over a diagram of operads parametrized by $\mathrm{BTop}(n)$ that sends the unique object to $\mathbb{E}_n$, to be a precise $\mathcal{D}isk(n)^{\sqcup} = (\mathbb{E}_n)_{h\,\mathrm{BTop}(n)}$ is given by taking homotopy coinvariants in $\mathcal{O}p_{\infty}$. Thus, an n-dimensional absolute factorization algebra is a homotopy invariant in $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{V})^{h\,\mathrm{BTop}(n)}$, i.e. an $\mathbb{E}_n$-algebra equipped with one automorphism for each homeomorphism from $\mathbb{R}^n$ to itself, with the caveat that (like in a semi-direct product), this automorphism interacts with the $\mathbb{E}_n$-structure. For example for $n = 1$, we have $\mathrm{Top}(1) \simeq \mathbb{Z}_2$ and we obtain the statement above.
- If we restrict to absolute factorization algebras on oriented manifolds, we should only consider orientation-preserving homeomorphisms in the previous statement; and for smooth manifolds we should only take homotopy invariants under $\mathrm{O}(n)$. Similarly for other kinds of restrictions; we introduce a general machinery in 4.2). If we restrict to framed manifolds, $\mathcal{D}isk^{fr}(n)^{\sqcup} \simeq \mathbb{E}_n^{\otimes}$.

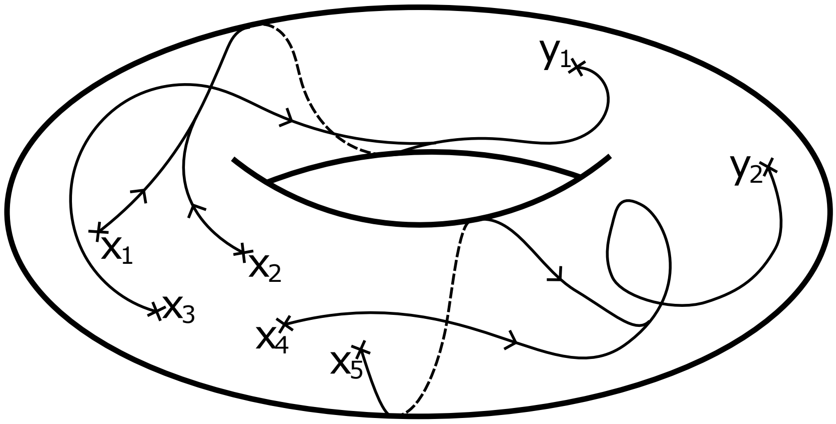
2.5. Digression: Exodromy in the Ran space

Since they encode observables of perturbative quantum field theories, factorization algebras must have a very close relationship to Feynman Diagrams and Operator Product Expansions. We give a short discussion of one particular way to see this that also sheds light on how these mathematical structures were originally discovered. Since we do not further use the following results and precise statements are highly technical, this section can be skipped on first reading. We assume familiarity with both Appendices.

The main idea is very straightforward and follows the lines of B.5.5: Given a locally constant sheaf $\mathcal{F}$ on a topological space X, and a continuous path $p : [0, 1] \to X$, we obtained a monodromy map between the stalks $\mathcal{F}_{p(0)} \to \mathcal{F}_{p(1)}$. In fact, $\mathcal{F}$ is uniquely determined by its monodromy for any homotopy class of path between any two points, as explained in the proof of B.5.5.

So locally constant sheaves admit monodromy along paths, but what kind of monodromy do locally constant factorization algebras admit? If our intuition about a close relationship to operator products is right, we should replace individual paths by finite sets of paths that start and end at a finite set of points, respectively, and are allowed to join together but not to split (mimicking a finite product of vertex operators, whose insertion points can be moved around inside the manifold and joined together, yielding an OPE). We call such complexes of paths *multipaths*, since their joining resembles the composition of multimorphisms in a multicategory (see A.6.1).

Figure 2.1.: Multipath from $\{x_1, \ldots, x_5\}$ to $\{y_1, y_2\}$ in T^2



The first step in proving that locally constant factorization algebras satisfy monodromy along multipaths is to find a nice mathematical description of the latter.

Definition 2.5.1. For X a topological space, we define the *Ran space* $\mathrm{Ran}(X)$ as the set of nonempty finite subsets of X.

For subsets $A, B \subseteq \mathrm{Ran}(X)$ we define $A \star B := \{S \cup T | S \in A, T \in B\} \subseteq \mathrm{Ran}(X)$. Also, for $(U_i)_{i=1,\ldots,n}$ subsets of X, we define the subset $\mathrm{Ran}(\{U_i\}) := \mathrm{Ran}(U_1) \star \cdots \star \mathrm{Ran}(U_n) \subseteq \mathrm{Ran}(X)$. Then, a topology of $\mathrm{Ran}(X)$ is induced by taking sets of the form $\mathrm{Ran}(\{U_i\})$ as a basis, where the U_i are pairwise disjoint open subsets of X.

Also, $\mathrm{Ran}(X)$ admits a natural $\mathbb{N}_{>0}$-stratification by cardinality of the finite subsets. We call the strata $\mathrm{Ran}(X)_n =: \mathrm{Conf}_n(X)$ the *configuration spaces* of n (indistinguishable) points in X. They carry the induced topology from $X^{\times n}$, as one can show.

Proposition 2.5.2. For M a topological manifold, $\mathrm{Ran}(M)$ is paracompact and Hausdorff, since for a fixed metric d on M it obtains a metric

$$d(S, T) := \max\{\sup_{x \in S} \inf_{y \in T} d(x, y), \sup_{x \in T} \inf_{y \in S} d(x, y)\} \; . \tag{2.27}$$

As shown in [CL21], it is even a C^0-stratified space in the sense of B.1.9. While the closed strata $\mathrm{Ran}(M)_{\leq n}$ are of finite covering dimension, the whole space $\mathrm{Ran}(M)$ generally is not, and ∞-sheaves on it therefore don't have to be hypercomplete (see A.5.9).

Definition 2.5.3. We define a modified Ran space as a colimit in the category of topological spaces (hence, carrying the final topology):

$$\mathrm{Ran}(M)_{<\omega} := \operatorname*{colim}_{n\in N} \mathrm{Ran}(M)_{\leq n} \tag{2.28}$$

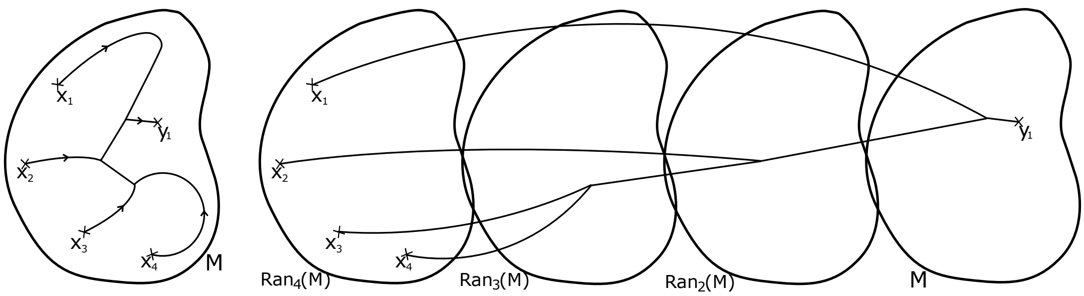
This space has the same set of points as $\mathrm{Ran}(M)$, and is $\mathbb{N}_{>0}$-stratified in the same way. Generally, it however has a different topology and is not conically stratified, in particular not C^0-stratified by [Lej21, 2.14].

Proposition 2.5.4 ([Lej21, 3.3])**.** The exit path categories of $\mathrm{Ran}(M)$ and $\mathrm{Ran}(M)_{<\omega}$ are equivalent and given by a colimit over closed strata:

$$\mathrm{Sing}^{\mathbb{N}_{>0}} \mathrm{Ran}(M) = \mathrm{Sing}^{\mathbb{N}_{>0}} \mathrm{Ran}(M)_{<\omega} = \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Sing}^{\{1<\cdots<n\}} \mathrm{Ran}(M)_{\leq n} \tag{2.29}$$

This already solves our problem: The opposite ∞-category of enter-paths $\mathrm{Sing}^{\mathbb{N}_{>0}} \mathrm{Ran}(M)^{op}$ contains as objects finite subsets of M, and as morphisms paths in $\mathrm{Ran}(M)$ that start in a higher cardinality stratum and stay in it, until they finally end in a lower cardinality stratum. Put in different words, we start with n distinct points that move around in M without colliding, until at the final time some of their paths join together to yield $m \leq n$ distinct points; see the figure for an impressionistic illustration. We have constructed an ∞-category of multipaths!

Figure 2.2.: Multipath and depiction of the associated concatenation of enter paths in the Ran space



Warning. As we define exit paths in B.2, they are only allowed to start in one stratum and live the rest of their life in a single, different stratum – however in the diagram, we let exit paths pass through multiple strata. It turns out that this does not matter as it yields equivalent categories in our case (every path going through multiple strata can be slightly deformed so it only visits the beginning, and the lowest, stratum), but it should still be noted.

But what does it mean to satisfy monodromy with respect to multipaths? In analogy with B.5.6 and B.5.11, we would expect that a locally constant factorization algebra with values in $\mathcal{V}$ yields a functor $\operatorname{Sing}^{\mathbb{N}_{>0}} \operatorname{Ran}(M)^{op} \to \mathcal{V}$. As a short digression, let us characterize such functors via the exodromy correspondence. For $\mathcal{V}$ a presentable stable or compactly generated ∞-category, B.5.11 tells us

$$\mathcal{Sh}^{hypcbl}(\operatorname{Ran}(M); \mathcal{V}) \simeq \mathcal{Sh}^{cbl}(\operatorname{Ran}(M)_{<\omega}; \mathcal{V}) \simeq \operatorname{Fun}\left(\operatorname{Sing}^{\mathbb{N}_{>0}} \operatorname{Ran}(M), \mathcal{V}\right) \tag{2.30}$$

where we have to work with constructible hypersheaves since the stratification poset $\mathbb{N}_{>0}$ does not satisfy the ascending chain condition. We can restrict to constructible sheaves on $\operatorname{Ran}(M)_{<\omega}$ by the treatment of $\mathbb{N}$-stratified spaces in [Lej21]; in analogy to section 4.3 therein.

Relating this result to our previous characterizations of factorization algebras requires the following at first glance miraculous statement:

Proposition 2.5.5 ([AFT14a, 2.20])**.** Let M be connected and $\mathcal{Disk}^{surj,\leq i}(M)$ denote the subcategory of $\mathcal{Disk}(M)$ on finite disjoint unions of $\leq i$ disks in M, together with morphisms that are surjective on connected components. Then,

$$\operatorname{Sing}^{\{1,\ldots,i\}} \operatorname{Ran}(M)^{op}_{\leq i} \simeq \mathcal{Disk}^{surj,\leq i}(M) \tag{2.31}$$

Using 2.5.4 and the observation that $\mathcal{Disk}^{surj}(M)$ is the union over the above full subcategories, we can conclude

$$\operatorname{Sing}^{\mathbb{N}_{>0}} \operatorname{Ran}(M)^{op} \simeq \mathcal{Disk}^{surj}(M) \tag{2.32}$$

Remark. Compare this statement with 2.2.6.

Corollary 2.5.6. Every locally constant factorization algebra yields a monodromy representation on multipaths via the composition

$$\begin{aligned} \operatorname{FA}^{lc}(M, \mathcal{V}) \simeq \operatorname{Fun}^{\otimes}(\mathcal{Disk}(M), \mathcal{V}) \to \operatorname{Fun}^{\otimes}(\mathcal{Disk}(M), \mathcal{V}) \simeq \operatorname{Fun}^{\otimes}(\mathcal{Disk}(M), \mathcal{V}) \simeq \\ \simeq \operatorname{Fun}\left(\operatorname{Sing}^{\mathbb{N}_{>0}} \operatorname{Ran}(M), \mathcal{V}^{op}\right) \simeq co\mathcal{Sh}^{cbl}(\operatorname{Ran}(M)_{<\omega}) \end{aligned} \tag{2.33}$$

Remark. This map is not an equivalence of categories, like for locally constant sheaves:

- We go from symmetric monoidal functors to arbitrary functors, which forgets the factorization axiom. This can be remedied by restricting to *factorizable* constructible hypersheaves on the Ran space; those that are symmetric monoidal with respect to the $\star$ operation above.
- We restrict from $\mathcal{Disk}(M)$ to $\mathcal{Disk}(M)^{surj}$, forgetting about the situation where an empty disjoint union of disks is embedded into a disk. This yields a map $A(\emptyset) = 1_{\mathcal{V}} \to A(D)$ acting as an unit object in A. Therefore, the monodromy equivalence we derived only holds for *non-unital* factorization algebras.

There does not seem to be an extension of the equivalence to the unital case yet, although there are a few ideas on how to do it in [CL21] and [HA, 5.5.4.12].

2.6. Equivalent Perspectives on Factorization Algebras

Let us conclude this chapter by summarizing the equivalent characterizations of (relative) factorization algebras we have obtained. Let M be a topological manifold.

Theorem 2.6.1. A factorization algebra on M with values in a sifted complete ∞-category $(\mathcal{V}, \otimes)$ is equivalently given by:

- A factorizable Weiss cosheaf on M
- A symmetric monoidal functor $(\mathrm{Disk}(M), \sqcup) \to (\mathcal{V}, \otimes)$

$$\mathrm{FA}(M, \mathcal{V}) = co\mathcal{S}h^{\otimes}(M_{\mathrm{Weiss}}, \mathcal{V}) \simeq \mathrm{Fun}^{\otimes}(\mathrm{Disk}(M)^{\sqcup}, \mathcal{V}^{\otimes})$$

Theorem 2.6.2. A locally constant factorization algebra on M with values in a sifted complete ∞-category $(\mathcal{V}, \otimes)$ is equivalently given by:

- A locally constant factorizable Weiss cosheaf on M
- A symmetric monoidal functor $(\mathrm{Disk}(M), \sqcup) \to (\mathcal{V}, \otimes)$ that sends disk inclusions to isomorphisms
- A symmetric monoidal functor $\mathcal{D}isk(M) \to \mathcal{V}$, where $\mathcal{D}isk(M) = \mathbb{E}_M$ is the little cubes operad on M
- A homology theory in $\mathcal{H}(M, \mathcal{V})$, see 2.4.11.

We have seen (but will not further use) that this induces:

- A factorizable constructible hypersheaf on $\mathrm{Ran}(M)$
- A factorizable constructible sheaf on $\mathrm{Ran}(M)_{<\omega}$

$$\mathrm{FA}^{lc}(M, \mathcal{V}) = co\mathcal{S}h^{\otimes, lc}(M_{\mathrm{Weiss}}, \mathcal{V}) = \mathrm{Fun}^{\otimes, lc}(\mathrm{Disk}(M)^{\sqcup}, \mathcal{V}^{\otimes}) = \mathrm{Fun}^{\otimes}(\mathcal{D}isk(M)^{\sqcup}, \mathcal{V}^{\otimes})$$

Technical Remark. Another equivalent definition of factorization algebras, that is even more conceptual as the above, arises from Goodwillie-Weiss-Calculus. Given a functor $\mathrm{Mfd}_n \to \mathcal{S}$ (one can also allow fairly general ∞-categories, see [HA, Chapter 6]), this formalism allows to construct an associated Taylor-Expansion, i.e. a unique decomposition into so called n-excisive functors that behave in some sense like monomials. Absolute factorization algebras are precisely the analytic functors (those that agree with their Taylor expansion) satisfying a factorization property, see [dBW12, Theorem 5.2].

3. Abelian Chern-Simons Theory and other Physical Examples

The methods of the last chapter will now be put to use to describe field theories on manifolds without boundary. After explaining a new way to think about the BV formalism that relies on the path integral and therefore makes sense for quantum (and not only classical) field theories, we construct the corresponding factorization algebras for general free field theories, in particular the free scalar field and Chern-Simons theory among many other examples. We loosely follow the discussion in [CG16, Chapter 4].

3.1. Homological Path Integral

Let us begin by giving a different point of view on the BV-BRST formalism that will be helpful to make contact with more standard concepts in (quantum) field theories, especially when we later look at factorization algebras of quantum observables.

One of the main purposes of a quantum field theory is to assign, to any observable $O[\phi]$ depending on the field configuration ϕ, an expectation value. For example, take a look at the scalar field $\phi \in C^\infty(M)$ on a smooth manifold M with Riemannian metric g, with action

$$S_{KG}[\phi] = \int_M \frac{1}{2}\phi(\Delta_g + m^2)\phi \,\mathrm{vol}_g \tag{3.1}$$

where Δ_g is the Laplace operator, and vol_g the volume form associated to g; and $m^2 \in \mathbb{R}$ is the mass of ϕ. Then, the expectation value of $O[\phi]$ is determined by the path integral

$$\langle \mathcal{O} \rangle_S = \frac{\int D\phi\, O[\phi] \cdot e^{-S/\hbar}}{\int D\phi\, e^{-S/\hbar}} \;, \tag{3.2}$$

where the denominator may be set to 1 by adding a suitable constant to S. In other words, the path integral determines a linear map $\langle - \rangle_S : \{\text{observables}\} \to \mathbb{R}$. Up to the arbitrary normalization factor, this map is uniquely determined by its kernel via the homomorphism theorem. Since there is no mathematically rigorous way to define the path integral yet, this observation opens up a different way to proceed:

- Elaborate what exactly is meant by the term *observable* in this context. On the one hand, $\langle - \rangle_S$ should a priori be defined on all off-shell observables; but we should pose restrictions to make a mathematical discussion of this map actually feasible.

- Find a large subspace of the space observables (if possible of codimension 1) where $\langle - \rangle_S$ vanishes.

Let us start with the first point. An off-shell observable for the scalar field on an open subset $U \subseteq M$ is, by definition, a function on the space of field histories $\mathcal{F} = C^\infty(U)$. We restrict ourselves to polynomial functions (one could also use power series) in the field ϕ, since we only want to work perturbatively in the end anyway. As a first step, linear off-shell observables are just continuous linear functionals on $\mathcal{F}$:

$$\mathcal{F}^\vee = C^\infty(U)^\vee := \mathrm{Hom}_{cnt}(C^\infty(U), \mathbb{R}) = \bar{C}_c^\infty(U) \tag{3.3}$$

Here $\mathcal{F}^\vee$ is the strong dual of a topological vector space (the space of continuous linear forms), and $\bar{C}_c^\infty(U)$ are the compactly supported distributions on U. The most straightforward way to obtain polynomial observables would be to form the symmetric algebra, but this operation uses the tensor product and we must remember that we are working with topological vector spaces, so we need to complete it:

$$P(\mathcal{F}) := \bigoplus_{i=0}^{\infty} C_c^\infty(U^i)_{S^i} = \bigoplus_{i=0}^{\infty} (C_c^\infty(U) \hat{\otimes} \dots \hat{\otimes} C_c^\infty(U))_{S^i} \tag{3.4}$$

is the space of polynomial observables that we will consider, where we went back from distributional to smooth sections for convenience; we will explain this in 3.1.15. The subscript S^i means taking coinvariants with respect to the action of the symmetric group (taking invariants yields an isomorphic result) and $\hat{\otimes}$ is the (completed) tensor product of *convenient vector spaces* that contains the algebraic tensor product as a dense subspace – see [CG16] and the references therein for more. In particular, a homogeneous element $O \in C_c^\infty(U^i)_{S^i}$ acts on ϕ as

$$O[\phi] = \int_U O(x_1, x_2, \dots, x_i) \cdot \phi(x_1)\phi(x_2) \dots \phi(x_i) \ \mathrm{vol}_g(x_1) \dots \mathrm{vol}_g(x_n) \ . \tag{3.5}$$

We will not care much for functional analytic subtleties, we only need, for vector bundles E, F the equality

$$\Gamma(U, E) \hat{\otimes} \Gamma(U, F) = \Gamma(U, E \boxtimes F) \tag{3.6}$$

where $E \boxtimes F$ is the exterior tensor product bundle $\mathrm{pr}_1^* E \otimes \mathrm{pr}_2^* F \to M \times M$. We have used this identity for the case of trivial bundles above. Note that it is *not* the definition of $\hat{\otimes}$ since this operation is defined in more general situations than sections of vector bundles, but a very nice result. We also need:

Proposition 3.1.1 ([CG16, Appendix B])**.** The category of *convenient vector spaces* CVS is a full subcategory of locally convex topological vector spaces and continuous linear maps that has several good properties: It is abelian (but not Grothendieck abelian!), possesses a symmetric tensor product $\hat{\otimes}$ and an associated internal Hom. Finally, it admits all colimits and $\hat{\otimes}$, being a left adjoint, preserves colimits separately in its arguments.

Both CVS and the category of locally convex topological vector spaces can be fully faithfully embedded into another category DVS of *differentiable vector spaces*. They are defined as vector-space-valued sheaves on the site of smooth manifolds Man from 1.4.5 that are modules over the algebra object $C^\infty : \mathrm{Man}^{op} \to \mathrm{Vec}_\mathbb{R}$ and possess some kind of a flat connection.

This category is Grothendieck abelian, possesses an internal Hom and a notion of multimorphisms (i.e. it is a colored operad), and the embedding CVS $\hookrightarrow$ DVS is additive, preserves kernels, multimorphism spaces (remember that symmetric monoidal categories can be regarded as operads) and internal Homs; but not cokernels and quasi-isomorphisms.

Corollary 3.1.2. The ∞-category of chain complexes $\mathcal{Ch}(\mathrm{CVS})$ and its derived category $D(\mathrm{CVS})$, as defined in A.3.14, are stable, sifted complete symmetric monoidal ∞-categories; and $\mathcal{Ch}(\mathrm{DVS})$ as well as $D(\mathrm{DVS})$ are stable ∞-categories underlying respective ∞-operads. In fact, $D(\mathrm{DVS})$ is even presentable.

Proof Sketch. Stability follows by construction, the symmetric monoidal structure is induced from $\hat{\otimes}$ (which does not need to be derived since it can be shown to preserve kernels), and still preserves colimits separately in its arguments since the internal Hom induces a right adjoint to it. To conclude, $D(\mathrm{DVS})$ is presentable by [HA, 1.3.5.13] as DVS is Grothendieck abelian. $\square$

Warning. As is made clear by this statement, we can not have everything – either we work with a category that is not presentable, or with a different category that has no symmetric monoidal structure. In particular, our definition of factorization algebras does not extend to values in differentiable vector spaces – we will ignore this and similar problems, since most of them can be fixed by switching around between CVS and DVS since the former is a full subcategory of the latter (be aware that only homotopy equivalences are preserved under this operation though, not general quasi-isomorphisms) or hoping that an even better setting will be found in the future.

Example 3.1.3. For E any vector bundle on M and $U \subseteq M$ open, the spaces

$$\begin{aligned} &\mathcal{E}(U), && \bar{\mathcal{E}}(U) := \mathcal{E}(U) \otimes_{C^\infty(M)} \bar{C}^\infty(M) \\ &\mathcal{E}_c(U), && \bar{\mathcal{E}}_c(U) := \mathcal{E}_c(U) \otimes_{C^\infty(M)} \bar{C}^\infty(M) \end{aligned}$$

of (compactly supported and/or distributional) sections are convenient, and hence also differentiable, vector spaces.

After this digression, let us return to our second point. We can regard the path integral $\int D\phi\, O[\phi] e^{-S/\hbar}$ as an integral of O with respect to the modified "measure" $e^{-S/\hbar} D\phi$, so we have immediate access to a large space where $\langle - \rangle_S$ must vanish – total derivatives.

Example 3.1.4. Let us, for a moment, assume that $\mathcal{F}$ is an n-dimensional vector space. If we denote by $\mathrm{Vect}(\mathcal{F})$ the space of vector fields on it, by $\Omega^\bullet(\mathcal{F})$ the space of differential forms, and by $\mathrm{vol}'_{\mathcal{F}} = \exp(-S/\hbar)dx_1 \dots dx_n$ the modified volume form on $\mathcal{F}$, then a function on $\mathcal{F}$ is a *total derivative* iff it is in the image of the *divergence* map Div_S, defined via the commuting diagram

$$\begin{tikzcd} \Omega^{n-1}(\mathcal{F}) \arrow[r, "d"] & \Omega^n(\mathcal{F}) \\ \mathrm{Vect}(\mathcal{F}) \arrow[u, "\cong"] \arrow[r, "\mathrm{Div}_S"] & \Omega^0(\mathcal{F}) \arrow[u, "\cong"] \end{tikzcd}$$

where the vertical isomorphisms are induced by contracting with the non-vanishing top-form $\mathrm{vol}'_{\mathcal{F}}$. Explicitly, it can be written in coordinates as

$$\mathrm{Div}_S\left(\sum_i v_i \frac{\partial}{\partial x_i}\right) = -\sum_i v_i \frac{\partial S}{\hbar \partial x_i} + \sum_i \frac{\partial v_i}{\partial x_i}\,. \tag{3.7}$$

For a total derivative,

$$\int_{\mathcal{F}} \mathrm{Div}_S(v)\,\mathrm{vol}'_{\mathcal{F}} = \int_{\mathcal{F}} d(\iota_v\,\mathrm{vol}'_{\mathcal{F}}) = 0 \tag{3.8}$$

Extending this to the infinite-dimensional case amounts to first fixing a space of polynomial vector fields. A generating system for them is determined by the set of possible directions in $\mathcal{F}$; since this is a vector space we identify each direction with a $\phi \in \mathcal{F}$ and write it as $\frac{\partial}{\partial \phi}$. It acts on above homogeneous observable O as

$$\frac{\partial}{\partial \phi} O(x_1, \dots, x_n) = \sum_j \int_U \phi(x_j) O(x_1, \dots, x_i)\,\mathrm{vol}_g(x_j) \tag{3.9}$$

A general vector field is then an element of

$$\mathrm{Vect}(\mathcal{F}) := P(\mathcal{F}) \hat{\otimes} C_c^\infty(U) = \bigoplus_{i=0}^{\infty} C_c^\infty(U^{i+1})_{S^i} \tag{3.10}$$

where coinvariants act on the first n factors, and can be approximated by linear combinations of $\frac{\partial}{\partial \phi}$ where ϕ has compact support to ensure that our divergence operator

$$\mathrm{Div}_S\left(O \otimes \frac{\partial}{\partial \phi}\right) := -O \cdot (\Delta + m^2)\phi + \frac{\partial}{\partial \phi} O \tag{3.11}$$

(extended linearly and continuously to non-homogeneous elements of $P(\mathcal{F})$ and the completed tensor product) is a map $\mathrm{Div}_S : \mathrm{Vect}(\mathcal{F}) \to P(\mathcal{F})$.

Definition 3.1.5. For any open $U \subseteq M$, denote by

$$H^0\,\mathcal{O}bs^q(U) := \frac{P(\mathcal{F})}{\operatorname{im}\operatorname{Div}_S} \tag{3.12}$$

the zeroth cohomology of the *space of quantum observables* of the scalar field. We have seen that the map $\langle - \rangle_S$ should factor as $H^0\,\mathcal{O}bs^q(U) \to \mathbb{R}$.

We have thus restricted our problem of understanding the expectation value from all off-shell observables to a usually small factor space. However, we cannot be satisfied yet since $\operatorname{Vect}(\mathcal{F})$ that enters this quotient is again a big space. There are many vector fields v for which $\operatorname{Div}_S v = 0$ identically, and it would be nice to factor those out when finding functions that are total derivatives – in order to do this, we want to extend the divergence operator to a differential in a chain complex, so that its value vanishes in particular on all cocycles (and our hope is that the homology groups of this chain complex are small). This is definitely possible for finite-dimensional $\mathcal{F}$ – the commutative diagram

$$\begin{tikzcd}
\Omega^0(\mathcal{F}) \arrow[r, "d"] & \dots \arrow[r, "d"] & \Omega^0(\mathcal{F}) \arrow[r, "d"] & \Omega^{n-1}(\mathcal{F}) \arrow[r, "d"] & \Omega^n(\mathcal{F}) \\
\mathrm{PV}^n(\mathcal{F}) \arrow[u, "\cong"] \arrow[r, "\operatorname{Div}_S"] & \dots \arrow[u, "\cong"] \arrow[r, "\operatorname{Div}_S"] & \mathrm{PV}^2(\mathcal{F}) \arrow[u, "\cong"] \arrow[r, "\operatorname{Div}_S"] & \mathrm{PV}^1(\mathcal{F}) \arrow[u, "\cong"] \arrow[r, "\operatorname{Div}_S"] & \mathrm{PV}^0(\mathcal{F}) \arrow[u, "\cong"]
\end{tikzcd}$$

defines divergence operators on arbitrary polyvector fields $\mathrm{PV}^n(\mathcal{F}) := \Gamma(U, \bigwedge^n T\mathcal{F})$, where the vertical isomorphisms are again contractions with the modified volume form. This also works in the functional analytic case if, using completed versions of the symmetric and antisymmetric tensor product obtained by taking coinvariants with respect to the evident action, we define:

$$\mathrm{PV}^n(\mathcal{F}) := \operatorname{Sym} C_c^\infty(U)\hat{\otimes}{\textstyle\bigwedge^n} C_c^\infty(M) := \bigoplus_{i=0}^{\infty} C_c^\infty(U^{i+n})_{S^i \times S^n} \tag{3.13}$$

Definition 3.1.6. The *divergence complex*, or *algebra of quantum observables*, of the scalar field is the chain complex of convenient (or differentiable) vector spaces

$$\begin{aligned}
\mathcal{O}bs^q(U) &:= (\mathrm{PV}^*(\mathcal{F})[\hbar], \hbar\operatorname{Div}_S) = \\
&= \left(\dots \xrightarrow{\hbar\operatorname{Div}_S} \mathrm{PV}^2(\mathcal{F})[\hbar] \xrightarrow{\hbar\operatorname{Div}_S} \mathrm{PV}^1(\mathcal{F})[\hbar] \xrightarrow{\hbar\operatorname{Div}_S} \mathrm{PV}^0(\mathcal{F})[\hbar] \right).
\end{aligned} \tag{3.14}$$

where Div_S is extended to polyvector fields as a derivation, in particular it can still be written using the same formula as above. We have also shifted from viewing $\hbar$ as a number to a formal parameter. If we set $\hbar \to 0$, the differential becomes $-\hbar\operatorname{Div}_S \to \Delta + m^2$ acting solely on the antisymmetric tensor product in the definition of $\mathrm{PV}^n(\mathcal{F})$.

To be precise, this is the Schouten-Nijenhuis bracket $\{-, S\}$ with the action. We obtain the *algebra of (mollified, see below) classical observables*:

$$\begin{aligned} \mathcal{O}bs^{cl}_{md}(U) &:= \big(\mathrm{PV}^*(\mathcal{F}), \Delta + m^2\big) = \\ &= \left(\ \dots \xrightarrow{\Delta+m^2} \mathrm{PV}^2(\mathcal{F}) \xrightarrow{\Delta+m^2} \mathrm{PV}^1(\mathcal{F}) \xrightarrow{\Delta+m^2} \mathrm{PV}^0(\mathcal{F}) \ \right) \ . \end{aligned} \tag{3.15}$$

Observation 3.1.7. We can write the complex of polyvector fields as the symmetric algebra (with respect to the completed tensor product) of a chain complex:

$$\begin{aligned} \mathcal{O}bs^{cl}_{md}(U) &= \big(\mathrm{Sym}\, C^\infty_c(U) \hat{\otimes} \textstyle\bigwedge^* C^\infty_c(U), \Delta + m^2\big) = \\ &= \mathrm{Sym}\left(0 \to C^\infty_c(U)[1] \xrightarrow{\Delta+m^2} C^\infty_c(U) \to 0\right) =: \mathrm{Sym}(\mathcal{E}^!_c) \end{aligned} \tag{3.16}$$

Similarly, the divergence complex is the Chevalley-Eilenberg (see 1.3.3) algebra of a differential graded Lie algebra

$$\begin{aligned} \mathcal{O}bs^{q}(U) &= \big(\mathrm{Sym}\, C^\infty_c(U) \hat{\otimes} \textstyle\bigwedge^* C^\infty_c(U) \hat{\otimes} \mathbb{R}[\hbar], \mathrm{Div}_S\big) = \\ &= \mathrm{CE}_*\left(0 \to C^\infty_c(U)[1] \xrightarrow{\Delta+m^2} C^\infty_c(U) \oplus \hbar\mathbb{R} \to 0\right) = \mathrm{CE}_*(\mathcal{E}^!_c \oplus \hbar\mathbb{R}) \end{aligned} \tag{3.17}$$

where, in order to yield an appropriate extra term in the Chevalley-Eilenberg differential, the Lie bracket on $\mathcal{E}^! \oplus \hbar\mathbb{R}$ is defined as the degree 1 integration pairing

$$\langle \alpha, \beta \rangle := \hbar \int_U \alpha\beta \tag{3.18}$$

where $\alpha, \beta \in C^\infty_c(U)$, one in degree -1 and one in degree 0, landing in the summand $\hbar\mathbb{R}$. In other words, we perform a central extension of the (trivial) Lie algebra $\mathcal{E}^!_c$.

Warning. When we had defined the homological Chevalley-Eilenberg complex CE_*, it involved not only taking the symmetric algebra but also shifting by 1. We ignore this shift in the formula above, and all following formulae to not clutter the notation, trusting the reader to remember that not the BV-BRST complex itself carries a L_∞-structure that we will use in a moment, but $\mathcal{E}[-1]$.

The classical observables almost agree with the definition in 1.5, where we had set $\mathcal{O}bs^{cl}(U) := \mathrm{Sym}(\mathcal{E}(U)^\vee)$ with $\mathcal{E}$ the BV-BRST complex

$$\mathcal{E}(U) := \left(0 \to C^\infty(U) \xrightarrow{\Delta+m^2} C^\infty(U)[-1] \to 0\right) \ . \tag{3.19}$$

Notice $\mathcal{E}^!_c(U) \cong \mathcal{E}_c(U)[1]$ in this scalar field case – we will later see that this is due to the (-1)-shifted symplectic structure on covariant phase space. While $\mathcal{E}(U)^\vee = \bar{\mathcal{E}}_c(U)$ involves taking distributional sections of this complex, it turns out that this yields a homotopy equivalent complex, which is far from trivial:

Definition 3.1.8. For $E, F \to M$ vector bundles of rank n, m with local fiber coordinates (e^i, f^i) and sheaves of sections $\mathcal{E}, \mathcal{F}$, a *differential operator* $D : E \to F$ is an $\mathbb{R}$-linear map of sheaves $D : \mathcal{E} \to \mathcal{F}$ that can, in local coordinates (x^i) on M, be written as

$$D\left(\sum_i s_i(x)e^i\right)(x) = \sum_j \sum_{\alpha\in\mathbb{N}^n} d_\alpha^{ij} \frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1}\dots(\partial x^n)^{\alpha_n}}\ , \tag{3.20}$$

and the maximal $|\alpha| = \sum \alpha_j$ such that an $d_\alpha^{ij} \not\equiv 0$ exists is called the *degree* of D.

Technical Remark. In other words, a differential operator is a map of vector bundles $J^\infty E \to F$ and a differential operator of degree $\leq d$ is a map $J^d E \to F$, where $J^d E$ is the *jet bundle* with local basis the formal derivatives of order $\leq d$ of a local basis of E, and the *infinite jet bundle* $J^\infty E = \operatorname*{colim}_d J^d E$.

Definition 3.1.9. The *principal symbol* of a differential operator D of order d is the section of the bundle $\mathrm{Sym}^d(TX) \otimes E^\vee \otimes F \to M$, where $E^\vee$ denotes the dual bundle, that is in local coordinates determined by $(d_\alpha^{ij})_{\substack{i=1,\dots,n;\, j=1,\dots,m \\ \alpha\in\mathbb{N}^m;\, |\alpha|=d}}$.

Definition 3.1.10. For D as above, let $\pi : T^*M \to M$ be the cotangent bundle, and identify the principal symbol of D with a map of vector bundles $\sigma_D : \pi^*E \to \pi^*F$, locally a polynomial homogeneous of total degree d in the fiber coordinates of T^*M. We say that D is *elliptic* if restricted to $T^*M\backslash M$, the map $\sigma_D : \pi^*E| \to \pi^*F|$ is an isomorphism.

Example 3.1.11. On a Riemannian manifold, the Laplace operator is elliptic – however not the d'Alembert operator on a Lorentzian manifold of dimension ≥ 2.

Definition 3.1.12. A *differential complex* on a manifold M is a bounded chain complex $E = (E_i, D_i)$ of vector bundles on M, where the differentials are differential operators (in coordinates, $C^\infty(M)$-linear combinations of finite products of partial derivatives, as above). It is called *elliptic* if the associated complex $(\pi^*E_i|, \sigma_{D_i})$ on $T^*M\backslash M$ is exact.

Theorem 3.1.13 (Formal Hodge Theorem, see [CG16, A.6.4])**.**
If $E = (E_i, D)$ is an elliptic complex on a closed manifold M, then its homology groups $H^i(E)$ are finite dimensional.

Example 3.1.14. All BV-BRST complexes of classical field theories we have considered are elliptic complexes. In particular, the deRham and Dolbeault complexes are elliptic.

Theorem 3.1.15 (Atiyah-Bott 67; Tarkhanov 87 in the non-compact case)**.**
Let $E = (E_i, D)$ be an elliptic complex on M and $U \subseteq M$ open, then the inclusion $\mathcal{E}(U) \hookrightarrow \bar{\mathcal{E}}(U)$ of its sections into distributional sections is a homotopy equivalence. Similarly for compactly supported sections, $\mathcal{E}_c(U) \hookrightarrow \bar{\mathcal{E}}_c(U)$ is a homotopy equivalence.

Definition 3.1.16. For $E = (E^*, D)$ a chain complex of vector bundles, let $E^! := E^\vee \otimes \mathrm{Dens}_M$ be the Verdier dual complex with $(E^!)^i := (E^{-i})^\vee \otimes \mathrm{Dens}_M$ and formally adjoint differential operators as differentials. Here, $E^\vee$ denotes the dual vector bundle and Dens_M the sheaf of densities (i.e. canonical sheaf) of M – we will often ignore this factor, as on an orientable manifold it is trivial anyway. In particular, if $\mathcal{E}$ denotes the sheaf (in convenient vector spaces) of sections of E, then $\mathcal{E}^!_c = \Gamma_c(-, E^\vee \otimes \mathrm{Dens}_M)$ is the sheaf of compactly supported sections of $E^!$.

Corollary 3.1.17. For the scalar field, because

$$\mathcal{E}^!_c(U) \simeq \bar{\mathcal{E}}^!_c(U) = \left(0 \to \bar{C}^\infty_c(U) \stackrel{\Delta+m^2}{\longrightarrow} \bar{C}^\infty_c(U) \to 0\right) \tag{3.21}$$

is the continuous dual $\mathcal{E}^\vee$ of the complex of topological vector spaces $\mathcal{E}$, the subalgebra of smoothened, or mollified observables is homotopy equivalent to the full algebra of observables:

$$\mathcal{O}bs^{cl}_{md} = \mathrm{Sym}\, \mathcal{E}^!_c \simeq \mathrm{Sym}\, \mathcal{E}^\vee = \mathcal{O}bs^{cl} \ . \tag{3.22}$$

3.2. Factorization Algebras from Field Theories

Let us consider the case of more general field theories. The BV-BRST complex $\mathcal{E}(U)$ of a classical field theory on an open subset U of the spacetime manifold M is an L_∞-algebra, and (given that our theory is suitably local) our claim was that the map $U \mapsto \mathcal{E}(U)$ further is a sheaf on M. In all physically interesting cases, the underlying chain complex of $\mathcal{E}(U)$ can be written as the sheaf of sections of a complex of vector bundles, where the differentials are differential operators. Formalizing our claim amounts to the following theorem:

Theorem 3.2.1 ([CG16, 6.5.2])**.** For (E_i, D_i) a differential complex on a manifold M, the functor

- $\mathcal{E} : \mathrm{Open}(M)^{op} \to \mathrm{Ch}(\mathrm{DVS})$ mapping $U \mapsto \Gamma(U, E)$ is a sheaf,
- $\mathcal{E} : \mathrm{Open}(M)^{op} \to \mathcal{C}h(\mathrm{DVS})$ mapping $U \mapsto \Gamma(U, E)$ is an ∞-sheaf (remember that we also just call those sheaves, which should not lead to confusions since the target is a true ∞-category here),
- $\mathcal{E}_c : \mathrm{Open}(M) \to \mathrm{Ch}(\mathrm{DVS})$ mapping $U \mapsto \Gamma_c(U, E)$ is a cosheaf,
- $\mathcal{E}_c : \mathrm{Open}(M) \to \mathcal{C}h(\mathrm{DVS})$ mapping $U \mapsto \Gamma_c(U, E)$ is an ∞-cosheaf.

Similarly for distributional sections of $\mathcal{E}$. Also, since all spaces involved are sections of vector bundles, they lie in the full subcategory $\mathrm{CVS} \subseteq \mathrm{DVS}$, so we could use both settings interchangeably.

Proof. The case of sheaves and cosheaves is standard, using partitions of unity. We follow Costello in considering the case of ∞-cosheaves, as it is the most complicated one. Let $U \subseteq M$ be open, and $\mathfrak{U} = (U_i \subseteq U)_{i\in I}$ an open cover of it – we want to show that

$$\mathcal{E}_c(U) \cong \operatorname*{colim}_{\Delta^{op}} \left(\coprod_i \mathcal{E}_c(U_i) \longleftarrow \coprod_{i,j} \mathcal{E}_c(U_i \cap U_j) \Lleftarrow \cdots \right) . \tag{3.23}$$

It is a standard fact that the (homotopy) colimit over such a simplicial diagram is the total complex of the associated bicomplex (via the Dold-Kan correspondence). Spelling this out, we claim that the canonical map

$$\mathcal{E}_c \to \check{C}^*(\mathfrak{U}, \mathcal{E}_c) \tag{3.24}$$

into the Čech complex of the precosheaf $\mathcal{E}_c$ is a homotopy equivalence. For simplicity, we assume that $\mathcal{E}_c$ is concentrated in degree 0; the general case can be Left-Kan-extended from this or alternatively, we can apply the following construction column-wise in a double complex and use a staircase-argument.

Explicitly, the $(-r)$-th component of the Čech complex is then given by

$$\bigoplus_{\underline{i}=(i_0,\ldots,i_r)} \mathcal{E}_c(U_{i_0} \cap \cdots \cap U_{i_r}) \tag{3.25}$$

while all positive components vanish, with differential of $\nu = (\nu_{\underline{i}}) \in \check{C}^{-r}(\mathfrak{U}, \mathcal{E}_c)$ given by

$$(d\nu)_{(j_0,\ldots,j_{r-1})} = \sum_{m=0}^{r} (-1)^j \sum_{k\in I} \nu_{(j_0,\ldots,j_{m-1},k,j_m,\ldots,j_{r-1})} . \tag{3.26}$$

To show that this is homotopy equivalent to $\mathcal{E}_c(U)$, we can augment it by the natural map from above, putting $\mathcal{E}_c(U)$ in degree 1, and give a contracting homotopy of the resulting augmented Čech complex $\check{C}^*_{aug}(\mathfrak{U}, \mathcal{E}_c)$. For (λ_i) a partition of unity subordinate to the cover (U_i), a possible choice is the degree (-1) map

$$K : \check{C}^*_{aug}(\mathfrak{U}, \mathcal{E}_c) \to \check{C}^*_{aug}(\mathfrak{U}, \mathcal{E}_c)[1] \tag{3.27}$$

that sends $(\nu_\emptyset) \in \mathcal{E}_c(U)$ in degree 1 to $(K\nu)_i = \rho_i \nu_\emptyset$, and $\nu_{\underline{i}}$ in degree r to $(K\nu)_{(j_0,\ldots,j_{r+1})} := \rho_{j_0} \nu_{(j_1,\ldots,j_{r+1})}$. Then, $((dK + Kd)\nu)_{\underline{i}}$ can be written out as

$$\sum_{k\in I} \rho_k \nu_{\underline{i}} + \rho_{i_0} \left(\sum_{m=1}^{r+1} \sum_{k\in I} (-1)^m \nu_{(i_1,\ldots,i_{m-1},k,i_m,\ldots,i_r)} + \sum_{m=0}^{r} \sum_{k\in I} (-1)^m \nu_{(i_1,\ldots,i_m,k,i_{m+1},\ldots,i_r)} \right)$$

where the terms in the bracket cancel, leaving $\left(\sum_{k\in I} \rho_k\right) \nu_{\underline{i}} = \nu_{\underline{i}}$. □

Remark. The same proof also shows that $\mathcal{E}$ and $\mathcal{E}_c$ are (co)sheaves in the derived ∞-category $D(\mathrm{DVS})$. This is conceptually more useful, since the BV complex arises out of derived geometric derivations, and should therefore only be well-defined up to quasi-isomorphism.

Using this knowledge, let us construct factorization algebras for free field theories.

Theorem 3.2.2 ([CG16, 6.5.3])**.** Let (E_i, D_i) be a differential complex on a manifold M. Then, the functor

$$\mathrm{Sym}(\mathcal{E}_c) : U \mapsto \mathrm{Sym}\,\Gamma_c(U, E) \tag{3.28}$$

where Sym is taken with respect to $\hat{\otimes}$, is a factorization algebra on M, both when regarded as a functor from opens into the 1-category Ch(DVS) and into the ∞-category $\mathcal{C}h$(DVS) or D(DVS). Similarly for the functor $\mathrm{Sym}(\bar{\mathcal{E}}_c)$ using distributional sections.

Proof. We use a different proof than Costello, making use of the abstract theory we have developed in chapter 2. Since we know that $\mathcal{E}_c$ is a cosheaf, we can use 2.4.16 and reformulate this as saying that $\mathcal{E}_c$ is a factorization algebra with values in $\mathcal{C}h(\mathrm{DVS})^{\oplus}$. Further, the functor

$$\mathrm{Sym} : \mathcal{C}h(\mathrm{DVS})^{\oplus} \to \mathcal{C}h(\mathrm{DVS})^{\hat{\otimes}} \tag{3.29}$$

is symmetric monoidal and preserves sifted colimits, as we will see in the proof of 3.2.6. This means first of all that the composition

$$\mathrm{Disk}(M)^{\sqcup} \xrightarrow{\mathcal{E}_c} \mathcal{C}h(\mathrm{DVS})^{\oplus} \xrightarrow{\mathrm{Sym}} \mathcal{C}h(\mathrm{DVS})^{\hat{\otimes}} \tag{3.30}$$

is symmetric monoidal, and therefore a factorization algebra. Also, the value of it on a specific open subset can be written via the left Kan extension of this functor to Open(M), and by 4.3.4 this involves only sifted colimits. Therefore,

$$\mathrm{Lan}_{\mathrm{Disk}(M)}^{\mathrm{Open}(M)} \mathrm{Sym} \circ \mathcal{E}_c = \mathrm{Sym} \circ \mathrm{Lan}_{\mathrm{Disk}(M)}^{\mathrm{Open}(M)} \mathcal{E}_c = \mathrm{Sym} \circ \mathcal{E}_c \tag{3.31}$$

since $\mathcal{E}_c$ was a factorization algebra. □

Proposition 3.2.3. If the cosheaf $\mathcal{E}_c$ is locally constant, then the factorization algebra $\mathrm{Sym}(\mathcal{E}_c)$ is so as well.

Proof. If $\mathcal{E}_c$ is locally constant, then for a disk inclusion $D \subseteq D'$, the induced map $\mathcal{E}_c(D) \to \mathcal{E}_c(D')$ is an isomorphism by B.3.7. Therefore, $\mathrm{Sym}(\mathcal{E}_c(D)) \to \mathrm{Sym}(\mathcal{E}_c(D'))$ is an isomorphism as well. □

Now, we will sketch how this generalizes to field theories that are not free, and to the quantum BV-BRST complexes of free field theories.

Definition 3.2.4 ([CG21, 3.1.3.1])**.** A *local L_∞-algebra* on a smooth manifold is a differential complex $\mathcal{L} = (L_i, D_i)$ together with polydifferential (i.e. multilinear, factor-wise differential) operators

$$\ell_n : \mathcal{L}^{\otimes n} \to \mathcal{L} \tag{3.32}$$

for $n \geq 2$ that are of degree $2 - n$ respectively, graded antisymmetric, and equip the sections $\mathcal{L}(U)$ with the structure of L_∞-algebras for all opens U. Here, we identify the differential D on $\mathcal{L}$ with a bracket ℓ_1.

Proposition 3.2.5. Given a local L_∞ algebra $\mathcal{L}$, the sheaf of sections of the underlying differential complex (which we also denote by $\mathcal{L}$) is both a sheaf and an ∞-sheaf of L_∞-algebras. Similarly for distributional and compactly supported sections (the latter are a cosheaf).

The case of (co-)sheaves is not difficult to prove using what we have already seen in 3.2.1, however the case of ∞-(co)sheaves involves first of all defining an ∞-category of L_∞-algebras. We will not go into details since we later mostly restrict ourselves to the free case, however there is a model structure on the ordinary category of L_∞-algebras that is Quillen-equivalent to a model structure on the ordinary category of dg Lie-algebras (so for homotopy theorists, both constructions are essentially the same thing) which is combinatorial. Therefore, the homotopy category can be equipped with the structure of a presentable ∞-category (see the end of A.2). Checking the ∞-sheaf property then is similar to the above.

Theorem 3.2.6 ([CG16, 6.6.1]). Given a local L_∞-algebra $\mathcal{L}$ on a manifold M, the homological Chevalley-Eilenberg complex

$$\mathrm{CE}_*(\mathcal{L}_c) : U \mapsto \mathrm{CE}_*(\mathcal{L}_c(U)) \tag{3.33}$$

is a factorization algebra, and similarly for $\mathrm{CE}_*(\bar{\mathcal{L}}_c)$.

Remark. In principle, we should be interested in the Chevalley-Eilenberg algebra of $\mathcal{L}$, however the dualization in its explicit formula must be replaced by the strong dual. Then, up to completing the symmetric algebra, $\mathrm{CE}_*(\bar{\mathcal{L}}_c) \cong \mathrm{CE}^*(\mathcal{L})$ and $\mathrm{CE}_*(\mathcal{L}_c)$ is a mollified version of it.

Proof Sketch. The proof is analogous to the free case, all we have to show is that the functor CE_* from L_∞-algebras of differentiable vector spaces to $\mathcal{Ch}(\mathrm{DVS})$ is symmetric monoidal with respect to direct sum and $\hat{\otimes}$, and preserves sifted colimits. The first claim is due to the fact that for $V, W \in \mathrm{DVS}$,

$$\mathrm{Sym}(V \oplus W) \cong \mathrm{Sym}(V) \hat{\otimes} \mathrm{Sym}(W) \tag{3.34}$$

for the same reason that this equality holds for the usual tensor product, namely that $\hat{\otimes}$ preserves colimits in both variables, in particular direct sums (alternatively, one could use equation 3.6 since we only work with differential vector spaces of sections). This can be extended to chain complexes, and can be seen to be compatible with the higher brackets. The second claim also follows similarly to the case of ordinary vector spaces, see the proofs of [SAG, 13.2.4.1] and [SAG, 13.2.5.5].

Alternatively, Costello uses a spectral sequence argument that relies on the *Eilenberg-Moore comparison theorem*: Given suitable conditions on the abelian category we are working in, if a map f of complete filtered chain complexes induces an isomorphism on some page of the associated spectral sequences, then it is a quasi-isomorphism.

This can be applied to the filtration $F^{\leq n}\,\mathrm{CE}_*(\mathcal{L}_c) := \mathrm{Sym}^{\leq n}\,\mathcal{L}_c$. The first page of the associated spectral sequence (depending on the convention we use for numbering the pages) is the associated graded object to this complex, which agrees with $\mathrm{CE}_*(\mathcal{L}_c)$ as a bigraded differentiable vector space, but only sees the part of the brackets that preserves the degree in Sym^*, namely the part induced from the differential in $\mathcal{L}_c$ (and not the higher Lie brackets). Hence, this allows us to reduce to the case of differential complexes that we have discussed above. □

It is often said that the structure of a factorization algebra equips the space of operators with a so-called *factorization product*. In our case, this should be the multiplication in the symmetric algebra, as it corresponds to the pointwise product of operators. We formalize this in a way that avoids the slightly awkward use of transports and partitions of unity in e.g. [GR17]:

Construction 3.2.7. Let M be a connected n-manifold and $A \in \mathrm{Fun}^{\otimes}(\mathcal{D}isk_{/M}, \mathcal{V})$ a locally constant factorization algebra on M. Then, the ∞-groupoid of charts $\mathcal{C}harts_{/M} \simeq \mathrm{Sing}(M)$ (by 2.2.6) is connected as well. Since we can regard $\mathcal{C}harts_{/M} \subseteq \mathcal{D}isk_{/M}$ as a full subcategory, the values of A on any two charts are isomorphic, so choosing an arbitrary disk $D \subseteq M$, the value $A(D) \in \mathcal{V}$ is independent of D up to isomorphism.

In fact, for the same reason, for any two disks D and D', the restriction of A to $(\mathcal{C}harts(n)_{/M})_{/D} \simeq \mathcal{C}harts_{/D}$ and $\mathcal{C}harts_{/D'}$ are isomorphic since there is an isomorphism $D \to D'$ in $\mathcal{C}harts_{/M}$. This means that we can associate an essentially unique $\mathbb{E}_n$-algebra $A|_D$ to A; so by the factorization product, we mean the multiplication on the underlying $\mathbb{E}_1$-algebra (i.e. associative algebra) of $A|_D$.

Example 3.2.8. For the case of a free field theory on a connected n-manifold with notation as above, the factorization algebra $\mathcal{O}bs^{cl}_{md}(U)$ is given by $\mathrm{Sym}\,\mathcal{E}^!_c$. When embedding two disjoint smaller disks D', D'' into a disk D, we use the fact that a factorization algebra is a symmetric monoidal functor to obtain a canonical isomorphism

$$\mathrm{Sym}\,\mathcal{E}^!_c(D')\hat{\otimes}\,\mathrm{Sym}\,\mathcal{E}^!_c(D'') \cong \mathrm{Sym}\,\mathcal{E}^!_c(D) \tag{3.35}$$

sending observables a, b to their symmetric product $a\hat{\otimes}b$ in the symmetric algebra. Identifying this with polynomial functions on the BV-BRST complex, we see that the factorization product corresponds to the pointwise multiplication of classical observables. Similarly in the quantum case, meaning that under the identification of factorization algebras on $\mathbb{R}$ with associative algebras, the product on the latter is just the usual multiplication of observables.

This fails in the complex of quantum observables, since the differential obtains an extra term from the shifted symplectic structure. However one can use the Green's function of the equations of motion to give an isomorphism of chain complexes $W : \mathcal{O}bs^{cl}(U)[\hbar] \to$

$\mathcal{O}bs^q(U)$, shifting the discrepancy between classical and quantum setting into the factorization product. In particular, W is not a map of factorization algebras as it equips $\mathcal{O}bs^q(U)$ with the deformed *Moyal star product* – see [CG16, Section 4.6.2] and [GR17] for more.

3.3. More on the Scalar Field in 1D

For M an arbitrary Riemannian manifold and $U \subseteq M$ open, we have just found (both in the classical and quantum world) the BV-BRST complex and the factorization algebra of observables:

$$\mathcal{E}(U) = \Big(0 \to C^\infty(U) \xrightarrow{\Delta+m^2} C^\infty(U)[-1] \to 0\Big)$$
$$\mathcal{O}bs^{cl}(U) \simeq \mathcal{O}bs^{cl}_{md}(U) = \mathrm{Sym}\, \mathcal{E}^!_c(U) = \big(\mathrm{Sym}\, C^\infty_c(U) \hat{\otimes} {\textstyle\bigwedge}^* C^\infty_c(U), \Delta + m^2\big)$$
$$\mathcal{E}_q(U) = \Big(0 \to C^\infty(U) \oplus \mathbb{R}[\hbar] \xrightarrow{\Delta+m^2} C^\infty(U)[-1] \to 0\Big)$$
$$\mathcal{O}bs^q(U) = \mathrm{CE}_*\, \mathcal{E}^!_{q,c}(U) = \big(\mathrm{Sym}\, C^\infty_c(U) \hat{\otimes} {\textstyle\bigwedge}^* C^\infty_c(U) \hat{\otimes} \mathbb{R}[\hbar], \mathrm{Div}_S\big)$$

The quantum observables were obtained from the classical ones by performing a central extension of the underlying L_∞-algebra using the so-called (-1)-*shifted symplectic structure* $\langle -, - \rangle$:

Definition 3.3.1. The integration pairing on U induces an antisymmetric bracket of degree 1, called the (-1)-*shifted symplectic structure on the BV-BRST complex*, on $\mathcal{E}^!(U) = \Big(C^\infty_c(U)[1] \xrightarrow{\Delta+m^2} C^\infty_c(U)\Big)$ via

$$\langle -, - \rangle : \bar{\mathcal{E}}_c \otimes \mathcal{E}^! \to \mathbb{R}[1], \quad \langle \alpha, \beta \rangle := \int_U \alpha\beta \, \mathrm{vol}_g \tag{3.36}$$

for α, β in opposite degrees (-1) and 0 or the other way around, and vanishing for α, β in other degrees. In particular, it is called *symplectic* since it induces an isomorphism of differential complexes $\phi : E \cong E^![-1]$ that is antisymmetric in the sense that the formal dual of ϕ agrees with $-\phi$. More on this in 5.2.18.

Generally, every classical field theory admits such a (-1)-shifted symplectic structure as long as it is constructed from a variational principle, we will show this in 5.2.16. It induces on the factorization algebra of observables a P_0-*algebra structure*, and on the quantum observables a *BD-algebra structure*. We will not properly introduce them since we think studying them is easier on the BV-BRST complex itself, and refer to [CG16, A.3.2] for more.

From an abstract standpoint the above equations are all we need, as they tell us everything about the perturbative behavior of the theory. In fact, since the theory is free, the equations of motion cutting out the derived covariant phase space are linear so the stack is completely determined by its (shifted) tangent space $\mathcal{E}$, even non-perturbatively. However, much analysis is still required in order to draw insights from these expressions.

Let us restrict to the case of $U = (0,1) \subseteq \mathbb{R}$ and $m \geq 0$.

Proposition 3.3.2 ([CG16, 4.2.4])**.** The complexes $\mathcal{E}(U)$ and $\mathcal{E}^!_c(U)$ are homotopy equivalent to simpler complexes

$$\mathcal{E}(U) \simeq \mathbb{R}\langle q, p\rangle[0]\,, \quad \mathcal{E}^!_c(U) \simeq \mathbb{R}\langle Q, P\rangle[0]\;. \tag{3.37}$$

where Q, P are the duals of q, p.

Proof Sketch. Let us look at the first case. We begin by constructing a map $\psi : \mathbb{R}\langle q, p\rangle \to \mathcal{E}_0$ by sending $q \mapsto 1$, $p \mapsto x$ for $m = 0$, and $q \mapsto \cosh(mx)/m$, $p \mapsto \sinh(mx)/m$ for $m > 0$. This is a chain map since it lands in the kernel of the differential $\Delta + m^2$, in fact it even hits the whole kernel and therefore induces an isomorphism on 0th cohomology.

To show that ψ is a homotopy equivalence, a necessary condition is that $\mathcal{E}(U)$ has vanishing first cohomology, i.e. every smooth function lies in the image of $\Delta + m^2$. Informally, this is done by writing down a Greens function and using it to construct a preimage under this differential operator, but of course we run into functional analytic problems when convoluting with it. We have also argued with ordinary vector spaces instead of DVS until now. See the reference for a precise proof using homological algebra in such topological vector spaces; it turns out to be more convenient to prove the second statement since it only deals with compactly supported functions. $\square$

Corollary 3.3.3 ([CG16, 4.3.3])**.** The factorization algebras observables of a scalar field in one dimension are given, on any open interval $D \subseteq \mathbb{R}$, by

$$\begin{aligned} \mathcal{O}bs^{cl}_{md}(U) &\simeq \operatorname{Sym}\mathcal{E}^!_c(U) \simeq \operatorname{Sym}\mathbb{R}\langle Q, P\rangle = \mathbb{R}[Q, P] \\ \mathcal{O}bs^{q}(U) &\simeq \mathrm{CE}_*(\mathbb{R}\langle Q, P\rangle \oplus \hbar\mathbb{R}, [Q, P] = \hbar) =: W[Q, P] \end{aligned} \tag{3.38}$$

where the *Weyl algebra* $W[Q, P]$ is defined as the associative $\mathbb{R}[\hbar]$-algebra freely generated by Q and P modulo the relation $QP - PQ = \hbar$.

Proof Sketch. Again, see the reference for the functional analytic details. The first statement is clear from our previous result since Sym preserves homotopy equivalences. For the second statement, all we need to show is that the commutator $QP - PQ = \hbar$ when using the factorization product, since we already know those generate all solutions. In other words, if we choose an operator with support in a small disk representing the homology class of Q, and an operator in a disjoint disk representing P, we must show that moving these disks past each other introduces an $\hbar$ because of the twisted differential. This subtle calculation is carefully carried out in the reference. $\square$

Proposition 3.3.4 ([CG16, 4.3.1]). The factorization algebras $\mathcal{O}bs^{cl}(U) \simeq \mathcal{O}bs^{cl}_{md}(U)$ and $\mathcal{O}bs^{q}(U)$ are locally constant.

Proof. The classical case follows immediately from 3.3.2, since from the description there it is not hard to see that disk inclusions induce isomorphisms on $\mathcal{E}^!_c$ so that this is a locally constant cosheaf.

For the quantum case, we again employ the Eilenberg-Moore comparison theorem from the proof of 3.2.6 to reduce to the associated graded object of the natural filtration on $\mathrm{Sym}^{\leq n}$. Since the deformation from the classical to the quantum case involves a central extension by a Lie bracket, the associated term in the Chevalley-Eilenberg differential does not preserve the degree of Sym^* and therefore vanishes in the associated graded. This allows us to reduce to the classical case. $\square$

Corollary 3.3.5. Under the identification of locally constant factorization algebras on $\mathbb{R}$ with associative algebras in 2.4.3, the factorization algebras of classical and quantum observables correspond to $\mathbb{R}[Q, P]$ and the Weyl-algebra, respectively. In particular, this allows us to understand the factorization product as the usual product of observables/ operators.

Construction 3.3.6. The partial derivative $\frac{\partial}{\partial x}$ is a derivation on the algebra of smooth functions $C^\infty(U)$, for any open $U \subseteq \mathbb{R}$. Since it commutes with the differential $\Delta + m^2$ in the BV-complex, we obtain a derivation on $\mathcal{E}(U)$ or $\mathcal{E}^!_c(U)$. Abstractly, this means that our theory is (time-)translation-invariant. We can uniquely extend it to a derivation on the symmetric algebra $\mathrm{Sym}\,\mathcal{E}^!_c(U) = \mathcal{O}bs^{cl}_{md}(U)$ compatible with the structure maps of this factorization algebra. Generally, we expect that every (infinitesimally) translation-invariant field theory allows for such a derivation on the associated factorization algebra – and similarly for other symmetries.

Proposition 3.3.7 ([CG16, 4.3.3]). The derivation on $\mathcal{O}bs^q \mathrel{\hat{=}} W$ induced by the infinitesimal translation $-\frac{\partial}{\partial x}$ is inner, i.e. it is given as the commutator $[H, -]$ where the *Hamiltonian* $H \in W$ is given by

$$H = \frac{1}{2\hbar}(P^2 - m^2 Q^2)\ . \tag{3.39}$$

In particular, the scalar field in one dimension agrees with a harmonic oscillator of imaginary frequency – we will explain this in 3.5.1.

Of course, $\mathbb{R}$ is not the only one-dimensional manifold – we should also be interested in describing the scalar field on S^1 (non-connected manifolds do not give us anything new because of 2.4.14).

Proposition 3.3.8. The scalar field determines locally constant absolute factorization algebras $\mathcal{O}bs^{cl}_{md}$, $\mathcal{O}bs^{cl}$ and $\mathcal{O}bs^{q}$ on all oriented 1-manifolds. In fact, the same result holds in any dimension.

Proof. On 1-manifolds, we had seen in 2.4 that $\mathcal{D}isk_1^{or} \simeq \mathcal{D}isk_1^{fr} \simeq \mathbb{E}_1$ as ∞-operads, so a locally constant absolute factorization algebra on oriented 1-manifolds is completely determined by the locally constant relative factorization algebra it induces on $\mathbb{R}$.

This argument however breaks down on n-manifolds unless we restrict to the framed case. We have shown above that the mentioned algebras of observables are locally constant relative factorization algebras on every oriented n-manifold, and they clearly assemble to a functor $\mathrm{Disk}_n^{or} \to D(\mathrm{DBS})$ so by 2.3.6, they form an absolute factorization algebra. Showing that it is locally constant would require a proof of our conjecture 2.2.13. $\square$

Remark. To be precise, we need to fix a background metric in order to define the scalar field on an oriented manifold. We ignore this since we have seen that the eventual algebra of observables is (up to homotopy equivalence) independent of it. In fact, choosing an $\mathrm{O}(n)$-structure on a smooth manifold as in 4.2.2 already more or less fixes a metric, so $\mathcal{D}isk_n^{or}$-algebras do have some knowledge in that regard.

Proposition 3.3.9 ([CG16, 8.1.2])**.** The global classical observables $\mathcal{O}bs^{cl}(S^1)$ of the scalar field on the sphere S^1 with circumference $L \neq 0$ are given

- For $m = 0$, by $\mathbb{R}[1] \oplus \mathbb{R}[0]$,
- For $Lm \in 2\pi\mathbb{N}_{\neq 0}$, by $\mathbb{R}[2] \oplus \mathbb{R}^2[1] \oplus \mathbb{R}[0]$,
- Otherwise, by $\mathbb{R}[0]$.

The quantum observables $\mathcal{O}bs^{q}(S^1)$ can be obtained in each case by adjoining $\hbar$. To be precise, in order to let m be an imaginary number, we would have to complexify our space of observables.

Remark. Our result in the second case differs from the reference.

Proof. As we are working with an absolute factorization algebra over all 1-manifolds, we can use the excision axiom in 2.4.11 or the argumentation in 2.4.13 to show that the factorization homology

$$\int_{S^1} \mathcal{O}bs^{q} \cong \mathrm{HH}_*(\mathcal{O}bs^{q}, {}^{\mu}\mathcal{O}bs^{q}) \tag{3.40}$$

is the Hochschild Homology of the module ${}^{\mu}\mathcal{O}bs^{q}$ where the identity action of $\mathcal{O}bs^{q}$ is twisted by the monodromy action μ of shifting an operator around the circle. Since translations act via our Hamiltonian, we can see that this monodromy just shifts the wave functions corresponding to Q, P by L, to the effect of multiplying them with $\exp(Lm)$ in the case $m \neq 0$. Calculating the Hochschild Homology of the Wely algebra is still fairly complicated, so it is better to treat this case using the underlying differential equations as is done in the reference (it can however be carried out along similar lines as the last case below). We therefore restrict to the classical observables:

Case $m \neq 0$ and $Lm \in 2\pi i\mathbb{N}$: In this case, the monodromy vanishes, so we can apply the Hochschild-Kostant-Rosenberg theorem $\mathrm{HH}_*(\mathbb{R}[Q,P]) \cong \Omega^*(\mathbb{A}^2_{\mathbb{R}})^\sharp \cong \wedge^*\mathbb{R}^2$. The $\sharp$ should be a warning that we equip the differential forms with a vanishing differential, not the exterior differential, since this would correspond to the *Connes operator.* Alternatively, use the proof of the second case and set $m = 0$.

Case $m \neq 0$ and $Lm \notin 2\pi i\mathbb{N}$: For nontrivial monodromy, use the resolution

$$\begin{aligned}
\mathrm{HH}_*(\mathbb{R}[Q,P], {}^\mu\mathbb{R}[Q,P]) &\cong \mathbb{R}[Q,P] \otimes^L_{\mathbb{R}[Q,P]\otimes^L\mathbb{R}[Q,P]^{op}} {}^\mu\mathbb{R}[Q,P] \cong \\
&\cong \left(\mathbb{R}[Q] \otimes^L_{\mathbb{R}[Q_1,Q_2]} {}^\mu\mathbb{R}[Q]\right) \otimes \left(\mathbb{R}[P] \otimes^L_{\mathbb{R}[P_1,P_2]} {}^\mu\mathbb{R}[P]\right) \cong \\
&\cong \left((\mathbb{R}[Q_1,Q_2][1] \xrightarrow{\cdot(Q_1-Q_2)} \mathbb{R}[Q_1,Q_2]) \otimes_{\mathbb{R}[Q_1,Q_2]} {}^\mu\mathbb{R}[Q]\right) \otimes (\dots) \cong \\
&\cong (\mathbb{R}[Q][1] \xrightarrow{\cdot(1-e^{Lm})Q} \mathbb{R}[Q]) \otimes (\mathbb{R}[P][1] \xrightarrow{\cdot(1-e^{Lm})P} \mathbb{R}[P]) \cong \mathbb{R}[0] \otimes \mathbb{R}[0] = \mathbb{R}[0]
\end{aligned}$$

since we assume $e^{Lm} \neq 1$.

Case $m = 0$: Since Q corresponds to the function 1 and P to x in this case, $\mu(Q) = Q$ and $\mu(P) = LQ + P$. The monodromy of P now explicitly depends on Q, so we can not factor the Hochschild complex as above. A similar calculation yields

$$\mathrm{HH}_*(\mathbb{R}[Q,P], {}^\mu\mathbb{R}[Q,P]) \cong \left(\mathbb{R}[Q,P][2] \overset{0\otimes LQ}{\to} \mathbb{R}[Q,P]^{\oplus 2}[1] \overset{LQ\,\mathrm{pr}_1}{\to} \mathbb{R}[Q,P]\right) \cong \mathbb{R}[1] \oplus \mathbb{R}[0] \, .$$

□

Remark. Our result tells us that in the massless case, there is always a 1-dimensional space of solutions, namely the constant functions; while in the massive case the space is spanned by either 0 or 2 basic solutions depending on whether or not the radius of the circle allows for an integer number of waves to fit (which may both in around the circle in both directions). More generally, the factorization homology of $\mathcal{O}bs^q$ on n-manifolds therefore knows about spectral properties of the differential operator $\Delta + m^2$.

Finally, let us take a look at how the higher-dimensional case can be reduced to one dimension. Let N be a closed oriented n-manifold and $M = N \times \mathbb{R}$, with projection $\pi : M \to \mathbb{R}$. As usual, we can define the factorization algebra $\mathcal{O}bs^q$ of quantum observables for the free scalar field on M.

Lemma 3.3.10. This induces a *pushforward factorization algebra* $\pi_* \mathcal{O}bs^q$ on $\mathbb{R}$ via

$$\pi_* \mathcal{O}bs^q(U) := \mathcal{O}bs^q(\pi^{-1}(U)) \; \forall U \subset \mathbb{R} \, , \tag{3.41}$$

which is locally constant since $\mathcal{O}bs^q$ was locally constant.

Proof. The factorization algebra on M is completely determined by the underlying symmetric monoidal functor $\mathcal{O}bs^q : \mathrm{Disk}(M)^{\sqcup} \to \mathcal{V}^{\otimes}$. We note that the inverse image functor $\pi^{-1} : \mathrm{Disk}(\mathbb{R})^{\sqcup} \to \mathrm{Disk}(M)^{\sqcup}$ is well-defined and symmetric monoidal as well, so that their composition again yields a factorization algebra. Since π^{-1} sends disk inclusions to disk inclusions, it is locally constant as well. Finally, we need to show that $\pi_* \mathcal{O}bs^q$ and $\mathcal{O}bs^q \circ \pi^{-1}$ agree on all open subsets $U \subseteq \mathbb{R}$, not only disks:

$$\pi_* \mathcal{O}bs^q(U) = \mathcal{O}bs^q(\pi^{-1}(U)) \overset{!}{\cong} \mathrm{Lan}_{\mathrm{Disk}(\mathbb{R})}^{\mathrm{Open}(\mathbb{R})}(\mathcal{O}bs^q \circ \pi^{-1}) = \operatorname*{colim}_{D \in \mathrm{Disk}(U)} \mathcal{O}bs^q(\pi^{-1}(D))$$

We are finished when we realize that $(\pi^{-1}(D))_{D\in \mathrm{Disk}(U)}$ form a Weiss cover (in fact, we even need it to be a factorization SFK cover by 4.3.2), so $\mathcal{O}bs^q$ satisfies descent with respect to it. $\square$

Remark. This argument holds for all projections from product spaces and even for projections of arbitrary vector bundles; see [AFT14a, 2.24].

Let now $(e_i)_{i\in\Lambda}$ be an orthonormal basis of eigenvectors of the operator $\Delta + m^2$ on $C^\infty(N)$ with eigenvalues $(\lambda_i)_{i\in\Lambda}$. In particular, their span $\bigoplus_i \mathbb{R}e_i$ forms a dense subspace of $C^\infty(N)$ (the latter is a sort of completion of the former). We can use this fact to give an alternative description of the factorization algebra $\pi_* \mathcal{O}bs^q$, which in the physics literature is known as *canonical quantization*: For this, denote by A_{m^2} the Weyl algebra associated to the 1-dimensional free scalar field with mass m. We define an associative algebra

$$A_N := \bigotimes_{i\in\Lambda} A_{\lambda_i} \tag{3.42}$$

where $\otimes$ denotes the tensor product of $\mathbb{R}[\hbar]$-algebras.

Theorem 3.3.11 (Canonical Quantization, [CG16, 4.4.1])**.** The associative algebra object in $D(\mathrm{DVS})$ associated to the locally constant factorization algebra $\pi_* \mathcal{O}bs^{cl}$ on $\mathbb{R}$ contains a dense subalgebra that is quasi-isomorphic to A_N. This means that the free scalar field on a compact oriented space manifold N is essentially (up to functional analytic subtleties) equivalent, as a physical system, to a collection of harmonic oscillators (see 3.5.1 for their factorization algebras) with frequencies corresponding to the eigenvalues of the Klein-Gordon-operator $\Delta + m^2$.

Proof. From the above discussion, we know that $\bigoplus_i \left(C_c^\infty(U)e_i[1] \xrightarrow{\Delta+\lambda_i} C_c^\infty(U)e_i\right) =: \bigoplus_i \mathcal{E}^!_{i,c}(U)$ is a dense subcomplex of $\mathcal{E}^!_c(U)$. Therefore,

$$\mathcal{O}bs^{cl}_{md}(U) = \mathrm{Sym}\,\mathcal{E}^!_c(U) \supseteq \hat{\bigotimes_i} \mathrm{Sym}\,\mathcal{E}^!_{i,c}(U) =: \hat{\bigotimes_i} \mathcal{O}bs^{cl}_{i,md}(U) \tag{3.43}$$

is also a dense subspace, since Sym sends $\oplus$ to $\hat{\otimes}$, and preserves the property of being a dense subspace. Similarly for CE_* but with a different differential. We also know that the factorization algebras $\mathcal{O}bs_i^q$ correspond to the associative algebras A_{λ_i} since $\mathcal{E}_{i,c}^!$ corresponds to a free scalar field of mass $\sqrt{\lambda_i}$. From their explicit representation as Weyl algebras, we know it is finitely generated, so the completed tensor product agrees with the usual tensor product. This implies that

$$\bigotimes_i \mathbb{R}[q_{\lambda_i}^*, p_{\lambda_i}^*] \subseteq \mathcal{O}bs^{cl}(U)\,, \qquad \bigotimes_i A_{\lambda_i} \subseteq \mathcal{O}bs^q(U) \tag{3.44}$$

are dense sub-factorization algebras. □

3.4. Abelian Chern-Simons Theory and its Quantization

The case of Chern-Simons Theory works similarly. For M a smooth oriented 3-manifold and $\mathfrak{g}$ a Lie algebra with fixed ad-invariant (e.g. Killing) form, we start with the BV-BRST complex from 1.5:

$$\mathcal{E}(U) = \Big(0 \to \Omega^0(M,\mathfrak{g})[1] \xrightarrow{d} \Omega^1(M,\mathfrak{g}) \xrightarrow{d} \Omega^2(M,\mathfrak{g})[-1] \xrightarrow{d} \Omega^3(M,\mathfrak{g})[-2] \to 0\Big) \tag{3.45}$$

This is the case of vanishing background field A_0 and trivial G-bundle (as we have seen, this is always the case in 3 dimensions); a generalization is straightforward. To proceed, we use Poincaré duality – the integration pairing

$$\int (- \wedge -) : \bar{\Omega}_c^{n-i}(M) \otimes \Omega^i(M) \to \mathbb{R} \tag{3.46}$$

exhibits $\Lambda^{n-i}T^*M = (\Lambda^i T^*M)^!$ as the Verdier dual vector bundle, and $\bar{\Omega}_c^{n-i}(M) \cong \Omega^i(M)$ as the strong dual topological vector space. Using partial integration, we see that (up to a sign we ignore) d is its own formal dual under this pairing, so these identities extend to chain complexes. The factorization algebra of classical observables is then given as the algebra of polynomial functions on $\mathcal{E}(U)$, in the sense of:

$$\mathcal{O}bs^{cl}(U) = \mathrm{CE}_*\,\mathcal{E}(U)^\vee = \mathrm{CE}^*\left(\begin{array}{l} 0 \longrightarrow \bar{\Omega}_c^0(U,\mathfrak{g})[2] \xrightarrow{d} \bar{\Omega}_c^1(U,\mathfrak{g})[1] \\ \xrightarrow{d} \bar{\Omega}_c^2(U,\mathfrak{g})[-1] \xrightarrow{d} \bar{\Omega}_c^3(U,\mathfrak{g})[-2] \longrightarrow 0 \end{array}\right)$$

Since $\mathcal{E}(U)$ is an elliptic complex, the Atiyah-Bott lemma 3.1.15 again allows us to replace $\mathcal{E}(U)^\vee = \bar{\mathcal{E}}_c^!(U)$ with $\mathcal{E}_c^!(U) \cong \overline{\mathcal{E}}_c(U)[1]$, where we ignore the trivial factor Dens_M. This yields the homotopy equivalent subalgebra of smeared observables:

$$\mathcal{O}bs_{md}^{cl}(U) = \mathrm{CE}_*\,\mathcal{E}_c^!(U) = \mathrm{CE}^*\left(\begin{array}{l} 0 \longrightarrow \Omega_c^0(U,\mathfrak{g})[2] \xrightarrow{d} \Omega_c^1(U,\mathfrak{g})[1] \\ \xrightarrow{d} \Omega_c^2(U,\mathfrak{g})[-1] \xrightarrow{d} \Omega_c^3(U,\mathfrak{g})[-2] \longrightarrow 0 \end{array}\right)$$

At this point, it is important to remember that $\mathcal{E}$ is not just a chain complex, but an L_∞-algebra. In particular, the brackets ℓ_n unite to form the Chevalley-Eilenberg differential, which in the classical case was given by a Schouten-Nijenhuis-bracket with S:

$$\{-, S\} = \iota_{dS} \overset{!}{=} \ell_1^\vee + \frac{1}{2!}\ell_2^\vee + \frac{1}{3!}\ell_3^\vee + \dots \tag{3.47}$$

This observation leads us to the following insight on how to calculate ℓ_n:

Observation 3.4.1. Given a local L_∞-algebra $\mathcal{L}$ with (-1)-shifted structure $\langle -, - \rangle$, fiber coordinates $\phi = (\phi_i)$ and brackets ℓ_n, we can write down the *gauge fixed action*

$$S_{gf}[\phi_i] := \frac{1}{2!}\langle \phi, \ell_1(\phi)\rangle + \frac{1}{3!}\langle \phi, \ell_2(\phi, \phi)\rangle + \frac{1}{4!}\langle \phi, \ell_3(\phi, \phi, \phi)\rangle + \dots \tag{3.48}$$

This allows to retrospectively find the higher Lie brackets on $\mathcal{L}$, since the terms in the gauge-fixed action not involving ghosts, antifields and antighosts should agree with the usual action.

For Chern-Simons theory, the (-1)-shifted symplectic structure is the tensor product of our fixed ad-invariant inner product and the integration pairing. Also, the action only contains interactions of order 3, so comparing with the gauge-fixed action we see that $\ell_n = 0$ for $n \geq 3$. To obtain the right expressions for the quadratic and cubic terms (up to an overall normalization factor of 2), we find $\ell_1 = d$ as expected and $\ell_2 = [- \wedge -]$.

Proposition 3.4.2. The factorization algebra $\mathcal{O}bs^q$ for classical Chern-Simons theory is locally constant, both when regarded as valued in $\mathcal{C}h(\mathrm{DVS})$ and $D(\mathrm{DVS})$.

Proof. The complex $\mathcal{E}_c^!(U) = \Omega_c^*(U)[2] \in D^b(\mathrm{DVS})$ has homology $H_{c,dR}^*(U)$ the compactly supported deRham cohomology of U. This sends disks inclusions to isomorphisms by the Poincaré-Lemma, so that $\mathcal{E}_c^!$ sends disk inclusions to quasi-isomorphisms. In fact, they are even sent to homotopy equivalences; constructing the involved homotopies involves "contracting" a compactly supported differential form in a bigger disk into a smaller disk which can just be done by a scalar rescaling of the argument (we leave the details to the reader). □

Quantization using our simple methods is only possible for abelian Chern-Simons theory, since $\ell_2 = [-, -] = 0$ must vanish for a free field theory. Quantum operators are then given by

$$\mathcal{O}bs^q(U) = \mathrm{CE}_*(\Omega_c^*(U)[2] \oplus \mathbb{R}\hbar[0]) \tag{3.49}$$

where $\Omega_c^*(U)[2] \oplus \mathbb{R}\hbar[0]$ is the strong dual of the central extension of $\Omega^*(U)[1]$ carrying the modified L_∞-brackets $\ell_1 = d \oplus \mathrm{id}$, $\ell_2 = \hbar\langle -, - \rangle$ and $\ell_i = 0$ for $i > 2$. Of physical interest is mainly the homology of these complexes, since it yields the categories of physical observables:

Proposition 3.4.3 ([CG16, 4.5.1])**.** For M a smooth oriented 3-manifold without boundary, physical observables on an open subset $U \subseteq M$ in abelian Chern-Simons theory are given by

$$H^k \mathcal{O}bs^{cl}(U) = \mathrm{Sym}(H^k_c(M)[2]) \ . \tag{3.50}$$

Proof. As we are only interested in homology, it does not matter whether we work with smeared or unsmeared observables; and because we are working over the field $\mathbb{R}$ we can apply the Künneth theorem without torsion terms to obtain the above result. □

Proposition 3.4.4 ([CG16, 4.5.1])**.** Let M be a closed connected oriented 3-manifold and b_2 its second Betti number. Then, if we localize at $\hbar$, the physical quantum observables of abelian Chern-Simons theory are

$$H^k \mathcal{O}bs^q(M) = \mathbb{R}(\hbar)[1 - b_2] \ , \tag{3.51}$$

the dg algebra over $\mathbb{R}(\hbar)$ spanned by a single generator of degree $1 - b_2$.

Technical Remark. In the reference, this result is stated using a polynomial algebra in $\hbar$, which seems problematic as the differential in $\mathcal{O}bs^q$ adds a factor of $\hbar$ making it impossible for cycles of order 0 in $\hbar$ to be boundaries.

Proof. Let us work explicitly with a basis $[\beta_1], \dots, [\beta_{b_2}]$ of $H^2(M)$ and its dual basis $[\alpha_1], \dots, [\alpha_{b_2}] \in H^1(M)$ under the integration pairing. Also, let $[\alpha_0] := [1] \cdot \mathrm{vol}(M)^{-1}$ and $\beta_0 := [M]$ be dual generators of $H^0(M)$ and $H^3(M)$, respectively – this is where our assumptions on M enter. The Chevalley-Eilenberg differential in $\mathcal{O}bs^q$ is made out of the exterior differential (which trivially vanishes on all cohomology classes), the Lie bracket in $\mathfrak{g}$ (vanishes in this abelian case), and the twist by $\hbar$ times the integration pairing. Therefore, d_{CE} vanishes on all quadratic terms except for

$$d_{\mathrm{CE}}(\alpha_i \otimes \beta_j) = \ d_{\mathrm{CE}}(\beta_i \otimes \alpha_j) = \hbar\delta_{ij} \quad \forall i, j = 0, \dots, b_2 \ . \tag{3.52}$$

This is extended to the whole complex as a derivation satisfying the Leibniz rule, which is expressed by the *BD identity*

$$d_{\mathrm{CE}}(a \otimes b) = d_{\mathrm{CE}} a \otimes b + (-1)^{|a|} a \otimes d_{\mathrm{CE}} b + (-1)^{|a|}\hbar\langle a, b\rangle \ . \tag{3.53}$$

Let us suppress the tensor product. For example, since all other contractions vanish,

$$d_{\mathrm{CE}}(\alpha_0\alpha_1\beta_1\beta_0) = -\alpha_0 d_{\mathrm{CE}}(\alpha_1\beta_1)\beta_0 + d_{\mathrm{CE}}(\alpha_0\beta_0)\alpha_1\beta_1 = -\hbar\alpha_0\beta_0 + \hbar\alpha_1\beta_1 \ . \tag{3.54}$$

We can understand from this example that an operator $O \in \mathbb{R}(\hbar)[\alpha_0, \dots, \beta_{b_2}]$ is a cycle iff it does not involve α_i and β_i of the same i. However, we can always multiply a pair $\alpha_j\beta_j$ in front of this cycle to obtain a new operator with boundary $d_{\mathrm{CE}}\alpha_j\beta_j O = O$, as long as α_j, β_j do not appear in O. Using this idea, one can show that the operator $\alpha_0\alpha_1 \dots \alpha_{b_2}$ of degree $1 - b_2$ generates the whole cohomology as every other combination containing one α_j or β_j of each kind is homologous to it. □

Remark. From this discussion, one can go further and use the language of factorization algebras to derive the connection between Wilson loop observables in abelian Chern-Simons theory and the Gauss linking number. We refer to section 4.5.4 in [CG16]. Also, one can construct a connection between Wilson lines in non-abelian Chern-Simons theories and quantum groups, see 8.2 in loc. cit. and [Cos13].

3.5. Further Examples

Now that we know how this works, let us write down the factorization algebras for more examples.

Example 3.5.1 (Harmonic Oscillator)**.** Covariant formalisms are a bit clunky when applied to mechanical problems instead of field theory, but let us still indicate how this would work. Given the manifold $M = \mathbb{R}$ and the space of fields $\mathcal{F} = C^\infty(\mathbb{R})$, the action of the *Harmonic oscillator* is given by

$$S_{HO}[q] := -\frac{m}{2}\int_{\mathbb{R}} q\frac{d^2}{dt^2}q + \omega^2 q^2 dx \tag{3.55}$$

with $m, \omega > 0$. Variation yields the equations of motion

$$m\frac{d^2}{dt^2}q - m\omega^2 q = 0 \tag{3.56}$$

so that (since there are no local gauge symmetries) the BV-BRST complex is given by

$$\mathcal{E}(U) = \left(0 \to C^\infty(\mathbb{R}) \xrightarrow{\frac{d^2}{dt^2}-\omega^2} C^\infty(\mathbb{R})[-1] \to 0\right) \simeq \mathbb{R}^2[0]\ . \tag{3.57}$$

This complex is homotopy equivalent to $\mathbb{R}\langle s, c\rangle[0]$ with basis solutions $s = \sin(\omega t)$ and $c = \cos(\omega t)$, as can be seen similarly to the scalar field in one dimension. In particular, notice that (up to a sign that is due to us working in Euclidean signature before) the free scalar field in $0+1$ dimensions is equivalent to the harmonic oscillator; we have also seen in 3.3.11 that on a space manifold N, a dense subspace of the BV-BRST complex of the scalar field is given by independently taking one harmonic oscillator corresponding to every energy eigenvalue.

The factorization algebra of classical observables also corresponds to the algebra of polynomials $\mathbb{R}[s, c]$, while quantization amounts to twisting with the (-1)-shifted symplectic structure induced by integration. We again obtain, as corresponding associative algebra, the Weyl algebra spanned by $s, c, \hbar$. This may also be rewritten as a Weyl algebra on the ladder operators $a := \frac{s+c}{\sqrt{2}}$, $a^\dagger = \frac{s-c}{\sqrt{2}}$, and we will see in section 5.3 how the Fock space arises from this.

Example 3.5.2 (Yang-Mills Theory)**.** For Yang-Mills Theory with gauge group G on a trivial bundle with trivial background field, the BV-BRST complex is

$$\mathcal{E}(U) = \Big(0 \to \Omega^0(U,\mathfrak{g})[1] \xrightarrow{d} \Omega^1(U,\mathfrak{g}) \xrightarrow{d\star d} \Omega^{n-1}(U,\mathfrak{g})[-1] \xrightarrow{d} \Omega^n(U,\mathfrak{g})[-2] \to 0\Big) \tag{3.58}$$

with ℓ_2 the tensor product of Lie bracket and exterior product, and $\ell_n = 0$ for $n > 2$. As it is elliptic and (up to signs) the differentials are self-adjoint,

$$\mathcal{O}bs^{cl}(U) \simeq \mathcal{O}bs^{cl}_{md}(U) = \mathrm{CE}_* \left(\begin{tikzcd} 0 \arrow[r] & \Omega^0_c(U,\mathfrak{g})[2] \arrow[r, "d"] & \Omega^1_c(U,\mathfrak{g})[1] \arrow[dll, "d\star d"] & \\ \Omega^{n-1}_c(U,\mathfrak{g})[-1] \arrow[r, "d"] & \Omega^n_c(U,\mathfrak{g})[-2] \arrow[r] & 0 \end{tikzcd} \right)$$

where the differential is the sum of exterior differential and ℓ_2.

Example 3.5.3 (Abelian Yang-Mills)**.** In the special case where $\mathfrak{g} = \mathbb{R}$ with the trivial Lie bracket, this is a free theory and quantization amounts to twisting with the (-1)-shifted symplectic structure on $\mathcal{E}(U)$ induced by the integration pairing

$$\bar{\Omega}^i_c(U) \otimes \Omega^{4-i}(U) \to \mathbb{R}\ . \tag{3.59}$$

exhibiting Poincaré duality. This yields the algebra of quantum observables, with underlying graded vector space the symmetric algebra on

$$0 \to \Omega^0_c(U)[2] \xrightarrow{d} \Omega^1_c(U)[1] \xrightarrow{d\star d} \Omega^3_c(U) \oplus \hbar\mathbb{R} \xrightarrow{d} \Omega^4_c(U)[-1] \to 0 \tag{3.60}$$

and differential generated by d and the twist by $\hbar\int(- \wedge -)$.

Example 3.5.4 (Abelian B-Field)**.** The usual procedure tells us that

$$\mathcal{O}bs^{cl}_{md}(U) = \mathrm{Sym} \left(\begin{tikzcd} 0 \arrow[r] & \Omega^0_c(U)[3] \arrow[r, "d"] & \Omega^1_c(U)[2] \arrow[r, "d"] & \Omega^2_c(U)[1] \arrow[dlll, "d"] \\ \Omega^{n-2}_c(U) \arrow[r, "d"] & \Omega^{n-1}_c(U)[-1] \arrow[r, "d"] & \Omega^n_c(U)[-2] \arrow[r] & 0 \end{tikzcd} \right)$$

where we use $E^! \cong E[1]$ due to the shifted symplectic structure given by the integration pairing $\langle -,-\rangle$, so that $\mathrm{Sym}\,\mathcal{E}^!_c \cong \mathrm{Sym}\,\mathcal{E}_c[1]$ and expand around zero B-field. Note that all higher Lie brackets vanish as the theory is free, so quantization only amounts to a central extension by $\hbar\langle -,-\rangle$.

Example 3.5.5 ($\boldsymbol{\phi^4}$-theory)**.** Finally, let us write down the observables for classical ϕ^4-theory (to treat the quantum case, we would have to introduce Feynman graph combinatorics and renormalization). The space of fields $\mathcal{F} = C^\infty(M)$ is the same as for the scalar field on the spacetime manifold M, but the action is given by

$$S_{\phi^4}[\phi] := \int_M \left(\frac{1}{2}\phi(\Delta + m^2)\phi + \frac{\lambda}{4!}\phi^4\right)\ . \tag{3.61}$$

Perturbing around $\phi = 0$, this yields the same Euler-Lagrange equations as for the free theory since the ϕ^3-term appearing becomes negligible. This means that the BV-BRST complex is still given by

$$\mathcal{E}(U) = \left(0 \to C^\infty(U) \overset{\Delta+m^2}{\to} C^\infty(U)[-1] \to 0\right) \ , \tag{3.62}$$

but we see from 3.4.1 that the higher Lie brackets are different: While $\ell_i = 0$ for $i \neq 3$, we have $\ell_3(\alpha_1, \alpha_2, \alpha_3) = \lambda\alpha_1\alpha_2\alpha_3$ where α_i are smooth functions of degree 0 in this complex (this is indeed graded antisymmetric on $\mathcal{E}[-1]$). Since ℓ_3 is of degree $2 - 3 = -1$, and the arguments are of degree 1 in $\mathcal{E}[-1]$, this expression lands in degree $3 \cdot 1 - 1 = 2$ of $\mathcal{E}[-1]$ and can therefore, as an antifield, be contracted with a fourth field. Therefore,

$$S_{gf}[\phi] = \frac{1}{2!}\langle\phi, (\Delta + m^2)\phi\rangle + \frac{1}{4!}\langle\phi, \ell_3(\phi, \phi, \phi)\rangle = S_{\phi^4}[\phi] \tag{3.63}$$

as expected. From this, we may easily calculate the factorization algebra of classical observables as

$$\mathcal{O}bs^{cl}_{md}(U) = \left(\operatorname{Sym} C^\infty_c(U)\hat{\otimes}\bigwedge{}^* C^\infty_c(U), \Delta + m^2 + \ell_3^\vee\right) \tag{3.64}$$

where $\Delta + m^2$ acts on the fields in the exterior product, and

$$\ell_3^\vee : (C^\infty_c(U)[1] \to C^\infty_c(U)) \to \operatorname{Sym}^3(C^\infty_c(U)[1] \to C^\infty_c(U)) \tag{3.65}$$

sends $\alpha \in C^\infty_c(U)[1]$ to $\alpha\frac{\lambda}{3!}(1 \otimes 1 \otimes 1) \in \operatorname{Sym}^3 C^\infty_c(U)[0]$.

4. Factorization Algebras on Stratified Spaces

In this chapter, we follow [AFT14a] in extending the definitions and results of Chapter 2 to the case where instead of a topological manifold, the factorization algebras live on a stratified space – a space that is glued together from manifolds of different dimensions so that is locally looks like a cone. Again, we will see how homotopy theory and local structure of the space are mirrored in these algebraic objects. However, the connection to physical intuition is more subtle and less developed than in the non-stratified case; therefore we make several preparatory statements for their application in the next chapter. We assume the reader is familiar with Appendix B.

4.1. Categories of Stratified Spaces

Developing a theory of factorization algebras on stratified spaces amounts to finding suitable replacements for the categories and operads constructed in section 2.2. Let, for this purpose, $(M \to P)$ be a conically smooth stratified space and $\mathcal{V}$ a sifted complete symmetric monoidal ∞-category.

Definition 4.1.1. Denote by Sngl_n the ordinary category of conically smooth stratified spaces as defined in B.1 with smooth (stratified) open embeddings as morphisms, and define the full subcategories Bsc_n spanned by the conically smooth n-basics, and Disk_n on disjoint unions of such n-basics.

Remark. Most statements in this section make sense if we replace conically smooth by C^0-stratified spaces. We will generally ignore smoothness and work with the C^0-case instead as it is a lot easier to understand, pointing out situations where this leads to problems.

Construction 4.1.2. Let $r_{man} : \Delta \to \mathrm{Sngl}_n$ be the functor that sends $[n]$ to the smooth n-simplex

$$r_{man}([n]) := \Delta^n_{sm} = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum_i x_i = 0\} . \tag{4.1}$$

Definition 4.1.3 ([AFT14b, 4.1.4]). Define $\mathcal{Sngl}_n$ to be the ∞-category of conically smooth stratified spaces, where mapping spaces are defined as the simplicial sets

$$\mathrm{Map}_{\mathcal{Sngl}_n}(X,Y)_m := \left\{ f \in \mathrm{Hom}_{\mathrm{Sngl}_{n+m}}(\Delta^m_{sm} \times X, \Delta^m_{sm} \times Y) \,|\, \pi_{\Delta^m_{sm}} f = \pi_{\Delta^m_{sm}} \right\} \tag{4.2}$$

whose m-simplices are smooth maps $\mathbb{R}^m \times X \to \mathbb{R}^m \times Y$ that induce the identity on the Δ^m_{sm}-component (i.e. smooth $\mathbb{R}^m$-families of maps $X \to Y$). These simplicial sets are even Kan complexes, so we obtain the desired ∞-category after applying the homotopy-coherent nerve.

Denote by $\mathcal{Bsc}_n$ the full subcategory spanned by n-basics, and by $\mathcal{Disk}_n$ the full subcategory spanned by disjoint unions of n-basics.

Proposition 4.1.4. All of these categories admit ∞-operadic structures induced by disjoint union (defined precisely like in the manifold case), these are symmetric monoidal structures for $\mathcal{Sngl}^{\amalg}_n, \mathcal{Disk}^{\amalg}_n, \mathrm{Sngl}^{\amalg}_n, \mathrm{Disk}^{\amalg}_n$ and partially symmetric monoidal structures for $\mathcal{Bsc}^{\amalg}_n$ and $\mathrm{Bsc}^{\amalg}_n$.

Definition 4.1.5. *Absolute factorization algebras* on (conically smooth) stratified spaces are symmetric monoidal functors

$$\mathrm{FA}^{strat}(\mathcal{V}) := \mathrm{Fun}^{\otimes}(\mathrm{Disk}_n, \mathcal{V}) \simeq \mathrm{Alg}_{\mathrm{Bsc}_n}(\mathcal{V}) \;, \tag{4.3}$$

this equivalence of categories is as in the manifold case induced by the fact that elements of Disk_n are finite disjoint unions of elements of Bsc_n, so a symmetric monoidal functor on the former is determined by its values on the latter.

Definition 4.1.6. An absolute factorization algebra is *constructible* if it factors through the canonical symmetric monoidal functor $\mathrm{Disk}^{\sqcup}_n \to \mathcal{Disk}^{\sqcup}_n$; we write

$$\mathrm{FA}^{cbl}(\mathcal{V}) := \mathrm{Fun}^{\otimes}(\mathcal{Disk}_n, \mathcal{V}) \simeq \mathrm{Alg}_{\mathcal{Bsc}_n}(\mathcal{V}) \;. \tag{4.4}$$

In particular, such a functor sends inclusions of basics into themselves, that are isotopic to the identity, to isomorphisms in $\mathcal{V}$. We however do not know whether this statement has a converse.

Definition 4.1.7. To study relative factorization algebras over a (conically smooth) stratified space M, we need the slice 1-categories $\mathrm{Sngl}_{/M}, \mathrm{Disk}_{/M}, \mathrm{Bsc}_{/M}$ as well as the ∞-categories $\mathcal{Sngl}_{/M}\,\mathcal{Disk}_{/M}, \mathcal{Bsc}_{/M}$. Remember that we generally set $\mathcal{Disk}_{/M} = \mathcal{Disk}_n \times_{\mathcal{Sngl}} \mathcal{Sngl}_{/M}$ and so on.

Remark. As in the manifold case, $\mathrm{Sngl}_{/M} \simeq \mathrm{Open}(M)$ is equivalent to the poset of open subsets of M, and $\mathrm{Bsc}_{/M} \simeq \mathrm{Bsc}(M), \mathrm{Disk}_{/M} \simeq \mathrm{Disk}(M)$ for the subcategories of basics and disjoint unions of basics. In particular, all of these 1-categories possess operadic structures determined by disjoint union $\sqcup$.

Construction 4.1.8. Since the ∞-operad $\mathrm{Disk}_n^{\sqcup}$ is unital, there is a canonical map into the coCartesian operad $\mathrm{Disk}_n^{\amalg}$, see A.7.13. As described in A.7.14, the slice projection $\mathcal{D}isk_{/M} \to \mathcal{D}isk_n$ induces a pullback diagram

$$\begin{tikzcd}
\mathcal{D}isk_{/M}^{\sqcup} \arrow[r] \arrow[d] & \mathcal{D}isk_{/M}^{\amalg} \arrow[d] \\
\mathcal{D}isk_n^{\sqcup} \arrow[r] & \mathcal{D}isk_n^{\amalg}
\end{tikzcd}$$

giving $\mathcal{D}isk_{/M}^{\sqcup}$ the structure of an ∞-operad (that is not coCartesian) with underlying ∞-category $\mathcal{D}isk_{/M}$. Similarly, we can define $\mathcal{S}ngl_{/M}^{\sqcup}$ and $\mathcal{B}sc_{/M}^{\sqcup}$. Keep in mind that $\mathcal{D}isk_{/M}^{\sqcup}$ and $\mathcal{S}ngl_{/M}^{\sqcup}$ carry a weak symmetric monoidal structure in the sense of A.7.14.

Remark. This is equivalent to the construction 2.2.8 we had used in the classical case. Multimorphisms in $\mathcal{B}sc_{/M}^{\sqcup}$ from $(j_i : B_i \hookrightarrow M)_{i=1,\dots,m}$ to $j : B \hookrightarrow M$ are, for example, given by smooth stratified open embeddings $k_i : B_i \hookrightarrow B$ with disjoint images, and smooth isotopies $j_i \cong j \circ k_i$ since those are the 2-morphisms in $\mathcal{B}sc_n$:

$$\mathrm{Mul}_{\mathcal{B}sc_{/M}}(j_1, \dots, j_m; j) := \mathrm{Map}_{\mathcal{S}ngl_n}(B_1 \sqcup \dots \sqcup B_m; B) \times_{\prod_i \mathrm{Map}_{\mathcal{S}ngl_n}(B_i, B)} \prod_i \mathrm{Map}_{\mathcal{S}ngl_{/M}}(j_i, j)$$

Definition 4.1.9. A *(relative) factorization algebra* on M is a symmetric monoidal functor

$$\mathrm{FA}(M; \mathcal{V}) := \mathrm{Fun}^{\otimes}(\mathrm{Disk}_{/M}, \mathcal{V}) \simeq \mathrm{Alg}_{\mathrm{Bsc}_{/M}}(\mathcal{V}) \ . \tag{4.5}$$

It is called *constructible* iff it factors through $\mathrm{Disk}_{/M} \to \mathcal{D}isk_{/M}$, yielding

$$\mathrm{FA}^{lc}(M; \mathcal{V}) := \mathrm{Fun}^{\otimes}(\mathcal{D}isk_{/M}, \mathcal{V}) \simeq \mathrm{Alg}_{\mathcal{B}sc_{/M}}(\mathcal{V}) \ . \tag{4.6}$$

Proposition 4.1.10 ([AFT14a, 2.22])**.** Let $\mathcal{J}_M$ be the class of *isotopy equivalences* in $\mathrm{Disk}_{/M}$, namely precisely those morphisms sent to isomorphisms in $\mathcal{D}isk_{/M}$ by the canonical map. This exhibits $\mathcal{D}isk_{/M} \simeq \mathrm{Disk}_{/M}[\mathcal{J}_M^{-1}]$ as a localization, so that a relative factorization algebra $A : \mathrm{Disk}_{/M}^{\sqcup} \to \mathcal{V}^{\otimes}$ is constructible iff it sends the class $\mathcal{J}_M$ to isomorphisms in $\mathcal{V}$.

Warning. As in the manifold case, we could define $\mathcal{B}sc(M)$ as the full subcategory of $\mathcal{S}ngl_{/M}$ on the open subsets of M that are basics. There also is a canonical functor $\mathcal{B}sc_{/M} \to \mathcal{B}sc(M)$ sending an open embedding $(j : B \to M)$ to its image $j(B)$, which is an equivalence of categories by the same argument. While it is more convenient to work in the slice category, we hope this construction allows for better intuition, in particular when we later study field theories.

4.2. Tangent Classifier and Local Structures

When we want to associate to a field theory on stratified spaces an absolute factorization algebra of observables, the above definition is not very helpful. Even though conically smooth stratified spaces have very nice properties, we know of no physical theory that can be made sense of on every single one. They can only defined on subclasses, for example manifolds with corners, oriented manifolds with boundary, or maybe even particularly simple cones – all of these are restrictions on the *local* or *tangential structure* of our spaces. Mathematically speaking, such local structure should be related to the ∞-category $\mathcal{B}sc_{/M}$, which contains local trivializations/ charts of M in the sense of smooth basics.

Theorem 4.2.1 ([AFT14b, 1.2.10]). For $M \to P$ a conically smooth stratified space, there is an equivalence of ∞-categories

$$\mathcal{B}sc_{/M}^{op} \simeq \mathrm{Sing}^P(M) \ . \tag{4.7}$$

In other words, $\mathcal{B}sc_{/M}$ knows precisely about the stratified homotopy type of M. This theorem generalizes the analogous result 2.2.6 for manifolds.

In the case of smooth n-manifolds, tangential structures are restrictions on (the transition functions of) the tangent bundle $TM \to M$. Let $\tau_M : M \to \mathrm{BO}(n)$ be the associated continuous map into the classifying space of vector bundles, the *tangent classifier* of M. Then, for $G \to \mathrm{O}(n)$ a Lie group homomorphism, a *G-structure* on M is defined as a lift

$$\begin{tikzcd} & BG \arrow[d] \\ M \arrow[r] \arrow[ur] & \mathrm{BO}(n) \end{tikzcd}$$

Example 4.2.2.

- If $G = \mathrm{SO}(n) \subseteq \mathrm{O}(n)$, a $\mathrm{SO}(n)$-structure is an orientation.
- If $G = *$ so that $\mathrm{BG} = \Delta^0$, a G-structure is a framing.
- If $G = \mathrm{SO}(k, \mathbb{C})$ for $n = 2k$, a G-structure is a complex structure.
- For the double covering $\mathrm{Spin}(n) \to \mathrm{O}(n)$, a $\mathrm{Spin}(n)$-structure is a spin structure. Similarly for other lifts of the structure group, e.g. string- or fivebrane-structures.
- Many other interesting examples exist, e.g. the map $G_2 \hookrightarrow O(7)$ induces G_2-structures, that are equivalent to a choice of orientation and spin structure

- Following 2.2.7, we can define a topological manifold with G-structure for every homomorphism of topological groups $G \to \mathrm{Homeo}(\mathbb{R}^n) = \mathrm{Top}(n)$ in the same way (letting the tangent microbundle factor through G). An $O(n)$-structure is the same thing as a smooth structure, by smoothing theory.
- For $G = \mathrm{Homeo}^+(\mathbb{R}^n)$ the subgroup on orientation-preserving homeomorphisms, a G-structure is an orientation on a topological manifold
- One can define a map of classifying spaces $\mathrm{BPL}(n) \to \mathrm{BTop}(n)$. Factoring the tangent microbundle of a topological manifold M through this is equivalent to equipping M with a PL structure.

Let us generalize this to stratified spaces. In this section, let us always work in the C^0-stratified case, meaning that $\mathcal{B}sc_n$, $\mathcal{D}isk_n$, $\mathcal{S}ngl_n$ refer to C^0-basics and so on.

Definition 4.2.3. For $(M \to P)$ an n-dimensional C^0-stratified space, the *tangent classifier* is the slice projection

$$\tau_M : \mathrm{Sing}^P(M) \simeq \mathcal{B}sc_{/M} \to \mathcal{B}sc_n \ . \tag{4.8}$$

In particular if we form the stratified realization on both sides, B.2.1 allows us to interpret $|\,\mathcal{B}sc\,|_{\mathrm{strat}}$ as a (stratified) classifying space:

$$|\tau_M|_{\mathrm{strat}} : M \simeq |\,\mathrm{Sing}^P(M)|_{\mathrm{strat}} \to |\,\mathcal{B}sc_n\,|_{\mathrm{strat}} \ . \tag{4.9}$$

By A.2.14, slice projections are always right fibrations and hence classify presheaves in $\mathcal{PSh}(\mathcal{B}sc_n)$. Let us denote by $\tau : \mathcal{S}ngl_n \to \mathcal{PSh}(\mathcal{B}sc_n)$ the functor that sends a stratified space to its tangent classifier.

Theorem 4.2.4 ([AFT14b, 4.4.8])**.** In particular, if $(M \to *)$ is a smooth manifold, this map factors through the full subcategory $\mathrm{BO}(n)$ spanned by $(\mathbb{R}^n \to *) \in \mathcal{B}sc_n$, where it agrees with the classifying map of the tangent bundle of a smooth manifold. Similarly, for M a topological manifold, $\mathcal{C}harts(M) \to \mathrm{BTop}(n) \subseteq \mathcal{B}sc_n$ classifies the tangent microbundle of M, compare 2.2.7.

While this is difficult to prove, it is extremely intuitive: Every point x on M has a neighborhood U that is homeomorphic to $\mathbb{R}^n$, and our claim is that we may identify this neighborhood with the tangent space T_xM. The morphisms in $\mathcal{B}sc_{/M} \simeq \mathcal{C}harts_{/M}$ then apparently encode how these charts glue together to form a vector bundle.

Definition 4.2.5. A *tangential structure* for C^0-stratified spaces is a right fibration $\mathcal{B} \to \mathcal{B}sc$. By abuse of notation, it is often simply denoted by $\mathcal{B}$, and the presheaf in $\mathcal{PSh}(\mathcal{B}sc_n)$ classified by it according to A.2.13 is called $\mathcal{B}$ as well.

Let us in the following fix a tangential structure $\mathcal{B} \to \mathcal{B}sc_n$.

Definition 4.2.6. A $\mathcal{B}$-structure on a C^0-stratified space $(M \to P)$ is a factorization

$$\operatorname{Sing}^P M = \mathcal{B}sc_{/M} \to \mathcal{B} \to \mathcal{B}sc_n \tag{4.10}$$

of its tangent classifier. Using the pullback diagram

$$\begin{tikzcd}
\operatorname{Sngl}(\mathcal{B}) \arrow[r] \arrow[d] & \mathcal{S}ngl(\mathcal{B}) \arrow[r] \arrow[d] & \mathcal{P}Sh(\mathcal{B}sc_n)_{/\mathcal{B}} \arrow[d] \\
\operatorname{Sngl}_n \arrow[r] & \mathcal{S}ngl_n \arrow[r, "\tau"] & \mathcal{P}Sh(\mathcal{B}sc_n)
\end{tikzcd}$$

we construct a 1-category $\operatorname{Sngl}(\mathcal{B})$ and a ∞-category $\mathcal{S}ngl(\mathcal{B})$ of *$\mathcal{B}$-structured stratified spaces*, or *$\mathcal{B}$-manifolds*. Similarly, we construct $\mathcal{D}isk(\mathcal{B})$ and $\operatorname{Disk}(\mathcal{B})$, note that we do not need $\mathcal{B}sc(\mathcal{B})$ as this is just $\mathcal{B}$. All of these categories come equipped with symmetric monoidal structures induced by $\sqcup$, since $\tau : \mathcal{S}ngl_n^{\sqcup} \to \mathcal{P}Sh(\mathcal{B}sc_n)^{\amalg}$ is a symmetric monoidal functor, making the above a pullback diagram in symmetric monoidal ∞-categories if we also equip $\mathcal{P}Sh(\mathcal{B}sc_n)^{\amalg}_{/\mathcal{B}}$ with the coproduct.

This means that we can define absolute factorization algebras on $\mathcal{B}$-structured stratified spaces as symmetric monoidal functors $\operatorname{Disk}(\mathcal{B})^{\sqcup} \to \mathcal{V}^{\otimes}$, and call them locally constant iff they factor through $\mathcal{D}isk(\mathcal{B})^{\sqcup}$.

Example 4.2.7. Unlike in the manifold case, the category $\mathcal{B}sc_n$ has multiple objects, so choosing a tangential structure can not only pose restrictions on the transition maps, or force the transition maps to lift to a bigger group (like in the spin structure case); but can also restrict how the stratified space is locally allowed to look like (i.e. which kinds of links we are allowed to use), or equip it with extra data like colorings:

- Let $\operatorname{BTop}(n)$ be the full subcategory of $\mathcal{B}sc_n$ spanned by $\mathbb{R}^n \to *$, then $\operatorname{BTop}(n)$-manifolds are topological manifolds. Using topological subgroups $G \subseteq \operatorname{Top}(n)$, we can compose right fibrations

$$\mathrm{BG} \to \operatorname{BTop}(n) \to \mathcal{B}sc_n \tag{4.11}$$

  to incorporate G-structured topological (or, for $G \subseteq \mathrm{O}(n)$, even smooth) manifolds into this context.

- For $\mathcal{B}$ the full subcategory spanned by $\mathbb{R}^n$ and $\mathbb{R}^{n-1} \times C(*)$, we obtain manifolds with boundary.

- For $\mathcal{B}$ the full subcategory spanned by $\mathbb{R}^{n-m} \times \mathbb{R}^m_{\geq 0} \to [m]$ for all $0 \leq m \leq n$, we obtain manifolds with corners.

- For $\mathcal{B}$ the full subcategory spanned by basics $\mathbb{R}^i \times C(L)$ where $i \neq n-1$, we obtain topological pseudomanifolds up to the condition that the top stratum mush be dense.

- As 1-dimensional C^0-stratified spaces are the same thing as graphs, if we restrict to $\mathcal{B}sc_1^{or}$ with only orientation-preserving inclusions of basics, we obtain directed graphs
- For $\mathcal{B} = \mathcal{B}sc_1 \times C \to \mathcal{B}sc_1$ with C a discrete set of colors, we obtain colored graphs.
- For $\mathcal{B}$ the full subcategory of $\mathcal{B}sc_n$ on those basics $\mathbb{R}^i \times C(L)$ with $n-i$ even (so the dimension of L is odd), the category of $\mathcal{B}$-manifolds contains precisely those C^0-stratified spaces that only consist of strata of even codimension. In particular, every complex variety has this property.
- To generalize the last point, we could only allow basics $\mathbb{R}^i \times C(L)$ where, if $\dim(L) = 2k$ is even, the middle-dimensional middle-perversity intersection homology $\mathrm{IH}_k^{\overline{m}}(L, \mathbb{Q})$ vanishes. This yields (after also restricting to topological pseudomanifolds) an ∞-category of *Witt spaces.* Similarly for *Intersection-Poincaré spaces.*

One may imagine that the presheaf $\mathcal{B}sc_n \to \mathcal{S}$ classified by $\mathcal{B}$ associates to every basic a space of possible "colorings" of it, in particular an empty space of coloring would mean that this local structure is not allowed at all. Similarly, to every inclusion of basics it associates the induced changes of color/ whether such inclusions are allowed at all, among other data.

Remark. Absolute factorization algebras on $\mathcal{B}$-structured stratified spaces hence allow us to describe field theories that can be defined simultaneously on all stratified spaces with $\mathcal{B}$-structure, e.g. Chern-Simons theory which can be defined on closed orientable 3-manifolds, and will later be extended to oriented 3-manifolds with corners.

Observation 4.2.8 ([AFT14a, 2.7])**.** For any C^0-stratified space $M \to P$, its tangent classifier $\mathcal{B}sc_{/M} \to \mathcal{B}sc_n$ is a right fibration and can hence be interpreted as a tangential structure. However, it classifies a representable presheaf by A.2.14, so

$$\mathcal{S}ngl(\mathcal{B}sc_{/M}) = \mathcal{S}ngl_n \times_{\mathcal{P}Sh(\mathcal{B}sc_n)} \mathcal{P}Sh(\mathcal{B}sc_n)_{/h(M)} \cong \mathcal{S}ngl_{/M} \tag{4.12}$$

since $\mathcal{S}ngl_n \subseteq \mathcal{P}Sh(\mathcal{B}sc_n)$ is a full subcategory by B.1.10. Be *very* careful, since the symmetric monoidal structure this induces on $\mathcal{S}ngl_{/M}$ is not the one given by disjoint union that we are interested in! Relative factorization algebras are *not* just absolute factorization algebras with respect to this tangential structure.

Still, this is helpful to know: If $\mathcal{B}$ is an arbitrary tangential structure and $M \to P$ is $\mathcal{B}$-structured, we can factor the tangent classifier $\mathcal{B}sc_{/M} \to \mathcal{B} \to \mathcal{B}sc_n$ yielding a natural transformation of associated presheaves $h(M)|_{\mathcal{B}sc_{/M}^{op}} \to \mathcal{B}$. Applying the pasting lemma to the pullback diagram

$$\begin{tikzcd}
\mathcal{S}ngl(\mathcal{B})_{/M} \arrow[r] \arrow[d] & \mathcal{S}ngl(\mathcal{B}) \arrow[r] \arrow[d] & \mathcal{S}ngl_n \arrow[d, "\tau"] \\
\left(\mathcal{PS}h(\mathcal{B}sc_n)_{/\mathcal{B}}\right)_{/(h(M)\to\mathcal{B})} \arrow[r] & \mathcal{PS}h(\mathcal{B}sc_n)_{/\mathcal{B}} \arrow[r] & \mathcal{PS}h(\mathcal{B}sc_n)
\end{tikzcd}$$

and noticing the double slice category in the lower left is just $\mathcal{PS}h(\mathcal{B}sc_n)_{/h(M)}$ tells us that $\mathcal{S}ngl(\mathcal{B})_{/M} \simeq \mathcal{S}ngl_{/M}$ and similarly $\mathcal{D}isk(\mathcal{B})_{/M} \simeq \mathcal{D}isk_{/M}$ and $\mathcal{B}_{/M} \simeq \mathcal{B}sc_{/M}$, so tangential structures do not matter in the relative setting.

4.3. Weiss Descent

Analogously to factorization algebras on manifolds, we show that factorization algebras on stratified spaces can be thought of as factorizable Weiss cosheaves. This relies on an important descent property of $\mathcal{D}isk_{/M}$ that generalizes the Seifert-van-Kampen theorem from algebraic topology. Compare our discussion in B.4.

Definition 4.3.1 ([Mat13, 2.10])**.** For $(M \to P)$ a C^0-stratified space and $\mathcal{C}$ an ordinary category, a functor $U : \mathcal{C} \to \mathrm{Open}(M)$ is called a *factorization SFK cover* if, for any nonempty finite subset $S \subseteq X$, the full subcategory $\mathcal{C}_x$ spanned by those $C \in \mathcal{C}$ such that $S \subseteq U(C)$, is weakly contractible as in A.8.4.

Theorem 4.3.2 (Seifert-van-Kampen for Weiss covers)**.** Let $(M \to P)$ be a C^0 stratified space, $\mathcal{C}$ an ordinary category and $U : \mathcal{C} \to \mathcal{S}ngl_{/M}$ a factorization SFK cover. Then,

$$\mathcal{D}isk_{/M} \cong \operatorname*{colim}_{\mathcal{C}} \mathcal{D}isk_{/U} \ . \tag{4.13}$$

Proof Sketch. Ideally, we should be able to prove this by applying stratified SFK B.4.4 to the Ran space, since a factorizing SFK cover is a SFK cover on the Ran space. We however only know that $\mathcal{D}isk^{surj}_{/M} \cong \mathrm{Sing}^P \mathrm{Ran}(M)^{op}$ by 2.5.5, ignoring disk inclusions that are not surjective on connected components. This issue can either be fixed by adding the inclusions of $\emptyset$ into a disk in by hand; or by hoping that an enhanced Ran space, as in the references at the end of 2.5, allows for a more general statement. solve this problem by hand; or rely on this alternative proof following [Mat13, 2.25] (who hover only shows that this map is cofinal, not an equivalence):

As discussed in A.2.14, the left fibration $\mathcal{D}isk_{/M} \to \mathcal{D}isk_n$ classifies the representable functor $\mathrm{Map}_{\mathcal{D}isk_n}(-, M) : \mathcal{D}isk^{op} \to \mathcal{S}$, so by A.9.2 we have

$$\mathcal{D}isk_{/M} \cong \operatorname*{laxcolim}_{D \in \mathcal{D}isk_n} \mathcal{E}mb(D, M) \ . \tag{4.14}$$

where $\mathcal{E}mb$ denotes the mapping space in $\mathcal{D}isk_n$. A similar expression holds for each $U(C)$ with $C \in \mathcal{C}$. If one explicitly proves that this lax colimit commutes with the colimit over $\mathcal{C}$, it suffices to show that for each $D \in \mathcal{D}isk_n$,

$$\mathcal{E}mb(D, M) \cong \operatorname*{colim}_{C \in \mathcal{C}} \mathcal{E}mb(D, U(C)) \ . \tag{4.15}$$

Along the lines of [AFT14a, 2.21], one can show that there is a canonical map $\mathcal{E}mb(D, M) \to \mathcal{E}mb(\{1, \ldots, k\}, M) = \operatorname{Sing}\operatorname{Conf}_k(M)$ by choosing one point in every one of the k connected components (basics) in D. The fibers of this map consist of automorphisms of the respective basics. Since those arise on both sides of the expression, we can equivalently just show

$$\operatorname{Sing}\operatorname{Conf}_k(M) \cong \operatorname*{colim}_{C \in \mathcal{C}} \operatorname{Sing}\operatorname{Conf}_k(U(C)) \ . \tag{4.16}$$

But our assumption that U is a factorization SFK cover assures that $\operatorname{Conf}_k(U(C))$ form an SFK cover of $\operatorname{Conf}_k(M)$, so we can just apply generalized Seifert-van-Kampen B.4.3. $\square$

Corollary 4.3.3. The localization map $\mathrm{Disk}_{/M} \to \mathcal{D}isk_{/M}$ is left cofinal.

Proof. Similar to [HA, 5.5.2.13]. By Quillen's Theorem A A.8.3, it suffices to show that for every object $(j : D \to M) \in \mathcal{D}isk_{/M}$, the pullback

$$\mathcal{P} = \mathrm{Disk}_{/M} \times_{\mathcal{D}isk_{/M}} (\mathcal{D}isk_{/M})_{j/} \tag{4.17}$$

is weakly contractible. The projection $\mathcal{P} \to \mathrm{Disk}_{/M}$ is a right fibration as it is the pullback of a slice projection, and thus by A.2.13 classified by a functor $\chi : \mathrm{Disk}_{/M} \to \mathcal{S}$ that sends $D' \in \mathrm{Disk}_{/M}$ to

$$\operatorname{fib}\left(\mathcal{E}mb(D', D) \to \mathcal{E}mb(D', M)\right) \tag{4.18}$$

where by $\mathcal{E}mb$ we denote the mapping space in $\mathcal{S}ngl_n$. Conversely,

$$\mathcal{P} = \operatorname*{colim}_{\mathrm{Disk}_{/M}}(\chi) = \operatorname{fib}\left(\operatorname*{colim}_{D' \in \mathrm{Disk}_{/M}} \mathcal{E}mb(D', D) \to \mathcal{E}mb(M, D)\right) \tag{4.19}$$

since filtered colimits in $\mathcal{S}$ commute with finite limits; which vanishes by the previous proposition. $\square$

Proposition 4.3.4 ([AFT14a, 2.28])**.** For M a conically smooth stratified space, the simplicial set $\mathcal{D}isk(\mathcal{B})_{/M}$ is sifted.

Theorem 4.3.5. Let $\mathcal{V}$ be a sifted complete symmetric monoidal ∞-category. Restricting along the inclusion $\mathrm{Disk}^{\sqcup}_{/M} \hookrightarrow \mathrm{Sngl}^{\sqcup}_{/M} \simeq \mathrm{Open}(M)^{\sqcup}$ induces an equivalence of categories

$$\mathrm{FA}(M; \mathcal{V}) \simeq \mathcal{S}\mathrm{h}^{\otimes}_{\infty}(M_{\mathrm{Weiss}}, \mathcal{V}^{op}) \simeq \mathrm{Fun}^{\otimes}(\mathrm{Disk}_{/M}, \mathcal{V}) \ , \tag{4.20}$$

with inverse functor given by left Kan extension. This means factorization algebras on M with values in $\mathcal{V}$ are the same thing as factorizable $\mathcal{V}$-valued Weiss cosheaves on M.

Proof. Restriction a priori induces a functor into $\mathrm{Fun}(\mathrm{Disk}_{/M}, \mathcal{V})$. This factors trough the symmetric monoidal functors because we only plug in factorizable sheaves, whose underlying functor sends disjoint unions of embeddings to tensor products, i.e. is symmetric monoidal itself.

Further, the inclusion $\mathrm{Disk}_{/M} \hookrightarrow \mathrm{Sngl}_{/M}$ is fully faithful, so left Kan extension $\mathrm{Fun}(\mathrm{Disk}_{/M}, \mathcal{V}) \to \mathrm{Fun}(\mathrm{Open}(M), \mathcal{V})$ along it is so as well. It stays like that on the full subcategories of symmetric monoidal functors, however we must show that we actually land inside factorizable sheaves.

For this, let $A' : \mathrm{Disk}^{\sqcup}_{/M} \to \mathcal{V}^{\otimes}$ be a disk algebra; we need to show that $\mathrm{Lan}^{\mathrm{Open}(M)}_{\mathrm{Disk}_{/M}} A'$ is a Weiss cosheaf. This Kan extension can be calculated as the colimit

$$\mathrm{Lan}^{\mathrm{Open}(M)}_{\mathrm{Disk}_{/M}} A'(U) \cong \operatorname*{colim}_{D \in \mathrm{Disk}_{/U}} A'(D) \cong \operatorname*{colim}_{D \in \mathcal{D}isk_{/U}} A'(D)$$

by above cofinality statement, which we want to satisfy descent with respect to every Weiss cover $(U_i \subseteq U)_{i \in I}$. By adding all finite intersections (as they are contained in the Čech nerve incorporating this descent), we can make such a Weiss cover into a factorization SFK cover $U : \mathcal{C} \to \mathrm{Open}(M)$ with objects of $\mathcal{C}$ given by finite tuples in I. This leaves us with

$$\operatorname*{colim}_{C \in \mathcal{C}} \operatorname*{colim}_{D \in \mathcal{D}isk_{/U(C)}} A'(D) \overset{!}{\cong} \operatorname*{colim}_{D \in \mathcal{D}isk_{/U}} A'(D) .$$

To show this, we apply transitivity of (left) Kan extensions to the composition

$$\mathcal{D}isk_{/U} \to \mathcal{C} \to *$$

where using 4.3.2, the first map is the coCartesian fibration classified by $C \mapsto \mathcal{D}isk_{/U(C)}$. Left Kan extension along the terminal morphism is just a colimit, and left Kan extension along the middle morphism by [KER, Tag 02ZM] is a fiber-wise colimit along the fibers $\mathcal{D}isk_{/U(C)}$, so we are finished.

Finally, we need to show that the left Kan extension functor is essentially surjective, or in other words that for A a factorizable Weiss cosheaf,

$$\mathrm{Lan}^{\mathrm{Open}(M)}_{\mathrm{Disk}_{/M}} \left(A|_{\mathrm{Disk}_{/M}} \right) (U) \cong \operatorname*{colim}_{D \in \mathrm{Disk}_{/U}} A(D) \overset{!}{\cong} A(U)$$

Because $\mathrm{Disk}_{/U}$ clearly form a Weiss cover of U, this isomorphisms follows from Weiss descent. We do not have to use a simplicial diagram including all intersections since every such intersection can again by covered by disjoint unions of basics. □

Remark. Under this equivalence, constructible factorization algebras correspond to constructible Weiss cosheaves, i.e. Weiss cosheaves that send disk isotopies to isomorphisms.

To compare factorization algebras with cosheaves, assume that $\mathcal{V}$ contains all colimits and is equipped with the coCartesian symmetric monoidal structure $\mathcal{V}^{\amalg}$, i.e. the coproduct is the tensor product.

Proposition 4.3.6 ([Gin13, Lemma 11])**.** Factorization algebras in $\mathrm{FA}(M, \mathcal{V})$ are the same thing as cosheaves on M. In fact, factorizable Weiss cosheaves on every Hausdorff topological space are the same thing as cosheaves.

Proof. Using 4.3.5, the proof of 2.4.16 translates to this generality. □

Corollary 4.3.7. Constructible factorization algebras in $\mathrm{FA}^{lc}(M, \mathcal{V})$ are, via the above identification, the same thing as constructible cosheaves on M.

Proof. Combine the above proposition with B.3.6. □

Corollary 4.3.8. Given a (constructible) cosheaf $F \in \mathcal{coSh}^{cbl}(M; \mathcal{V})$ and a symmetric monoidal functor $S : \mathcal{V}^{\amalg} \to \mathcal{W}^{\otimes}$ into a sifted complete symmetric monoidal ∞-category, the composition $S \circ F : \mathrm{Disk}_{/M} \to \mathcal{W}$ is a (constructible) factorization algebra on M. If S preserves sifted colimits, then $S \circ F : \mathrm{Open}(M) \to \mathcal{W}$ is the associated factorizable Weiss cosheaf to this factorization algebra.

Proof. The first statement is immediately clear since we have seen that F is a (constructible) factorization algebra itself, with associated symmetric monoidal functor $F : \mathrm{Disk}_{/M} \to \mathcal{W}$ so that $S \circ F$ is again symmetric monoidal, i.e. a (constructible) factorization algebra. For the second statement, note that the associated Weiss cosheaf is recovered from this symmetric monoidal functor using a left Kan extension, which is pointwise given by sifted colimits because of 4.3.4. If S preserves sifted colimits, it commutes with this Kan extension; compare the argument of 3.2.2. □

4.4. Examples in Dimension 1

Let us, for simplicity, start by writing down some examples involving 1-dimensional stratified spaces. Remember that the only 1-basics are $\emptyset$, $\mathbb{R}$ with trivial stratification, and cones on a finite set of points $C(\{1, \ldots, k\}) \to [1]$.

We begin with $M = (C(*) \to [1]) = (\mathbb{R}_{\geq 0} \to [1])$, a manifold with boundary with its canonical stratification sending the boundary to 0, and the interior to 1. It contains basics of the form $C(*)$ and of the form $\mathbb{R}$, so the exit-path category is given by $\mathrm{Sing}^{[1]}(M) = \mathcal{Bsc}^{op}_{/M} = \Delta^1$ as all charts in the bulk are isotopic, and there is an inclusion $\mathbb{R} \hookrightarrow C(*)$. See B.2.20 for other ways to calculate it. Similarly, $\mathcal{Disk}_{/M}$ consists of finite sets of disjoint basics in M, possibly including several of them in the interior of the form $\mathbb{R}$, but there can only be one basic $C(*)$ containing the boundary $\partial M = \{0\}$.

Now, what is the operadic structure on $\mathcal{B}sc^{\sqcup}_{/M}$? We will denote the basic $C(*)$ in M as l, and the basic $\mathbb{R}$ as a. Then, the only non-degenerate simplices in $\mathcal{B}sc_{/M}$ are the objects l, a and one morphism $a \to l$. Multimorphisms are given by

$$\begin{aligned} \mathrm{Mul}(\{a, a, \ldots, a\}, a) &= \{\text{total orders on 1, ..., k}\} = \\ &= \mathrm{Mul}(\{l, a, \ldots, a\}, l) = \mathrm{Mul}(\{a, \ldots, a\}, l) \end{aligned}$$

where each source contains $k \in \mathbb{N}_0$ arguments a. All other multimorphism spaces are empty. Intuitively, we can either include k disks into one disk in the interior; or we can include k disks, possibly together with an interval $[0, a)$, into a bigger interval $[0, b)$.

Definition 4.4.1 ([HA, 4.2.1.1])**.** We call the ∞-operad with objects (i.e. colors) l, a and the discrete multimorphism spaces above by $\mathcal{LM}^{\otimes}$. Composition is induced by inserting an ordering into the respective position of another ordering, as for $\mathbb{E}_1^{\otimes}$. There is a canonical map of operads $\mathbb{E}_1^{\otimes} \to \mathcal{LM}^{\otimes}$ sending the unique object to a, and there is a map of operads $\mathbb{E}_0^{\otimes} \to \mathcal{LM}^{\otimes}$ sending the unique object to l.

Proposition 4.4.2. The above discussion shows that $\mathcal{B}sc^{\sqcup}_{/\mathbb{R}_{\geq 0}} \simeq \mathcal{LM}^{\otimes}$, and $\mathcal{D}isk_{/\mathbb{R}_{\geq 0}}$ can expressed in a similar way.

Corollary 4.4.3. Constructible factorization algebras on $\mathbb{R}_{\geq 0} \to [1]$ are the same thing as algebras over $\mathcal{LM}^{\otimes}$, which are also called *left modules* on $\mathcal{V}$. Precomposing with $\mathbb{E}_1^{\otimes} \to \mathcal{LM}^{\otimes}$ associates to every left module an underlying associative algebra object A in $\mathcal{V}$; and precomposing with $\mathbb{E}_0^{\otimes}$ yields a pointed object L in $\mathcal{V}$. The images of the multimorphisms described above in $\mathcal{V}$ are morphisms

$$\begin{aligned} A \otimes A \otimes \cdots \otimes A &\to A \\ L \otimes A \otimes \cdots \otimes A &\to L \\ A \otimes A \otimes \cdots \otimes A &\to L \end{aligned}$$

that equip L with a homotopy coherent module structure over A. In particular, the first morphism type describes the algebra structure on A, the second the module structure of L, and the third type describes for $k = 0$ the pointing of L, while for higher k it agrees with the composition of pointing and module multiplication.

Similarly, we could have defined an operad $\mathcal{RM}^{\otimes}$ with colors a, r, such that the multimorphisms in it encode a right module structure (here, the multimorphism space is only non-vanishing if r appears only as the rightmost source, or not at all). A similar argument would have allowed us to identify constructible factorization algebras over $\mathbb{R}_{\geq 0}$ with right module objects, since as expected the functor sending left to right modules by reversing the algebra structure on A is an equivalence. Intuitively, this reverses the orientation of $\mathbb{R}_{\geq 0}$. We see that

$$\mathrm{FA}^{lc}(\mathbb{R}_{\geq 0}; \mathcal{V}) \simeq \mathrm{Alg}_{\mathcal{LM}} \mathcal{V} \simeq \mathrm{Alg}_{\mathcal{RM}} \mathcal{V} \,. \tag{4.21}$$

Example 4.4.4. Next, let $M = ([0,T] \to [1])$ for $T > 0$ be a compact interval in $\mathbb{R}$; a manifold with boundary stratified in the canonical way. Its enter-path category is

$$\mathcal{B}sc_{/M} \simeq \mathrm{Sing}^{[1]}(M)^{op} \simeq (* \leftarrow * \to *)\ , \tag{4.22}$$

so the operad $\mathcal{B}sc^{\sqcup}_{/M}$ now has three colors l, a, r. Multimorphisms have a similar description as above, their space vanishes unless there is either no argument l, r involved, or precisely either one l or one r, on the respective side. In other words, constructible factorization algebras on $[0,T]$ consist of

- an associative algebra object A in $\mathcal{V}$,
- a left module L over A,
- a right module R over A.

Putting this in different words, the operad $\mathbb{E}^{\otimes}_{[0,T]} := \mathcal{B}sc^{\sqcup}_{/[0,T]} \simeq \mathcal{LM}^{\otimes} \amalg_{\mathbb{E}_1^{\otimes}} \mathcal{RM}^{\otimes}$ so algebras over it consist of a left and a right module over the same algebra.

However, we are still not finished: Since $[0,T]$ is not itself a basic, we should still calculate the factorization homology of an $\mathbb{E}_{[0,T]}$-algebra determined by a triple (A, L, R).

Construction 4.4.5. We call the factorization homology $\int_{[0,T]}(A, L, R) =: L \otimes_A R \in \mathcal{V}$ the *relative tensor product* of the left module L with the right module R. It is calculated as a left Kan extension using the formula

$$\begin{aligned} L \otimes_A R = \mathrm{Lan}^{\mathrm{Open}(M)}_{\mathrm{Disk}_{/M}}(A, L, R)([0,T]) &\cong \operatorname*{colim}_{D \in \mathcal{D}isk_{/M}} (A, L, R)(D) \cong \\ &\cong \operatorname*{colim}_{\Delta^{op}} \left(\dots\ L \otimes A \otimes A \otimes R \mathrel{\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}} L \otimes A \otimes R \mathrel{\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}} L \otimes R \right) \end{aligned} \tag{4.23}$$

where we have replaced $\mathcal{D}isk_{/M}$ by the cofinal subdiagram $\mathcal{D}isk'(M) \to \mathcal{V}$ on disjoint unions of disks containing 0 and T, which can be checked using Quillen's Theorem A A.8.3, see [AF15, 3.11] for a precise argument. Also, we have noticed that $\mathcal{D}isk'(M) \simeq \Delta^{op}$ by sending a disjoint union of two disks around the boundary, and i disks in the interior, to the nonempty finite set of "free spaces" between the disks, equipped with the total order induced from $\mathbb{R}$. The resulting geometric realization (i.e. colimit over a Δ^{op}-diagram) is also called the *two-sided bar construction.*

Example 4.4.6. Next, let us look at the case where the basic $C(\{0,1\}, *) \to [1]$ is used. This can be imagined as a point with two lines coming out of it, or in other words, $M = \mathbb{R} \to [1]$ where the stratification indicates a marked point. The enter-path category in this case is

$$\mathcal{B}sc_{/M} \simeq \mathrm{Sing}^{[1]}(\mathbb{R} \to [1])^{op} \simeq \{* \to * \leftarrow *\} \tag{4.24}$$

so again, the operad $\mathbb{E}^{\otimes}_{M} = \mathcal{B}sc^{\sqcup}_{/M}$ has three different colors a', b, a. In this case, disks inclusions however involve

- including multiple disks of type a', i.e. inside $\mathbb{R}_{<0}$, into a disk of type a',
- including multiple disks of type a, i.e. inside $\mathbb{R}_{>0}$, into a disk of type a,
- including multiple disks of type a', at most one disk of type b that includes 0, and multiple disks of type a into a big disk of type b.

Again, we can write down an operad $\mathcal{BM}^{\otimes}$ incorporating these types of disk inclusions, such that $\mathbb{E}_M^{\otimes} \simeq \mathcal{BM}^{\otimes}$. A $\mathcal{BM}^{\otimes}$-algebra then consists of a triple (A', B, A), where

- A and A' are $\mathbb{E}_1^{\otimes}$-algebras because of the first two types of disk inclusions
- B is a pointed object equipped with a right-module structure over A', and a left module structure over A.

This fully characterizes constructible factorization algebras on $\mathbb{R}$ with a marked point, since every open subset is a disjoint union of basics.

Example 4.4.7. Finally, we discuss absolute constructible factorization algebras on framed 1-dimensional singular spaces $\mathcal{S}ngl_1^{fr} := \mathcal{S}ngl(* \to \mathcal{B}sc_1)$. As mentioned above, basics of dimension 1 are precisely $\mathbb{R}$ and $C(\{1, \ldots, k\})$; and one can see that fixing a framing is equivalent to choosing an orientation on the manifolds $\mathbb{R}$ and $C(\{1, \ldots, k\}) - \{-\infty\}$, respectively. Therefore, objects of $\mathcal{S}ngl_1$ are graphs, and objects in $\mathcal{S}ngl_1^{fr}$ are directed graphs.

An absolute factorization algebra is given by a functor $\mathcal{B}sc_1^{fr} \to \mathcal{V}$ preserving the operadic structure. To every flavor of basic, we thus have to associate on object of $\mathcal{V}$, such that inclusions of multiple basics into one induce algebraic structures on these objects. Figuring those out works similarly to the examples above; we find

- An associative algebra $A := A(\mathbb{R}) \in \mathcal{V}$
- For each $i, j \in \mathbb{N}_0$ that are not both zero, an object $A(i, j) \in \mathcal{V}$ together with i left and j right module structures over $A(\mathbb{R})$ that all commute with each other. This is given by evaluating A on $C(\{1, \ldots, i+j\})$ where i of the lines are directed away, and j towards the cone point.

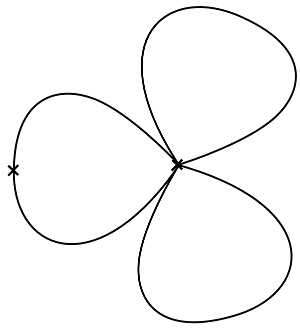
See the end of section 4.1 in [AFT14a] for more information. We can calculate the factorization homology of A on the bouquet of 3 circles

$$(M \to P) := ((S^1, 1, -1) \vee (S^1, 1) \vee (S^1, 1) \to \{0 < 1 > 0'\}) \,, \tag{4.25}$$

with one pointing inside and one pointing outside of the middle:

$$\int_M A = A(1,1) \otimes_{A \otimes A^{op}} A(3,3) \otimes_{A \otimes A^{op}} A \otimes_{A \otimes A^{op}} A \tag{4.26}$$

In other words, note that $A(3,3)$ possesses 3 different bimodule structures over A, and what we have to do is (in any order) to



- Form the derived tensor product of the first bimodule structure with the bimodule corresponding to the extra pointing,
- Take Hochschild Homology with respect to the second bimodule structure,
- Take Hochschild Homology with respect to the third bimodule structure.

4.5. Examples in Higher Dimension

For higher-dimensional stratified spaces, the module structures can become even more complicated.

Example 4.5.1. Let $M = \mathbb{R} \times \mathbb{R}_{\geq 0}$ with the usual stratification of a two-dimensional manifold with boundary. There are two kinds of basics in this space: Disks in the interior of M, and disks at the boundary of the form $\mathbb{R} \times C(*)$. This allows for similar classes of basic inclusions as in the case of $\mathbb{R}_{\geq 0}$, see the figure.

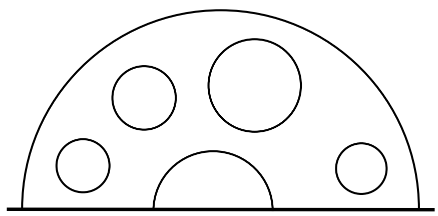


Figure 4.1.: Possible inclusion of multiple basics into one in $\mathbb{R} \times \mathbb{R}_{\geq 0}$

Arguing as in the last section, we may guess that for a constructible factorization algebra $A : \mathcal{D}isk^{\sqcup}_{/M} \to \mathcal{V}^{\otimes}$, the image $A(D)$ for a disk in the interior should be an $\mathbb{E}_2$-algebra, while $A(D')$ for a basic at the boundary is an $\mathbb{E}_1$-algebra because we can include such basics into each other in a similar way as we could include disks in $\mathbb{R}$ into each other. Also, there should be a module action $A(D') \otimes A(D) \to A(D')$, and inclusions of multiple disks into D' like in the figure introduce higher coherence relations. In other words, A

consists of an $\mathbb{E}_1$-algebra, an $\mathbb{E}_2$-algebra, and a module structure of the former over the latter that is compatible with both algebra structures.

An alternative way to see this is to prove a version of Dunn additivity 2.4.15 for constructible factorization algebras, making $\mathcal{D}isk^{\sqcup}_{/\mathbb{R}\times\mathbb{R}_{\geq 0}} \cong \mathbb{E}_1^{\otimes}\otimes\mathcal{LM}^{\otimes}$ so that a constructible factorization algebra on M consists of an associative algebra object in the ∞-category of pairs of algebras and left modules over them in $\mathcal{V}$; or conversely a pair of an algebra and a left module in the ∞-category $\mathrm{Alg}_{\mathbb{E}_1}(\mathcal{V})$.

Example 4.5.2. Similarly for the n-dimensional manifold with boundary $M = \mathbb{R}^{n-1}\times\mathbb{R}_{\geq 0}$, one finds that constructible factorization algebras on it agree with pairs (A, L) of an $\mathbb{E}_n$-algebra A and an $\mathbb{E}_{n-1}$-algebra L, together with a module action of A on L compatible with the respective algebra structures. Alternatively via Dunn additivity, an $\mathbb{E}_M$-algebra is a pair of an associative algebra and a left module over it inside $\mathrm{Alg}_{\mathbb{E}_{n-1}}(\mathcal{V})$.

There is even another way to express this. Remember that given a commutative ring R, an R-algebra is the same thing as another commutative ring R' equipped with a morphism $R \to R'$. Similarly, an R-module structure on a pointed set S is a morphism $R \to \mathrm{End}(S)$. We can generalize this to $\mathbb{E}_n$-algebras:

Proposition 4.5.3. Equipping an $\mathbb{E}_i$-algebra L with a module structure over an $\mathbb{E}_n$-algebra A is equivalent to equipping it with a morphism of $\mathbb{E}_n$-algebras

$$A \to \mathcal{Z}_n(L) \ , \tag{4.27}$$

where $\mathcal{Z}_n(L)$ is the $\mathbb{E}_n$-*center* of M. This generalizes classical constructions like the center of a monoid, the endomorphisms of a pointed set, Hochschild homology (this is also called the derived center) and the Drinfeld center.

Remark. While we do not properly define $\mathcal{Z}_n(L)$, it is uniquely determined by this universal property using the ∞-Yoneda Lemma.

Example 4.5.4. Given an n-manifold M with boundary, stratified in the usual way by [1], a constructible factorization algebra on it consists of

- An $\mathbb{E}_{\mathring{M}}$-algebra A,
- An $\mathbb{E}_{\partial M}$-algebra L,
- A module structure of L over A that is parametrized by the homotopy type of ∂M in a way that is hard to describe explicitly.

One should compare this with the recollement property of sheaves 5.5.4, which tells us that a cosheaf on M is the same thing as a triple consisting of a cosheaf $F_\circ$ on $\mathring{M}$, a cosheaf F_∂ on ∂M, and a map $i^* j_* F_\circ \to F_\partial$ for $j : \mathring{M} \hookrightarrow M, i : \partial M \hookrightarrow M$.

Example 4.5.5. Given a manifold M without boundary, with stratification $M \to [1]$ such that the preimage of 0 marks an embedded submanifold $N \hookrightarrow M$, describing constructible factorization algebras on it is relatively difficult as we need to be aware of the fact that N might cut M into two pieces, or do other topologically non-trivial things, that need to be captured in the algebraic structure. An easier approach to embedded submanifolds is via absolute factorization algebras, let us explain this for the case of knots in 3-manifolds.

We may define a tangential structure $\mathcal{B} \subseteq \mathcal{B}sc_3$ spanned by $\mathbb{R}^3$ itself and $\mathbb{R} \times C(S^1)$. The idea behind this is that $\mathcal{S}ngl(\mathcal{B})$ contains spaces which locally either look like a manifold, or like $\mathbb{R} \times C(S^1) \simeq (\mathbb{R}^3 \to [1])$ where the stratification marks one coordinate axis in $\mathbb{R}^3$. This are precisely 3-manifolds with marked lines and circles!

As there are precisely two flavors of basics, one can show that constructible absolute factorization algebras on $\mathcal{B}$-manifolds are determined by an $\mathbb{E}_3$-algebra for the bulk, an $\mathbb{E}_1$-algebra for the lines and a module structure of the latter over the former. Note we do not have to distinguish between left-, right- and bimodules over $\mathbb{E}_n$-algebras for $n \geq 2$ since the ordering of $\mathbb{R}$ is not present any more. This classification is very useful both for knot theory and for studying Wilson lines in e.g. Chern-Simons theory, compare [CG16, 8.2] and [Cos13].

Example 4.5.6. For M a compact smooth n-manifold with boundary, equip the quotient ${}^{M}\!/_{\partial M}$ with the stratification sending points in the interior to 1, and the collapsed boundary to 0. This space is C^0-stratified since it locally either looks like $\mathbb{R}^n$ or $C(\partial M)$ by a collar argument; it even has an atlas induced from M making it conically smooth since this property is stable under forming a cone and gluing. A constructible factorization algebra on this quotient is, intuitively, given by

- An $\mathbb{E}_{\mathring{M}}$-algebra A encoding inclusions of disks away from the singular point,
- A pointed object $L = A(C(\partial M)) \in \mathcal{V}$ associated to a basic around the singular point,
- A module action of A on L induced by the inclusion of disks of the first into basics of the second type. Just as A is not just an ordinary algebra, this is not just an ordinary action but should rather be thought of as a family of actions parametrized by (the homotopy type of) ∂M, see below.

Example 4.5.7. For $M = \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ stratified over [2] as a manifold with corner, a constructible factorization algebra on M consists of

- An $\mathbb{E}_2$-algebra A corresponding to disks living in the interior of M,
- Two left modules L and L' over A for the two boundary components,
- $\mathbb{E}_1$-algebra structures on L and L' that are compatible with the A-module structures,

- A pointed object $N \in \mathcal{V}$ that is a left module for both L and L' corresponding to the corner,
- Such that the two induced module structures of N over A are isomorphic.

Again, note that we can also write $\mathcal{D}isk^{\sqcup}_{/M} \cong \mathcal{LM}^{\otimes} \otimes \mathcal{LM}^{\otimes}$.

Example 4.5.8. Finally, let us look at the pinched torus from B.1.12 as a more complicated example. A constructible factorization algebra on it consists of

- An $\mathbb{E}_2$-algebra A for the interior,
- A pointed object $L \in \mathcal{V}$ for the pinch point,
- A left and a right module structure of L over A, corresponding to shifting disks from the interior to the singular point from both directions,
- Automorphisms σ and σ' of A, corresponding to moving disks around the torus in the direction of the small circle,
- An involution τ of A turning the left- into the right module structure, corresponding to moving disks around the big circle,
- Such that $\tau \circ \sigma \cong \sigma'$ in a homotopy coherent way.

We could say that L has a module structure over A that is parametrized by the link $S^1 \sqcup S^1$ of the singular point. Let us capture this intuitive understanding in a somewhat cryptic mathematical statement.

Conjecture 4.5.9. For a conically smooth stratified space $(M \to P)$, a constructible factorization algebra on it should consist of

- One $\mathbb{E}_n$-algebra A_p for each n-dimensional stratum M_p with $p \in P$,
- For $p \leq p'$ in P, a module structure of A_p over $A_{p'}$ that is parametrized over the link of M_p in $M_{p'}$
- Such that for $p \leq p' \leq p'$, the composite module structure of A_p over $A_{p''}$ should be homotopic to the one induced by $p \leq p''$ as in the corner case,
- Higher coherence relations for longer chains in P.

We conjecture that similarly to the manifold case 2.4.19, this can be captured in an elegant way by saying that

$$\mathrm{FA}^{lc}(M; \mathcal{V}) \simeq \operatorname*{laxlim}_{(j:\mathbb{R}^i \times C(L) \hookrightarrow M) \in \mathcal{B}sc_{/M}} \mathrm{Alg}_{\mathbb{E}_i}(\mathcal{V}) \tag{4.28}$$

Remember $\mathcal{B}sc_{/M} \simeq \mathrm{Sing}^P(M)^{op}$ is the category of enter-paths, its morphism spaces describe the *homotopy links* of points in strata encoding the second item above. We use a lax (co-)limit in this expression motivated by the analogous result for constructible sheaves B.24, and we use enter-paths because we are working with *co*sheaves. In the manifold case, this reduces to an ordinary colimit by A.9.6 as the exit-path category becomes a Kan complex.

The diagram $\mathcal{B}sc_{/M} \hookrightarrow \boldsymbol{Cat}_\infty$ to which we apply the Grothendieck construction is informally given by sending any basic parametrized by $\mathbb{R}^i \times C(L)$ to the ∞-category of $\mathbb{E}_i$-algebras $\mathrm{Alg}_{\mathbb{E}_i}(\mathcal{V})$, and any isotopy transport of basics in M making them factor through an embedding $\mathbb{R}^i \times C(L) \hookrightarrow \mathbb{R}^j \times C(L')$ where automatically $j < i$ to an induced transport map on $\mathbb{E}_i$-algebras, composed with the operation of sending an $\mathbb{E}_i$-algebra to its center $\mathbb{E}_j$-algebra. This is probably hard to describe explicitly, but can be checked to yield the correct result in simple cases. See A.9.2 and A.2.12 for more (note the pullback expression for the Grothendieck construction), we omit a more thorough discussion as it is both speculative and technical.

We invite the reader to figure out by himself what constructible factorization algebras over the other examples B.1.12 look like.

5. Applications for Field Theories

The techniques developed in the last chapter will now be put to use to study field theories. After an extensive motivation on why fields on stratified spaces are interesting at all, and some mathematical background on shifted symplectic structures, we go through a wealth of examples. Not only manifolds with boundaries or corners will be considered (in the first case, our discussion follows [GRW20]), but also triangulations and cell decompositions of manifolds. We end with a compilation of future prospects and applications.

5.1. Motivation

Just as (locally constant) factorization algebras are used to describe operator algebras on manifolds, constructible factorization algebras allow us to apply similar constructions to stratified spaces – in particular manifolds with corners, manifolds with marked embedded submanifolds, conifolds that arise in string theory and even general complex varieties. We have seen many examples that indicate how their algebraic structure captures the local structure and homotopy theory of a stratified space. But why should we even consider physics in these contexts?

5.1.1. Manifolds with boundary, bordisms, and Hamiltonian Field Theory

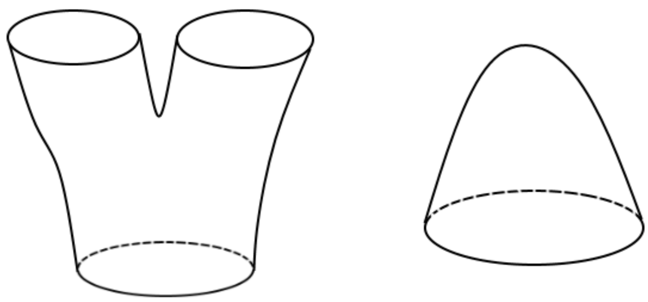
In 1.3, we have introduced a derived geometric formulation of Lagrangian field theory that fits together very nicely with the formalism of factorization algebras. To repeat the most important points: Given a spacetime manifold M, a space of (off-shell) field histories $\mathcal{F}' = [\mathcal{F}/\mathcal{G}]$ where we form a stacky quotient with respect to gauge symmetries, and an action functional $S : \mathcal{F}' \to \mathbb{R}$, we constructed a derived stack $X := \mathrm{dCrit}(S)$ and called it the *derived covariant phase space* of our theory. Given a fixed background field configuration $\phi \in X$, we wrote $\mathcal{E} := \mathbb{T}_\phi X$ for the BV-BRST complex, with $\mathcal{E}[-1]$ an L_∞-algebra that describes perturbative field theory around ϕ.

All of this is generally covariant – we did not need to fix a time direction or a Cauchy surface, not even a metric on spacetime. However, many physical concepts like states and time evolution, and most crucially our very perception of reality, rely on the concepts of a beginning and an end. In other words, we should try to incorporate the formalism of *Hamiltonian field theory* into this derived geometric description, since the Hamiltonian phase space captures precisely this notion of "physics at a time slice".

A skeptical reader might point out the fact that we mostly deal with topological field theories here, which are not actually dynamical. While it seems that this would make a time evolution picture redundant, the opposite is actually the case, and this becomes immediately clear when looking at Atiyah's axiomatic description:

Definition 5.1.1. Fix a symmetric monoidal 1-category $\mathcal{V}^{\otimes}$ (usually the category of chain complexes, vector spaces, or Hilbert spaces) and let $\mathrm{Bord}_d^{\sqcup}$ be a category where closed $(d-1)$-manifolds are objects, and morphisms from M to N are bordisms, i.e. compact d-manifolds W with boundary $\partial W = -M \sqcup N$, equipped with the symmetric monoidal structure given by disjoint union. A d-dimensional topological field theory, in the sense of Atiyah, is a symmetric monoidal functor $Z : \mathrm{Bord}_d^{\sqcup} \to \mathcal{V}^{\otimes}$.

Figure 5.1.: Examples of bordisms



The philosophy is that for every such bordism, we have a space of incoming states $Z(M)$, a space of outgoing states $Z(N)$ and a linear map $Z(W) : Z(M) \to Z(N)$ that acts as a time translation operator. This captures the essence of the Hamiltonian formalism!

Because this concept is so elegant and powerful, a similar description should be available to non-topological field theories as well. For those, we must distinguish:

- The (Hilbert) space of states of the theory, which depends on a choice of time-slice. For free field theories, this usually is a Fock space.
- The covariant phase space, i.e. the space of field theories that satisfy the Euler-Lagrange equations of motion.

By a system of time-slices, we mean a foliation by submanifolds in the Euclidean case and a foliation by Cauchy surfaces in the Minkowski case. In the simplest case where $M = N \times \mathbb{R}$, path integral quantization amounts to associating, for any two times $t_0, t_1 \in \mathbb{R}$ and two states $|\psi_0\rangle$ on $N \times \{t_0\}$ and $|\psi_1\rangle$ on $N \times \{t_1\}$, a transition amplitude $\langle\psi_1|U(t_0,t_1)|\psi_0\rangle$. We can understand the state $|\psi_0\rangle$ as associating an amplitude to any classical field configuration at time t_0, and similarly for $|\psi_1\rangle$. To understand how the above transition amplitude is calculated, we use the following diagram:

$$
\begin{tikzcd}
 & \text{field histories on } N \times [t_0, t_1] \arrow[dl] \arrow[dr] \arrow[loop above, "- \cdot e^{i\int_{t_0}^{t_1} L dt}"] & \\
\text{states on } N \times \{t_0\} & & \text{states on } N \times \{t_1\}
\end{tikzcd}
$$

Diagrams of this form are often called *spans* or *correspondences*. The arrows are given by restricting a field history to (its germs at) the fixed times t_0 and t_1. We can understand the path integral as a push-pull transformation, acting by:

- Precomposing $|\psi_0\rangle$ with the left arrow, pulling it back to an amplitude defined on all field histories
- Multiplying this amplitude by the weight factor e^{iS}
- Pushing the result forward to obtain an amplitude $U(t_0, t_1)\,|\psi_0\rangle$, using a sort of infinite-dimensional fiber integration.

This resembles the notion of a Fourier-Mukai transformation in algebraic geometry, generally push-pulls along spans are called *integral transformations* (for example, the usual Fourier transformation can also be written in this way, with weight/ integration kernel given by e^{ikx}).

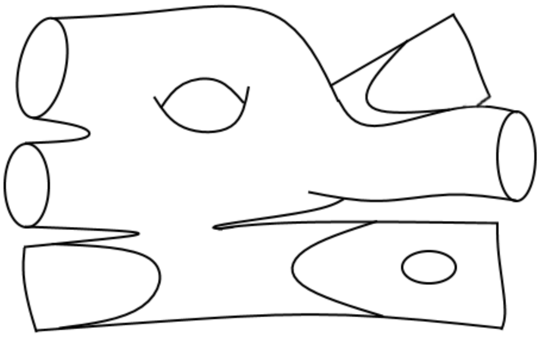
We will see, using language from derived symplectic geometry and Hermitian K-Theory, that (perturbative) classical field theories on manifolds with boundaries are generally given by such a span; and we will study the induced structure on factorization algebras.

5.1.2. Extended Topological Field Theories

A particular class of topological field theories that has been studied intensely is the family of *topological string theories*. One of their main features is that they not only have a state space for every closed 1-manifold, like a 2-dimensional topological field theory, but also for 1-manifolds with boundary. Since the functor Z associating to a manifold its space of states is monoidal, it is enough to specify its value on a circle, which is called the space of *closed strings*, and on intervals corresponding to *open strings*. In fact, one usually allows for the ends of open strings to be labeled by the D-branes they are restricted to, so that one can construct an enriched *category of branes* with objects D-branes and morphism spaces being the corresponding (Fock) spaces of open string states.

To axiomatize situations like this, Costello introduced definitions of open, closed, and open-closed *Topological Conformal Field Theories* (TCFTs) in [Cos06] as functors from

Figure 5.2.: Example of a 2-dimensional extended bordism



certain dg-categories $\mathcal{O}, \mathcal{C}, \mathcal{OC}$ of bordisms between open, closed and open-closed strings into the dg-category $\mathrm{Ch}(\mathbb{C})$ of chain complexes. Of course, there are canonical inclusions $\mathcal{C} \hookrightarrow \mathcal{OC} \hookleftarrow \mathcal{O}$, inducing restrictions between the different types of TCFTs. Costello shows that:

- Given an open TCFT $Z : \mathcal{O} \to \mathrm{Ch}(\mathbb{C})$, we can construct a universal associated closed TCFT as a homotopy Kan extension $Z_{cl} := \left(\mathrm{Lan}_{\mathcal{O}}^{\mathcal{OC}} Z\right)\big|_{\mathcal{C}}$.
- An open TCFT can equivalently be described by its category of branes, which is a Calabi-Yau A_∞-category. Similarly, it is a well-known fact that a closed TCFT (as a special case of 2-dimensional topological field theories) is described by the closed string space of states, a Frobenius algebra.
- The Frobenius algebra describing Z_{cl} is the Hochschild-Homology of the Calabi-Yau A_∞-category of branes.

These statements suggest a close connection to factorization algebras, where homotopy Kan extensions are used to calculate factorization homology and Hochschild Homology calculates factorization homology on S^1, corresponding to an incoming/ outgoing closed string in the bordism. We will not make this connection more explicit, since the concept of a TCFT is actually a bit dated: In his paper [Lur09], Lurie has sketched how to formalize a beautiful generalization of this construction to arbitrary dimensions. It relies on the theory of (∞, d)-categories, which we briefly touch upon in A.1 – an informal understanding is more than enough for our purposes.

Definition 5.1.2. The (∞, d)-category $\boldsymbol{Bord_d}$ of d-dimensional bordisms consists of:

- objects being 0-dimensional manifolds
- morphisms being 1-dimensional bordisms between the respective 0-manifolds
- 2-morphisms being 2-dimensional bordisms between bordisms
- ...
- d-morphisms being d-dimensional bordisms between bordisms between...

- $(d+1)$-morphisms being diffeomorphisms between the resulting manifolds with corners, restricting to the identity on the boundary
- $(d+2)$-morphisms being isotopies between those diffeomorphisms
- $(d+3)$-morphisms being isotopies between isotopies, and so on...

Note that this should indeed by an (∞, d)-category, since $(d+1)$, $(d+2)$-morphisms etc. are invertible. It is equipped with a symmetric monoidal structure given by disjoint union.

Definition 5.1.3. Fix a symmetric monoidal (∞, d)-category $\boldsymbol{\mathcal{V}}^{\otimes}$ with duals. A *fully extended topological field theory* is a symmetric monoidal functor $Z : \boldsymbol{Bord_d}^{\sqcup} \to \boldsymbol{\mathcal{V}}^{\otimes}$. More generally, one can define partially extended field theories that only associate values in $\boldsymbol{\mathcal{V}}$ to manifolds and bordisms of dimension $d-r, \ldots, d$ for $0 \leq r \leq d$.

Remark. A precise formulation of the above definitions is of course very difficult, see [Lur09] and [Sch14] for more. We have not explained what "having duals" means, and will not explain many terms in the next statement.

Theorem 5.1.4 (Cobordism Hypothesis, [Lur09])**.** There is a canonical equivalence of (∞, d)-categories

$$\mathrm{Fun}^{\otimes}(\boldsymbol{Bord_d}^{\sqcup}, \boldsymbol{\mathcal{V}}^{\otimes}) \simeq (\boldsymbol{\mathcal{V}}^{fd})^{h\,\mathrm{O}(d)} \tag{5.1}$$

between fully extended topological field theories and homotopy invariants in the (∞, d)-category of fully dualizable objects in $\boldsymbol{\mathcal{V}}$ with respect to the canonical $\mathrm{O}(d)$ action on it. If we restrict to framed bordisms, we obtain precisely $\boldsymbol{\mathcal{V}}^{fd}$ on the right.

Remark. In particular, this can be seen as a universal property that uniquely characterizes the (∞, d)-category of (framed) bordisms. One could say that (∞, d)-categories know about bordism theory, e.g. one can calculate the Thom spectrum MTSO as the geometric realization of the oriented bordism category:

$$\Omega_{\infty}\Sigma^{n}\,\mathrm{MTSO}(d) \simeq |\boldsymbol{Bord_d^{or}}| \tag{5.2}$$

This result by Galatius-Madsen-Tillmann-Weiss (see [Lur09, 2.5.7]) also bears striking similarity to the theory of Quinn spectra.

Example 5.1.5. A TCFT is essentially the same thing as a fully extended topological field theory of dimension 2 with values in chain complexes, and the cobordism hypothesis captures the fact that a TCFT is completely determined by the associated Calabi-Yau A_{∞}-category. In fact, one can show that this datum (together with a choice of splitting on the Hodge filtration of its periodic cyclic homology) is enough to calculate for example Gromov-Witten invariants in the topological B-model, see [CT20].

In particular, the definition of an extended topological field theory proclaims that such a theory associates an element of $\mathcal{V}$ to every d-manifold with corners. This is exploited in the proof of the cobordism hypothesis, that very roughly proceeds by triangulating arbitrary manifolds with corners by a simplicial complex and hence gluing the value of an extended field theory from its value on simplices, which are manifolds with corners themselves. We will mimic this idea in 5.7.

As an upshot, we conclude that extended field theories somehow generalize the philosophy of Hamiltonian field theory in the sense that instead of only cutting in the time direction, we are allowed to cut in every direction (e.g. for topological strings, we can cut along their spacial direction). This strong locality principle is powerful enough to allow for a full classification of possible field theories, by the cobordism hypothesis.

5.1.3. Types of Boundary Theories

This idea of viewing spacial and temporal boundaries on the same footing is a striking example of general covariance. However it a priori leads to some cognitive dissonance – on spacial boundaries, we often want to impose boundary conditions like Dirichlet- and Neumann-conditions on the bosonic string, but what is the interpretation of this on temporal boundaries? The answer: Temporal boundary conditions are *polarizations*, i.e. possible conventions to distinguish creation from annihilation operators. We are lead to distinguish distinct types, or philosophies, of field theories on a manifold M with boundary:

- **Atiyah-Segal-type/ Free boundary condition:** We do not impose any boundary conditions at all and do not fix any polarization, so that the theory on M does not become a well-defined Lagrangian field theory. Rather, we regard it as an individual patch, that we can glue together with other patches along the boundary to construct a well-defined theory on a manifold *without* boundary that we have cut into pieces.
- **Fixed boundary condition:** The boundary field theory that lives on ∂M induces a boundary condition, or polarization, on the bulk field theory on M, and only after restricting to the field histories satisfying this condition do we obtain a well-defined field theory on M. This is essentially the spirit of the BV-BFV theories in [CMR14].
- **Defects:** The boundary introduces an a priori arbitrary source term into the bulk field theory. In particular, we do not need to define an actual field theory on the boundary.

All of these concepts make sense not only for manifolds with boundary, but for manifolds with corners or embedded submanifolds as well. Even further, we argue that all of them are of immediate physical interest, meaning that we need to extend the definition of a factorization algebra to these more general kinds of spaces.

What we will see is that the interaction between bulk and boundary occurs, ideally, at a small neighborhood of the boundary looking like $\mathbb{R}^{d-1} \times \mathbb{R}_{\geq 0}$. Compare this to the neighborhood of a point inside the manifold, which always looks like $\mathbb{R}^d$. We might say that every interior point of a manifold with boundary is connected to the rest of the manifold via a small sphere S^{d-1} around it, called the *link* of the point, while any point of the boundary has link $D^1 \times S^{d-2}$. For a manifold with corner, the links will look like $S^{d-k-1} \times D^k$, as is easy to imagine.

What if we allow for more general local structures? For example, let us look at a double cone, like $\{x^2 + y^2 = z^2\} \subset \mathbb{R}^3$. The link of the singularity in the center is $S^1 \sqcup S^1$, and this in some way classifies the type of singularity that occurs there. Physical intuition would again tell us that if we have a physical theory living on the double cone (which we regard as a smooth manifold, forgetting about the singular point), and introduce a defect at the singularity, then the way it interacts with the theory would depend on a small neighborhood around the singularity that looks like an open cone of the link.

When calculating the factorization product of two physical observables, the algebraic properties of this product are determined by the space of different directions from which one of the corresponding operators can be shifted "into" the other, namely the link of the point where we calculate this factorization product. We had formulated this very philosophy for the module actions encoded in constructible factorization algebras in 4.5.9, giving us a first reason to assume that those are the right mathematical objects to describe such spaces of observables.

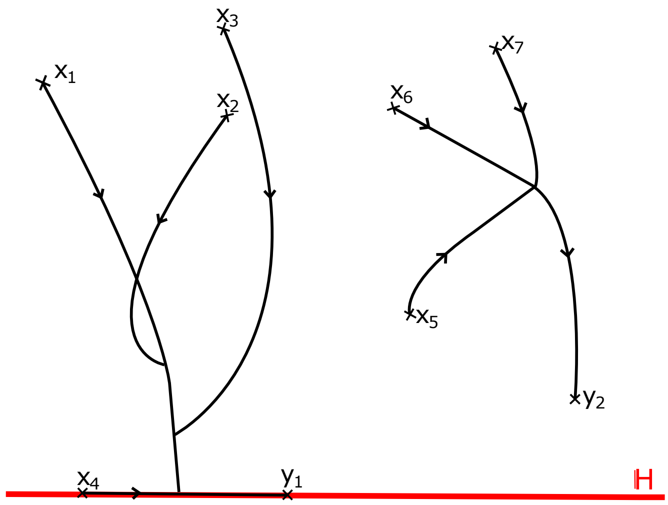
Unfortunately, there are view interesting field theories on general stratified spaces (that are not manifolds with corners). Some examples however arise in string theory: The open (topological) A-model on the conifold $\{(z_1, z_2, z_3, z_4 \subseteq \mathbb{C}^4 | \sum z_i^2 = 0)\}$ can be described both by the analogous model on the deformed non-singular space where $\sum z_i^2 = a^2$ with $a \in \mathbb{C}\backslash\{0\}$, or on the blow-up (the composition of this equivalences is known as *conifold transition*). Still, understanding the theory on the singular space itself might lead to new insights, for some ideas in this direction see [Ban10].

5.1.4. Observables and Constructible Sheaves

Finally, let us follow the intuition from 2.5 to find out how to define a BV theory on a stratified space. As a reminder, we had remarked that locally constant factorization algebras on a manifold M have *monodromy along multipaths*: Given a finite set of starting points $(x_i)_{1 \leq i \leq n}$ in M, and a finite set of end points $(y_j)_{1 \leq j \leq m}$ with $m \leq n$, that are connected by a finite set of continuous paths which are allowed to join together, but not split apart, we can transport local operators at the starting points to local operators at the end points by taking their product at the places where paths join. For example, this is how the operator product in a conformal field theory is recovered from the respective factorization algebra.

But what about a boundary CFT on a manifold with boundary M? Here, there are some operators that we can insert in the interior/ bulk, and some that live on the boundary. We can compute the operator product of a bulk with a boundary operator by moving the former closer and closer to the boundary; however we can not move a boundary operator off the boundary (to be precise, a boundary operator lives at a small half-disk at the boundary, which contains a disk in the interior where bulk operators live). This means that the factorization algebra of observables should also have monodromy with respect to *enter-multipaths*, multipaths that only move downwards in the stratification poset as in the picture – note that the boundary is sent to 0 by the canonical stratification on a manifold with boundary, while to interior is sent to 1.

Figure 5.3.: Enter-multipath in the upper half-plane



Definition 5.1.6 ([AFT14b, 3.7.1, 3.7.4])**.** Given a connected C^0-stratified space $(M \to P)$, we define its *Ran space* $\mathrm{Ran}(M)$ as the space of nonempty finite subsets $S \subseteq M$, equipped with a similar topology as in 2.5. It is C^0-stratified over the poset $P^{\mathbb{N}_0}$ of maps $P \to \mathbb{N}_0$ by counting the number of selected points in each stratum; we say that $c \leq c'$ in this poset iff the set $\{p \in P | c(p) > c'(p)\}$ contains no maxima of P.

Remark. This stratification poset is to coarse for out purposes; from the discussion in [AFT14a] one can however follow that $\mathrm{Ran}(M)$ is even stratified by the set of finite tuples in P. The partial order on it is generated by the classes of splittings $(p_1, \ldots, p_i, \ldots, p_r) \leq (p_1, \ldots, p_i, p_i, \ldots, p_r)$ and exits $(p_1, \ldots, p_i, \ldots, p_r) \leq (p_1, \ldots, p_j, \ldots, p_r)$ for $p_i \leq p_j$.

Saying that factorization algebras of boundary CFTs (and, analogously, topological field theories on manifolds with boundary) should have monodromy with respect to multipaths where all indivual segments are enter-paths is then, up to the same subtleties regarding the definition of the exit-path category we faced in 2.5, equivalent to saying that every such algebra should induce a functor $\left(\mathrm{Sing}^{\mathbb{N}_{>0}} \mathrm{Ran}(M \to P)\right)^{op} \to \mathcal{V}$, which is (using the exodromy correspondence) a constructible (hyper-)sheaf on the Ran space. As in the manifold case, the exit path category of the Ran space can be identified with the category of disks $\mathcal{D}isk^{surj}_{/M}$ with maps that are surjective on connected components, so that (adding a factorization condition, and units) we recover our definition of a constructible factorization algebras.

But how should we recover such a complicated object from an actual physical theory? As for the manifold case, we would expect it to be the Chevalley-Eilenberg algebra of a BV-complex that itself is a sheaf on M, i.e. ignoring its L_∞-structure, $\mathcal{O}bs^{cl} = \mathrm{Sym}\,\mathcal{E}^\vee$. In particular, the space of linear observables is given by $\mathcal{E}^\vee$, and by taking the symmetric algebra, we obtain polynomial observables.

Of course, if we multiply two linear observables, we obtain a quadratic observable – therefore, they should not have monodromy along multipaths (since two paths joining would yield the product in the symmetric algebra, as this is the factorization product). However, they should still have monodromy along enter-paths $\mathrm{Sing}^{[1]}(M) \to \mathcal{V}^{op}$, where we can freely switch the *op* from the source to the target. Linear observables in $\mathcal{E}^\vee$, using exodromy, should therefore form a $\mathcal{V}^{op}$-valued constructible sheaf on M, i.e. a constructible cosheaf. Consequently,

The BV complex of a free topological field theory on a C^0-stratified space $(M \to P)$ should be a constructible ∞-sheaf (whose stalks satisfy a finiteness condition), together with a sheafy kind of (-1)-shifted symplectic structure we sketch in 5.8.

Warning. One idea to define a symplectic structure on a constructible sheaf F is to equip it with an isomorphism to a shift of its Verdier dual sheaf. While this is useful on (pseudo-)manifolds, we will see that this is generally the *wrong* definition when it comes to boundaries and corners.

5.2. Shifted Symplectic Structures and Lagrangians

As a motivation, we remind the reader of the following definitions, for V a finite-dimensional $\mathbb{R}$-vector space:

Definition 5.2.1. A bilinear form $b : V \otimes V \to \mathbb{R}$ is called *non-degenerate* if for any $v \in V$, such that $b(v, -) \equiv 0$ identically, we must have $v = 0$ already. Equivalently, the induced linear map $b : V \to V^*$ is an isomorphism (or monomorphism or epimorphism, since the dimensions agree).

Definition 5.2.2. A non-degenerate antisymmetric bilinear form $\omega : \bigwedge^2 V \to \mathbb{R}$ is called *symplectic form*.

Definition 5.2.3. A non-degenerate symmetric bilinear form $b : S^2V \to R$ is called an *inner product*.

We want to generalize these definitions until they can capture the symplectic structure of the BV-BRST complex. As a first step, let us replace vector spaces by chain complexes of R-modules, for any ring R with $R = \mathbb{R}$ of primary interest to us. We need a notion of finite-dimensionality:

Definition 5.2.4. Denote by $D^{\mathrm{fp}}(R)$ the derived ∞-category of *finitely presented* chain complexes, which is the full (stable) subcategory of the derived ∞-category (see A.3.14) $D(R)$ generated by $R[0]$ under shifts, direct sums and (co-)fibers. Similarly, the derived ∞-category of *perfect* chain complexes $D^{\mathrm{perf}}(R)$ is its its closure under direct summands. To be explicit, $D^{\mathrm{fp}}(R)$ consists of bounded complexes of finitely free R-modules, and $D^{\mathrm{perf}}(R)$ of bounded complexes of finitely generated projective R-modules.

Remark. The BV-BRST complex of a finite-dimensional theory should lie in $D^{\mathrm{perf}}(\mathbb{R})$ because it is an object of derived geometry. However since projective $\mathbb{R}$-modules are free, $D^{\mathrm{fp}}(\mathbb{R}) \cong D^{\mathrm{perf}}(\mathbb{R})$, and since the abelian category of $\mathbb{R}$-vector spaces is semisimple, quasi-isomorphisms and homotopy equivalences agree so $D(\mathbb{R}) \simeq \mathcal{C}h(\mathbb{R})$. The latter does however not hold for CVS or DVS. In the following, while we work with $R = \mathbb{R}$, our results still hold for arbitrary R and in even more general situations that include these functional analytic contexts, see the end of this chapter.

Lemma 5.2.5. For $P \in D^{\mathrm{perf}}(\mathbb{R})$, the dual chain complex $P^\vee := \underline{\mathrm{Hom}}(P, \mathbb{R})$ is also in $D^{\mathrm{perf}}(\mathbb{R})$ and there is a canonical isomorphism $P \cong P^{\vee\vee}$.

Definition 5.2.6. A chain complex $P \in D^{\mathrm{perf}}(\mathbb{R})$ (or more generally, over any ring R) is called an *n-dimensional Poincaré complex* if it is equipped with an *inner product of degree n*, which is a symmetric map of chain complexes

$$\omega : P \otimes P \to \mathbb{R}[-n] \tag{5.3}$$

such that the associated chain map $P \to P^\vee[-n]$ by the tensor product/ internal Hom adjunction is a quasi-isomorphism. By symmetric, we mean that ω factors through the invariants $(P \otimes P)^{S_2}$ of the S_2-action on $P \otimes P$ that interchanges the arguments. More generally, we can define *Poincaré objects* in every stable ∞-category with a duality functor $(-)^\vee$ as defined at the end of this section.

Remark. If we replace $\mathbb{R}$ by an arbitrary ring, we need to set $P^\vee = \mathrm{RHom}(P, \mathbb{R})$, and use $\otimes^L$ and homotopy invariants since we work in the derived ∞-category. Over $\mathbb{R}$, the considered functors are exact and need not be derived.

Example 5.2.7 (Poincaré-Duality)**.** For M a closed oriented topological n-manifold, the cap product with the fundamental class induces a quasi-isomorphism

$$C^*(M; R) \cong C^*(M; R)^\vee[-n] \tag{5.4}$$

exhibiting the singular cochain complex of M with values in R as an n-dimensional Poincaré complex. Note that it indeed lives in the perfect derived category since closed manifolds have finite-dimensional homology.

Example 5.2.8. Given a suitably well-behaved derived ∞-category $\mathcal{D}$ of good topological vector spaces, containing in particular sections of differential complexes and admitting a duality functor $(-)^\vee$ that agrees with strong duality in this case (compare the end of this section), Poincaré duality again tells us (see [Cal21, 1.13]) that for a closed oriented smooth n-manifold M, the complex of differential forms $\Omega^*(M)$ is an n-dimensional Poincaré complex in $\mathcal{D}$.

Technical Remark. Just as Sylvester's theorem gives a classification of non-degenerate symmetric forms on $\mathbb{R}^n$, algebraic L-theory classifies Poincaré objects in a stable ∞-category with duality functor, or more generally in a *Poincaré ∞-category* [CDH+20a].

Proposition 5.2.9. An inner product on a finite-dimensional vector space V is the same thing as a pairing exhibiting $V[0]$ as a (0-dimensional) Poincaré object. Similarly, a symplectic structure on V is the same thing as a pairing exhibiting the complex $V[-1]$ concentrated in degree 1 as a 2-dimensional Poincaré object.

Proof. For the first claim, we note that such a pairing $V[0] \otimes V[0] \to \mathbb{R}[0]$ is precisely a symmetric bilinear map of vector spaces, and the condition that $V[0] \cong V[0]^\vee$ is a quasi-isomorphism implies on 0th homology groups that this map is non-degenerate, and conversely.

In the symplectic case, a non-degenerate symmetric pairing of degree 2 on $V[-1]$ is a symmetric bilinear map $V[-1] \otimes V[-1] \to \mathbb{R}[-2]$ that induces a quasi-isomorphism

$$V[-1] \simeq \mathbb{R}[-2] \otimes V^*[1] = V^*[-1] \tag{5.5}$$

Shifting by 1, this is an isomorphism $V \cong V^*$ corresponding to a non-degenerate bilinear map $\omega : V \otimes V \to \mathbb{R}$. Because of the Koszul sign rule, this map has to antisymmetric so that the shifted map on $V[-1] \otimes V[-1]$ is symmetric. □

Definition 5.2.10. An *n-shifted symplectic structure* on $P \in D^{\mathrm{perf}}(\mathbb{R})$ is a symmetric pairing exhibiting $P[-1]$ as an $(2-n)$-dimensional Poincaré complex.

Example 5.2.11. A symplectic structure on a finite-dimensional vector space V is precisely a 0-shifted symplectic structure on $V[0]$.

Example 5.2.12. For M a closed oriented topological 3-manifold, the shifted singular cochain complex $C^*(M; R)[1]$ admits a canonical $(2-n)$-shifted symplectic structure – similarly, up to functional analytic subtleties, for $\Omega^*(M)[1]$. In particular, the BV-BRST complex $\Omega^*(M)[1]$ for abelian Chern-Simons theory on a smooth oriented 3-manifold admits a (-1)-shifted symplectic structure.

Definition 5.2.13 ([CDH+20b, Chapter 2])**.** Let P be an n-dimensional Poincaré-complex with isomorphism $\omega : P \cong P^\vee[-n]$. A *Lagrangian* of it is a complex $L \in D^{\mathrm{perf}}(\mathbb{R})$ equipped with a chain map $f : L \to P$ such that the composition

$$L \xrightarrow{f} P \cong P^\vee[-n] \xrightarrow{f^\vee[-n]} L^\vee[-n] \tag{5.6}$$

is a distinguished triangle (or, in the nomenclature of stable ∞-categories, a fiber sequence) in $D^{\mathrm{perf}}(\mathbb{R})$. In fact, we have to consider a 2-morphism $\eta : \omega \circ f \simeq 0$ in the space of symmetric pairings as part of the datum. This means that we can identify $L^\vee[-n]$ with the mapping cone of f, in a way that makes the respective distinguished triangles isomorphic.

More generally, a *Lagrangian correspondence* between n-dimensional Poincaré complexes (P, ω) and (P', ω') is a diagram of the form

$$\begin{tikzcd} & L \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ P & & P' \end{tikzcd}$$

together with an isomorphism $\eta : \omega \circ f \to \omega' \circ f'$ inside the space of symmetric forms $\mathrm{Map}_{D^{\mathrm{perf}}(\mathbb{R})}(L \otimes L, \mathbb{R}[-n])^{hS_2}$, that induces (together with ω, ω') a map

$$L \to P^\vee[-n] \times_{L^\vee[-n]} P'^\vee[-n] \cong (P \amalg_L P')^\vee [-n] \tag{5.7}$$

that is required to by an isomorphism. Equivalently, the composite map

$$\begin{aligned} &\mathrm{fib}(L \to P) \cong \mathrm{fib}(P' \to P \amalg_L P') \longrightarrow \\ &\qquad \longrightarrow \mathrm{fib}(P' \to L^\vee[-n]) \cong \mathrm{fib}((L \to P')^\vee[-n]) \cong (\mathrm{cofib}\, L \to P')^\vee[-n] \end{aligned}$$

has to be an isomorphism.

This might seem very technical since we have to keep track of the explicit isomorphisms of P, P' and their duals, but we do not exactly have a lot of wiggle-room translating these classical definitions for symplectic vector spaces to the chain complex setting. A few examples should be helpful:

Example 5.2.14.

- A Lagrangian correspondence L from $P \cong P^\vee[-n]$ to the zero Poincaré object $0 \cong 0^\vee[-n]$ is a Lagrangian of P, since the above condition just reduces to $L \cong (P \amalg_L 0)^\vee[-n]$.
- A Lagrangian correspondence $0 \leftarrow L \to 0$ where we regard 0 as an n-dimensional Poincaré object, i.e. a Lagrangian of the zero object, is the same thing as an isomorphism $L \cong 0 \times_{L^\vee[-n]} 0 = \Omega L^\vee[-n] = L^\vee[-n-1]$, exhibiting L as an $(n+1)$-dimensional Poincaré object.

- If V is a vector space with inner product ω (in other words $V[0]$ is 0-dimensional Poincaré), and L a vector space such that $L[0] \to V[0]$ is Lagrangian, then the map $L \to V$ is injective since the sequence $0 \to L \to V \cong V^* \to L^* \to 0$ has to be exact, meaning that L must be a Lagrangian subspace of V. In particular, the signature of ω vanishes.

- If V is a symplectic vector space, i.e. $V[-1]$ is a 2-dimensional Poincaré object, then for a vector space L a map $L[-1] \to V[-1]$ is Lagrangian iff it witnesses L as a Lagrangian subspace of V, in particular it has to be injective.

- Given a compact orientable n-manifold M with boundary, the cap product with the relative fundamental class of M exhibits the restriction map

$$i^* : C^*(\partial M; R) \to C^*(M; R) \tag{5.8}$$

as Lagrangian of the $(n-1)$-dimensional Poincaré complex $C^*(\partial M, R)$. Similarly for differential forms, up to functional analytic subtleties. Taking (co-)homology groups of the defining equation

$$C^*(M, \partial M; R) := \mathrm{fib}(i^*) \cong C^*(M; R)^\vee[-n] = \mathrm{Hom}(C^{n-*}(M; R), R) \tag{5.9}$$

yields Poincaré-Lefschetz duality on a compact oriented manifold:

$$H^*(M, \partial M; R) \cong H_{n-*}(M; R) \tag{5.10}$$

Also, we can derive a theorem of Thom, stating that for $\dim(\partial M) = 4k$, the signature of ∂M must vanish: One can show (using e.g. algebraic surgery) that our statement implies that $i^* : H^{2k}(M; R) \to H^{2k}(\partial M; R)$ is also a Lagrangian subspace with respect to the intersection pairing on the right hand side, so the signature of this pairing vanishes.

- If W is a compact oriented $(n+1)$-manifold with $\partial W = M \sqcup -N$, i.e. a bordism between closed oriented manifolds M and N, then the restriction maps

$$C^*(M; R) \leftarrow C^*(W; R) \to C^*(N, R) \tag{5.11}$$

form a Lagrangian correspondence. Similar for differential forms.

Theorem 5.2.15. Let P, P', P'' be n-dimensional Poincaré complexes and $P \leftarrow L \to P'$ as well as $P' \leftarrow L' \to P''$ Lagrangian correspondences. Then, the span $P \leftarrow L \times_{P'} L' \to P''$ induced by the diagram

$$\begin{tikzcd}
& & L \times_{P'} L' \arrow[dl] \arrow[dr] & & \\
& L \arrow[dl, "f"'] \arrow[dr, "f'"] & & L' \arrow[dl, "g"'] \arrow[dr, "g'"] & \\
P & & P' & & P''
\end{tikzcd}$$

is also a Lagrangian correspondence.

Proof. We know that $P \cong P^\vee[-n]$ and similarly for P' and P'', and L, L' being Lagrangian correspondences amounts to isomorphisms

$$\begin{aligned} L \cong & P^\vee[-n] \times_{L^\vee[-n]} P'^\vee[-n] \, , \\ L' \cong & P'^\vee[-n] \times_{L'^\vee[-n]} P''^\vee[-n] \, . \end{aligned}$$

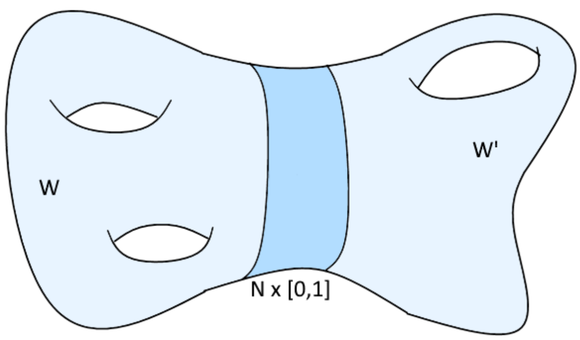
Consequently we can use the Pasting Lemma to dualize and extend above commutative diagram to

$$\begin{tikzcd} (L \times_{P'} L')^\vee[-n] & L^\vee[-n] \arrow[l] & P^\vee[-n] \arrow[l] \\ L'^\vee[-n] \arrow[u] & P'^\vee[-n] \arrow[l] \arrow[u] & L \arrow[l] \arrow[u] \\ P''^\vee[-n] \arrow[u] & L' \arrow[l] \arrow[u] & L \times_{P'^\vee[-n]} L' \arrow[l] \arrow[u] \end{tikzcd}$$

where every square is a pullback (and by stability also a pushout). This implies, using $P' \cong P'^\vee[-n]$ and pasting, that $L \times_{P'} L' \cong P^\vee[-n] \times_{(L \times_{P'} L')^\vee[-n]} P''^\vee[-n]$. Compatibility of the required witnessing 2-morphisms can be verified quickly. $\square$

Remark. Compare this with the observation that a bordism W between closed oriented manifolds M and M', and a bordism W' from M' to M'', can be glued along a collar to a bordism from M to M''. In particular, if P is an n-dimensional Poincaré complex with two distinct Lagrangians $L \to P$ and $L' \to P$, the pullback $L \times_P L'$ is a Lagrangian correspondence from 0 to 0, i.e. an $(n+1)$-dimensional Poincaré complex; just like two null-bordisms of an n-manifold can be glued to an $(n+1)$-manifold as in the picture or the proof of 1.9.

Figure 5.4.: Gluing two null-bordisms W, W' of N to a closed manifold



Example 5.2.16. We have defined covariant phase space X of a field theory with space of off-shell fields $\mathcal{F}$ and action $S : \mathcal{F} \to \mathbb{R}$ as a derived intersection of the graph of dS and the zero section in $T^*\mathcal{F} \to \mathcal{F}$. It turns out that $T^*\mathcal{F}$, as a derived stack, possesses a tautological symplectic structure (constructed similarly as for smooth manifolds) such that the the graph of any closed section is a Lagrangian substack. We haven't introduced these terms, but it is clear what they mean on tangent complexes: $\mathbb{T}T\mathcal{F}$ pointwise admits a 0-shifted symplectic structure such that the tangent complexes of the graphs are Lagrangian. We have already claimed in 1.4.13 that the tangent complex plays well with pullbacks, so for a fixed field configuration ϕ in X,

$$\mathbb{T}_\phi X = \mathbb{T}_\phi \operatorname{Graph}(dS) \times_{\mathbb{T}_\phi T^*\mathcal{F}} \mathbb{T}_\phi \mathcal{F} \tag{5.12}$$

which means by above theorem that the BV-BRST complex $\mathcal{E} = \mathbb{T}_\phi X$ possesses a natural (-1)-shifted symplectic structure, as expected.

Let us generalize another statement from bordism theory that will be useful later. Given a bordism W from M to M itself, we can glue W to itself along the boundary components to obtain a closed manifold. For example for $W = S^1 \times [0,1]$ the cylinder, the components of $\partial W = S^1 \sqcup -S^1$ can be glued to obtain a torus.

Proposition 5.2.17. Given an n-dimensional Poincaré complex (P, ω) and a Lagrangian correspondence $P \xleftarrow{f} L \xrightarrow{f'} P$, the equalizer

$$\operatorname{equ}(f, f' : L \to P) = \operatorname{fib}(f - f') \tag{5.13}$$

is an $(n+1)$-dimensional Poincaré complex.

Proof. Let P be any chain complex, then there are natural diagonal and codiagonal chain maps $\Delta : P \to P \oplus P$ and $\nabla : P \oplus P \to P$, and for chain maps $f, f' : L \to P$ the composition

$$L \xrightarrow{\Delta} L \oplus L \xrightarrow{f \oplus f'} P \oplus P \xrightarrow{\nabla} P$$

agrees with the sum $f + g$.

Since we assume $P \xleftarrow{f} L \xrightarrow{f'} P$ is a Lagrangian correspondence, we have

$$\begin{aligned} \operatorname{fib}(f) &\simeq \operatorname{fib}(f')^\vee[-n-1] \\ \operatorname{fib}(f') &\simeq \operatorname{fib}(f)^\vee[-n-1] \end{aligned}$$

where the second equation follows from the first by applying $(-)^\vee[-n-1]$ to both sides. Now, we take the direct sum of these equations and take homotopy invariants with respect to the S_2-action that exchanges direct summands:

$$(\operatorname{fib}(f) \oplus \operatorname{fib}(f'))^{hS_2} \simeq (\operatorname{fib}(f') \oplus \operatorname{fib}(f))^{\vee hS_2}[-n-1]$$

We can write the fiber on the right side as a shift of a cofiber, and pull the homotopy (co-)invariants inside:

$$\mathrm{fib}\left((L \oplus L)^{hS_2} \overset{(f\oplus f')^{hS_2}}{\longrightarrow} (P \oplus P)^{hS_2}\right) \simeq \mathrm{cofib}\left((L \oplus L)_{hS_2} \overset{(f'\oplus f)_{hS_2}}{\longrightarrow} (P \oplus P)_{hS_2}\right)^{\vee}[-n]$$

Now, we use the statement from the beginning combined with the fact that the diagonal map induces an isomorphism $C \simeq (C \oplus C)^{hS_2}$, just as the codiagonal induces an isomorphism $(P \oplus P)_{hS_2} \simeq P$:

$$\mathrm{fib}(L \overset{f+f'}{\longrightarrow} P) \simeq \mathrm{cofib}(L \overset{f+f'}{\longrightarrow} P)^{\vee}[-n] \simeq \mathrm{fib}(f + f')^{\vee}[-n-1]$$

Up to exchanging f' and $-f'$ which gives isomorphic Poincaré complexes (and is due to the different orientations in the manifold case), we have succeed since we have a quasi-isomorphism that exhibits $\mathrm{fib}(f + f')$ as an $(n + 1)$-dimensional Poincaré complex. To be precise, we would still have to check that this is induced by a symmetric pairing; this is more technical but in principle works analogously. □

Remark. For an alternative proof this and similar statements compare 5.7.15.

We can generalize the definitions of this chapter to the functional analytic setting:

Definition 5.2.18 ([BY16, 2.15], [CG21, 4.2.0.2])**.** Let $\mathcal{E}$ be a local L_∞ algebra with underlying differential complex (E, D). A map between chain complexes of vector bundles $\omega : E \to E^![n-2]$ is called an *n-shifted symplectic structure* on $\mathcal{E}$ if it

- is an isomorphism on fibers,
- induces a symmetric pairing $\langle -, - \rangle : E \otimes E \to \mathrm{Dens}_M$,
- and is compatible with the higher Lie brackets on $\mathcal{E}$ in the sense that the expressions $\int_M \langle -, \ell_K(-, \ldots, -)\rangle$ appearing in the gauge-fixed action are graded antisymmetric.

In particular, this induces bilinear maps $\omega_U \in \mathcal{E}(U)^\vee \otimes \mathcal{E}^!(U)[n-2]$ where $\mathcal{E}(U)^\vee = \bar{\mathcal{E}}^!_c(U)$ is the strong dual, that are closed under the Chevalley-Eilenberg differential.

Definition 5.2.19 ([GRW20, 2.2])**.** A *Lagrangian subbundle* of a differential complex (E, D) equipped with an n-shifted symplectic structure is a differential subcomplex $(L, D) \subseteq (E, D)$ such that

- The total rank of L is half of the total rank of E,
- The shifted symplectic structure vanishes when restricted to L.

Up to functional analytic subtleties, for any open subset $U \subseteq M$, this makes the induced map $\mathcal{L}(U) \to \mathcal{E}(U)$ Lagrangian.

In the following, we will ignore these functional analytic considerations and pretend that we are in the world of finite-dimensional vector spaces. Since every definition in this chapter works in much greater generality, namely the language of *Poincaré ∞-categories* we mentioned above, we hope that a version of functional analysis containing spaces of sections of finite dimensional vector bundles can in the future be formalized in this way, making this caveat unsubstantial. Candidates for such a formalism could be (finitely generated) differential cohomology theories (i.e. sheaves in $\mathcal{Sh}(\mathrm{Man}, \mathcal{S}p)$) that are modules over the algebra object $C^\infty(M)$; and more generally spectrifications of certain structured spaces as in [Wal16]. They should be connected to the differentiable vector spaces we have studied by a sort of stable Dold-Kan equivalence.

Assumption: Let in the following $\mathcal{D}$ be a stable ∞-category that behaves like a version of $D^{\mathrm{perf}}(\mathbb{R})$ for topological vector spaces in the sense that

- It contains spaces of sections of vector bundles on manifolds as a full subcategory
- Its morphism spaces are $\mathbb{R}$-linear (to be mathematically precise, they should be module spectra over the Eilenberg-MacLane spectrum $H\mathbb{R}$),
- There is an antiequivalence $(-)^\vee : \mathcal{D}^{op} \to \mathcal{D}$ that is a *dualizing functor* in the sense of [CDH$^+$20a], which means that there is a natural isomorphism $\phi : (-)^{\vee\vee} \cong \mathrm{Id}_\mathcal{D}$ satisfying higher coherence conditions. To again be precise, ϕ must exhibit the pair $(\mathcal{V}, (-)^\vee)$ as a homotopy fixed point of the ∞-category $\mathcal{C}at_\infty^{ex}$ of stable ∞-categories with exact functors under the S_2-action that sends an ∞-category to its opposite. This is equivalent to requiring $\mathcal{D}$ to be a Poincaré ∞-category, since 2 is invertible in all morphism spaces.
- On the subcategory of spaces of sections of vector bundles, $(-)^\vee$ should be similar to the strong dual, i.e. taking continuous linear forms.

Also, mimicking $D^{\mathrm{perf}}(\mathbb{R}) \subseteq D(\mathbb{R})$, we expect $\mathcal{D}$ to be contained in a larger ∞-category $\hat{\mathcal{D}}$ that is presentable like $D(\mathrm{DVS})$, as a subcategory of objects satisfying a finiteness condition.

5.3. Topological Quantum Mechanics and Polarizations

We begin our study of field theories on manifolds with boundary with one of the simplest cases imaginable: A theory called *topological quantum mechanics.*

Fix a symplectic vector space V with Darboux basis $V = \langle a_1, \dots, a_n, a_1^\dagger, \dots, a_n^\dagger \rangle$. This means that the symplectic form $\omega : V \otimes V \to V$ is defined by $\omega(\alpha_i, a_j^\dagger) = -\omega(a_i^\dagger, a_j) = \delta_{ij}$ for $i, j = 1, \dots, n$ on basis elements. Also, we identify ω with the induced duality isomorphism $\omega : V \to V^*$. Finally, $L := \langle a_1, \dots, a_n \rangle \subseteq V$ and $L' := \langle a_1^\dagger, \dots, a_n^\dagger \rangle$ are

Lagrangian subspaces complementing each other in the sense that $L \oplus L' = V$. Note such a complement is not unique.

Given the 1-dimensional manifold with boundary $M = \mathbb{R}_{\geq 0}$, the space of fields for topological quantum mechanics is the space $\mathcal{F} = C^\infty(M, V) = \Omega^0(M) \otimes V$ of smooth V-valued functions on M. The action is given by

$$S_{TQM}[\phi] := \int_M \frac{1}{2}\omega(\phi \wedge d\phi) = \langle \phi, d\phi \rangle \ , \tag{5.14}$$

where $\omega(- \wedge -) := \omega \otimes \wedge$, and $\langle -, - \rangle := \int \omega(- \wedge -)$ is the tensor product of ω with the Kronecker/ integration pairing of differential forms. Let us vary this action, remembering that $\partial M \neq \emptyset$:

$$\delta S_{TQM} = \frac{1}{2}\int_M \omega(\delta\phi \wedge d\phi) + \omega(\phi \wedge d\delta\phi) = \frac{1}{2}\int_{\partial M} \omega(\phi, \delta\phi) + \int_M \omega(\delta\phi \wedge d\phi) \tag{5.15}$$

Iff the boundary contribution $\langle \phi, \delta\phi \rangle_\partial := \int_{\partial M} \omega(\phi, \delta\phi)$ vanishes, the fact that ω is non-degenerate yields the equations of motion $d\phi = 0$. Further, since the functional Hessian d of the action is non-degenerate on 0-forms, there is no local gauge symmetry and from 1.3.4 we can read off the BV-BRST complex:

$$\mathcal{E}(U) = \left(\Omega^0(U) \otimes V \xrightarrow{d \otimes \mathrm{id}_V} \Omega^1(U) \otimes V[-1]\right) = \Omega^*(U) \otimes V \tag{5.16}$$

This comes equipped with an almost (-1)-shifted symplectic structure given by $\langle -, - \rangle$, up to boundary terms.

However, vanishing of the boundary contribution requires fixing a boundary condition – a subspace of the covariant phase space (since our theory is free, this is the same thing as the BV-BRST complex) where the form $\langle -, - \rangle_\partial$ vanishes on fields restricted to the boundary. Note that since ∂M is just a point, the restriction $\phi|_{\partial M}$ can be identified with an element of V, and $\langle -, - \rangle_\partial = \omega : V \otimes V \to V$.

We say that $\mathcal{E}_\partial := V$ is the *boundary BV-complex* or *BFV-complex*, and ω is its associated 0-shifted symplectic form. Projection on the 0-component, followed by restriction, induces a map of chain complexes $r : \mathcal{E}(M) \to \mathcal{E}_\partial$. A *boundary condition* or *polarization* is, following the above discussion, determined by a subspace L of V that restrictions of physical ϕ must lie in, where $\omega|_{L \times L}$ vanishes – to obtain a well-defined physical theory, we will see that L must even be a Lagrangian subspace, like the one we have fixed above.

Remark. Technically, our boundary conditions could also have depended not only on $\phi(0)$, but also $\partial\phi(0)$ and higher derivatives, maybe even the whole germ of ϕ at $x = 0$. The reason why we do not consider this is that our action only contains first derivatives, so the boundary term does not contain any. A general discussion of Hamiltonian field theories would indeed require the boundary complex to involve the pullback of the BV complex to the boundary, see 5.5.7.

Since the theory we have developed above is only for compact manifolds, let us for a moment set $M = [0, T]$ for $T > 0$. This means that $\mathcal{E}_\partial = V \oplus V$ consists of two components $\mathcal{E}_\partial^{(0)} = \mathcal{E}_\partial^{(T)} = V$ for initial and final time, and $L \oplus L'$ is a Lagrangian subspace (note how we choose opposite polarization for in- and outgoing states because of the different orientations). As explained in 5.2.14 the chain map $r : \Omega^*(M) \otimes V \to \Omega^*(\partial M) \otimes V \cong V$ given by restriction is also Lagrangian, so we know by 5.2.15 that the (homotopy) pullback

$$\begin{tikzcd}
\mathcal{E}_L(M) := \mathcal{E}(M) \times^h_{\mathcal{E}_\partial} (L \oplus L') \arrow[r] \arrow[d] & \Omega^*(M) \otimes V \arrow[d] \\
L \oplus L' \arrow[r] & V \oplus V = \Omega^*(\partial M) \otimes V
\end{tikzcd}$$

admits a (-1)-shifted symplectic structure, so that we can regard it as the BV-BRST complex of a physical theory. One must notice at this point that since the map $L \to V$ is injective, the homotopy pullback agrees with the ordinary pullback of chain complexes by model category theory; in particular

$$\mathcal{E}_L(M) = \{\alpha_0 \in \Omega^0(M) \otimes V, \alpha_1 \in \Omega^1(M) \otimes V \mid \alpha_0(0) \in L, \alpha_0(T) \in L'\} \tag{5.17}$$

equipped with the pairing $\int_M \omega(- \wedge -)$ that is actually (-1)-shifted symplectic when restricted to this subspace of $\mathcal{E}(M)$ since the boundary terms vanish, by the above.

Remark. To be more accurate, we should have regarded M as a bordism to see that $\mathcal{E}_\partial^{(0)} \leftarrow \mathcal{E}(M) \to \mathcal{E}_\partial^{(T)}$ is a Lagrangian correspondence. We have not taken this point of view since we are ultimately interested in $\mathbb{R}_{\geq 0}$.

Note that we have only calculated a single chain complex, not a whole sheaf; in particular we have formed the pullback of $\mathcal{E}(M)$ and not $\mathcal{E}(U)$. If we restrict to an open interval $(a, b) \subseteq [0, T]$ with $0 < a < b < T$, we can not see the boundary and in particular no boundary terms, so even after fixing the boundary conditions, we should expect $\mathcal{E}_L((a, b)) = \mathcal{E}((a, b)) = \Omega^*((a, b)) \otimes V \cong V[0]$ as there are only constant solutions (or use the Poincaré Lemma). For the same reason, on opens of the form $[0, b)$, we obtain $\mathcal{E}_L([0, b)) = \mathcal{E}([0, b)) \times_V L \cong L[0]$ and similarly for $(a, T]$. So if we define a sheaf $\mathcal{E}_\partial$ on $\partial M = \{0, T\}$ with value V at both points, and similarly $\mathcal{L}$ with values L and L', we can write

$$\mathcal{E}_L := \mathcal{E} \times_{i_* \mathcal{E}_\partial} i_* \mathcal{L} \tag{5.18}$$

as a pullback of homotopy ∞-sheaves, where $i : \partial M \hookrightarrow M$ is the inclusion. Since sheafification is left exact, this is a pullback on each open.

Proposition 5.3.1. Above considerations determine a constructible sheaf $\mathcal{E}_L$ on the manifold with boundary $[0, T]$, equipped with its canonical stratification.

Proof. We know by construction that $\mathcal{E}$ is a homotopy sheaf, and have given explicit formulae for the values of $\mathcal{E}_L$ on each open interval; its value on a general open is hence given by a direct product over its connected components. Restriction maps are then induced by the identity and diagonal map on the components in the bulk, and the projection $\mathcal{E}_L \to \mathcal{E}$ out of the pullback if we embed bulk intervals into a boundary interval. Checking that this yields a constructible sheaf is thus straightforward using B.3.6.

Alternatively, one can use the exodromy equivalence

$$\mathcal{S}h^{cbl}([0,T],\mathcal{V}) \cong \mathrm{Fun}(\mathrm{Sing}^{[1]}[0,T],\mathcal{V}) \tag{5.19}$$

with $\mathrm{Sing}^{[1]}[0,T] \simeq \mathrm{Sing}^{[1]}|\Delta^1| \simeq (* \to * \leftarrow *)$, so a constructible sheaf on $[0,T]$ is uniquely determined by its values on intervals of the form $[0,b),(a,b),(a,T]$ and morphisms between them, which are in this case given by $L[0] \to V[0] \leftarrow L'[0]$. $\square$

Since $[0,\frac{T}{2})$ is (stratified) homotopy equivalent to $\mathbb{R}_{\geq 0}$, the restriction of $\mathcal{E}_L$ to this interval induces a constructible sheaf on $\mathbb{R}_{\geq 0}$ with similar properties, describing our theory on the original manifold.

Corollary 5.3.2. The algebra of classical observables compatible with the boundary condition $\mathcal{O}bs_{cl,L}(U) := \mathrm{Sym}\,\mathcal{E}_L^\vee$ is a constructible factorization algebra on both $[0,T]$ and $\mathbb{R}_{\geq 0}$ respectively, with values

$$\begin{aligned} \mathcal{O}bs_{cl,L}((a,b)) &= \mathrm{Sym}(\bar{\Omega}_c^*((a,b)) \otimes V^*[1]) \cong \mathrm{Sym}(V)[0] \\ \mathcal{O}bs_{cl,L}([0,b)) &= \mathrm{Sym}\left(\bar{\Omega}_c^*([0,b)) \otimes V^*[1] \big/ \{\alpha \mid \alpha_1(0) \in L\}\right) \cong \mathrm{Sym}(L')[0] \end{aligned} \tag{5.20}$$

Proof. That $\mathrm{Sym}\,\mathcal{E}^\vee$ is a constructible factorization algebra is immediate from above proposition and 4.3.8. To obtain the first row, the first equality is clear since the integration pairing exhibits $\bar{\Omega}_c^*((a,b))$ as the strong dual of $\Omega^*((a,b))$. For the second quasi-isomorphism, we use the Atiyah-Bott Lemma 3.1.15 to get rid of the distributional properties, and then the Poincaré Lemma. Also, we identify $V \cong V^*$ through the symplectic pairing.

A completely rigorous proof of the second row follows from the main result of [GRW20]. We only give a sketch: The dual space of $\mathcal{E}_L([0,b))$ consists of forms n $\bar{\Omega}_c^*([0,b)) \otimes V$ again using the integration pairing, but since forms in $\mathcal{E}_L$ are restricted to lie in L at the boundary, our linear forms do not need to distinguish (using that pairing) between forms that take values in L at the boundary. This might seems a bit counter-intuitive, but compare with Poincaré-Lefschetz duality that includes relative cohomology with respect to the boundary: If there were no restrictions on forms to lie in L at the boundary, the dual space would not be able to distinguish boundary forms at all. Generally, dual to the space of the forms that lie in $W \subseteq V$ at the boundary are forms with value in the annihilator $\mathrm{Ann}(W) \subseteq V^*$ at the boundary. Since $L' \cong V/L$ as a complement to L, using Atiyah-Bott and Poincaré-Lemma again, we obtain the right side. $\square$

Remark. Compare this with our classification of constructible factorization algebras on $\mathbb{R}_{\geq 0}$ in 4.4.2. $\mathcal{O}bs_{cl,L}$ consists an associative algebra object $\mathrm{Sym}(V)[0] \in \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{D}) \simeq \mathrm{FA}^{lc}(\mathbb{R}_{>0}, \mathcal{V})$ and a pointed object $\mathrm{Sym}(L')[0]$ with a module structure over it. In fact, as we are in the classical case, both of these are commutative algebras.

Since our boundary condition assures the existence of a (-1)-shifted symplectic structure we can quantize this free theory:

Proposition 5.3.3 ([GRW20, Section 5.1])**.** A central extension by $\hbar \int \omega(-,-)$ yields the quantum BV complex $\mathcal{E}_L^q = \mathcal{E}_L \oplus \hbar\mathbb{R}$. The constructible factorization algebra of quantum observables $\mathcal{O}bs_L^q$ on $[0,T]$ is then quasi-isomorphic to

$$\begin{aligned}
\mathcal{O}bs_L^q((a,b)) &\cong W(V) \\
\mathcal{O}bs_L^q([0,b)) &\cong F(L') := \mathrm{Sym}(L')[\hbar][0] \\
\mathcal{O}bs_L^q((a,T]) &\cong F(L)
\end{aligned}$$

where $W(V)$ is the *Weyl algebra* on V, the free associative $\mathbb{R}[\hbar]$-algebra generated by $a_1, \ldots, a_n, a_1^\dagger, \ldots, a_n^\dagger$ modulo the relations $[a_i, a_j^\dagger] = \hbar\delta_{ij}$ while all other generators commute. Similarly, $F(L')$ is the *Fock space* associated to $L' \subseteq V$, the symmetric $\mathbb{R}[\hbar]$-algebra spanned by the creation operators; and $F(L)$ is the symmetric $\mathbb{R}[\hbar]$-algebra spanned by the annihilation operators.

Explicitly, we will follow the physical convention of writing elements of the initial time Fock space as

$$(a_1^\dagger)^{N_1} \ldots (a_n^\dagger)^{N_n} \left|0\right\rangle \in F(L') \tag{5.21}$$

with $N_1, \ldots N_n \in \mathbb{N}_0$, and elements of the final time Fock space as $\left\langle 0\right| a_1^{N_1} \ldots a_n^{N_n}$. Operators in a bulk interval (a,b) are arbitrary products of creation and annihilation operators, which also seems reasonable from the point of view of quantum field theory. If we again compare with our classification of constructible factorization algebras 4.4.4, we identify $W(V)$ with an associative algebra, $F(L')$ with a left and $F(L)$ with a right (pointed) module over it, where the module structure is induced by the multiplication of operators, and the $\mathbb{E}_0$-structure, i.e. pointing of $F(L), F(L')$ is determined by the vacuum state.

While we have identified the Fock space with the boundary operators, physicists interpret it as the Hilbert space of states in our theory. The observation that these concepts agree may be interpreted as a kind of *bulk-boundary correspondence* or *operator-state-correspondence* (it is also sometimes called holography, which we fear risks oversimplifying that concept). Choosing a boundary condition amounts to choosing a way to distinguish creation and annihilation operators, or positions and momenta – this is why we also call $\mathcal{L}$ a polarization.

Proposition 5.3.4. The inclusion $(a,b) \subseteq [0,T]$ induces, on $\mathcal{O}bs_L^q$, the map that sends an operator to its vacuum expectation value.

Proof. We have seen that $\mathcal{O}bs^q_L((a,b))$ is the Weyl algebra of operators, and by 4.4.5 the global sections $\mathcal{O}bs^q_L([0,T])$ are given by the relative tensor product $F(L')\otimes_{W(V)} F(L)$ of the initial and final state Fock space. For any pure tensor

$$(a_1^\dagger)^{N_1}\dots(a_n^\dagger)^{N_n}\,|0\rangle\otimes\langle 0|\,a_1^{M_1}\dots a_n^{M_n} \tag{5.22}$$

we can shift all the creation operators to the right side and, using $[a_i, a_j^\dagger]=\hbar\delta_{ij}$, commute them next to $|0\rangle$ where they vanish (strictly decreasing the amount of involved creators and annihilators), and the remaining annihilation operators are now also on the left side where they vanish as well. Only multiples of $|0\rangle\otimes\langle 0|$ survive in the end and $\mathcal{O}bs^q([0,T])\simeq\mathbb{R}[\hbar]$. This is precisely how vacuum expectations values are calculated in quantum field theory. The map $\mathcal{O}bs^q((a,b))\to\mathcal{O}bs^q([0,T])$ sends an operator $\mathcal{O}\in W(V)$ to the result of acting on the vacuum state to the left and right $\langle 0|\mathcal{O}|0\rangle := |0\rangle\otimes\langle 0|\,\mathcal{O}$, as it determines the pointing in both modules. $\square$

Finally, let us think about more general 1-dimensional manifolds with boundary.

Proposition 5.3.5. Let $\mathrm{Charts}^{or}_{1,\partial}$ be the tangential structure determined by the subcategory of Bsc_1 on the basic oriented 1-manifolds with boundary $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ and orientation preserving maps. The functors

- $\mathcal{O}bs^{cl}:\mathrm{Charts}^{\sqcup}_{1,\partial}\to D(\mathbb{R})^{\otimes}$ sending $\mathbb{R}\mapsto\mathrm{Sym}(V)[0]$ and $\mathbb{R}_{\geq 0}\mapsto\mathrm{Sym}(V/L)[0]\cong\mathrm{Sym}(L')[0]$, with action on morphisms induced by the quotient map, as well as $\mathbb{R}_{\leq 0}\mapsto\mathrm{Sym}(L)[0]$.
- $\mathcal{O}bs^{q}:\mathrm{Charts}^{\sqcup}_{1,\partial}\to D(\mathbb{R})^{\otimes}$ sending $\mathbb{R}\mapsto W(V),\mathbb{R}_{\geq 0}\mapsto F(L')$ and $\mathbb{R}_{\leq 0}\mapsto F(L)$.

are absolute constructible factorization algebras with respect to this tangential structure on conically smooth stratified spaces.

Proof. Following our discussion in 4.4.7, an absolute constructible factorization algebra A on oriented manifolds with boundary consists of

- An associative algebra $A(\mathbb{R})$,
- A pointed left module $A(\mathbb{R}_{\geq 0})$ over $A(\mathbb{R})$,
- A pointed right module $A(\mathbb{R}_{\leq 0})$ over $A(\mathbb{R})$.

The triples described above precisely fit this pattern. $\square$

5.4. CS-WZW Boundary Correspondence

A similar calculation can be done for classical Chern-Simons theory on a smooth oriented 3-manifold with boundary. Remember the variation of the Chern-Simons action in 1.13:

$$0 = \frac{4\pi}{k}\delta S = 2\int_M \langle \delta A \wedge F_A \rangle + \int_{\partial M} \langle \delta A \wedge A \rangle \tag{5.23}$$

This yields the equations of motion $F_A = 0$ iff the boundary term

$$\langle A, \delta A \rangle_\partial := \int_{\partial M} \langle A \wedge \delta A \rangle \tag{5.24}$$

vanishes. Again, this must be ensured by a boundary condition imposed on $A|_{\partial M} \in \Omega^1(\partial M, \mathfrak{g})$.

The BV-BFV complex of Chern-Simons theory is given by $\mathcal{E}(M) = (\Omega^*(M, \mathfrak{g})[1], d)$ where we again trivialize the underlying principal bundle and expand around the trivial background connection. We choose the boundary BFV complex $\mathcal{E}_\partial(\partial M) = (\Omega^*(\partial M, \mathfrak{g})[1], d)$, not including derivatives of the fields (or their whole germs) since the boundary bracket already lives on this complex and does not contain any derivatives:

$$\langle -, - \rangle_\partial : \bar{\Omega}^*_c(\partial M, \mathfrak{g}) \hat{\otimes} \Omega^*(\partial M, \mathfrak{g}) \to \mathbb{R}[-2] \tag{5.25}$$

is given as the tensor product of integration pairing and Killing form is a non-degenerate symmetric pairing of degree 2 by Poincaré-Duality, so it induces a 0-shifted symplectic structure on $\mathcal{E}_\partial$ by 5.2.8. Further, the natural restriction map $\mathcal{E} \to \mathcal{E}_\partial$ is Lagrangian, because of the half-lives-half-dies principle in 5.2.14. We need to find an additional Lagrangian subcomplex, taking the rôle of a polarization as in the last example, that fixes the boundary condition.

To do that, we need to choose a complex structure on the 2-manifold $\partial M =: \Sigma$, and we need to assume that $\mathfrak{g}$ is a *complex* vector space with Lie bracket and Killing form, e.g. by tensoring with $\mathbb{C}$. For example in the abelian case, we set $\mathfrak{g} = \mathbb{C}$ instead of $\mathbb{R}$.

Proposition 5.4.1. In this situation, $\mathcal{L}(\Sigma) := (\Omega^{1,*}(\Sigma, \mathfrak{g})[1], \bar{\partial}) \subseteq \Omega^*(\Sigma, \mathfrak{g})[1], d) = \mathcal{E}_\partial$ is Lagrangian, since the differential subcomplex $(\bigwedge^{1,*} T^*M, \bar{\partial}) \subseteq (\bigwedge^* T^*M, d)$ is a Lagrangian subbundle as in 5.2.19.

Proof. It is clearly a subcomplex since $\bar{\partial} = d$ on $\mathcal{L}(\Sigma)$. Also, the Hodge star operator being an isomorphism (or alternatively, complex conjugation being an isomorphism) shows it is of half rank, and $\langle -, - \rangle_\partial|_{\mathcal{L}\hat{\otimes}\mathcal{L}} = 0$ since the wedge product of two forms in $\mathcal{L}$ must lie in $\Omega^{2,*}(\partial M) = 0$ for dimensional reasons. □

Remark. Even though Chern-Simons theory is topological, we are allowed to equip it with a boundary condition that is not; like in this case where, as we will see, it induces a conformal field theory on the boundary.

Since $\mathcal{L}(\Sigma) \subseteq \mathcal{E}_\partial(\Sigma)$ is injective, the homotopy pullback $\mathcal{E}_\mathcal{L}(M) := \mathcal{E}(M) \times^h_{\mathcal{E}_\partial(\Sigma)} \mathcal{L}(\Sigma)$ agrees with the ordinary pullback

$$\mathcal{E}_\mathcal{L}(M) = (\{\mathbb{A} \in \Omega^*(M, \mathfrak{g})[1] \,|\, \mathbb{A}|_\Sigma \in \Omega^{1,*}(\Sigma, \mathfrak{g})[1]\}, d) \subseteq \mathcal{E}\ . \tag{5.26}$$

The induced (-1)-shifted symplectic structure by 5.2.15 is, as expected, given by the tensor product of integration pairing and Killing form since boundary terms vanish on this subspace.

Let us remark at this point that $\mathcal{E} = \Omega^*_M[1] \otimes \mathfrak{g}$ is even a (homotopy) sheaf on M, similarly $\mathcal{L}$ and $\mathcal{E}_\partial$ are sheaves on Σ and the pullback $\mathcal{E}_\mathcal{L} := \mathcal{E} \times_{i_*\mathcal{E}_\partial} i_*\mathcal{L}$ for $i : \partial \hookrightarrow M$ is a sheaf on M with global sections $\mathcal{E}_\mathcal{L}(M)$. Precisely as in the last section, one can show that $\mathcal{E}_\mathcal{L}$ is constructible.

Proposition 5.4.2 ([GRW20, Section 5.3])**.** The classical observables $\operatorname{Sym} \mathcal{E}^\vee_\mathcal{L}$ of Chern-Simons theory satisfying the chiral boundary condition $\mathcal{L}$ form a constructible factorization algebra determined by

$$\begin{aligned} \mathcal{O}bs^{cl}(U) &\cong \left(\operatorname{Sym} \bar{\Omega}^*_c(U, \mathfrak{g})[2], d_{CE}\right) \cong (\operatorname{Sym} \Omega^*_c(U, \mathfrak{g})[2], d) \\ \mathcal{O}bs^{cl}(V) &\cong \left(\operatorname{Sym}(\Omega^{0,*}_c(U) \otimes \mathfrak{g}[1]), \bar{\partial}\right) \end{aligned} \tag{5.27}$$

for open subsets $U \subseteq \mathring{M}$ and V intersecting the boundary. Here, d_{CE} is the Chevalley-Eilenberg differential induced by the exterior derivative and the Lie bracket.

Proof. That these are constructible factorization algebras again follows from 4.3.8, and the first row follows from Poincaré-Duality and the Atiyah-Bott Lemma. The second row works similarly as for Topological Quantum Mechanics 5.3.2 as well, by choosing a complement to the Lagrangian $\mathcal{L}$. We use $(\Omega^{0,*}(\Sigma)[1], \bar{\partial})$ for this purpose, even though it is notably *not* a subcomplex of $\mathcal{E}_\partial$ as the differential is different. See loc. cit. for more details. □

Remark. In 4.5.6, we had classified constructible factorization algebras on manifolds with boundary as consisting of a bulk factorization algebra, a boundary factorization algebra and a module structure of the latter over the former. We now see this in practice, as the second formula describes a factorization algebra on Σ by the formula

$$\mathcal{O}bs^{cl}_\partial = \operatorname{Sym}(\Omega^{0,*}_c \otimes \mathfrak{g}[1], \bar{\partial})\ . \tag{5.28}$$

This is the algebra of operators of the *chiral WZW model* on Σ, which is why the Lagrangian $\mathcal{L}$ we have chosen is also called the *chiral WZW boundary condition.*

In the case of abelian Chern-Simons theory, let us discuss the quantization carried out via a twist by $\hbar\langle -, - \rangle$.

Proposition 5.4.3 ([GRW20, Section 5.3])**.** The quantum observables of abelian Chern-Simons theory over $\mathbb{C}$ satisfying the WZW boundary condition form the constructible factorization algebra $\mathcal{O}bs^q \simeq \mathrm{Sym}(\mathcal{E}_{\mathcal{L},c}[1][\hbar], d + \hbar\Delta)$ twisted by the shifted symplectic structure Δ, associating

$$\begin{aligned} \mathcal{O}bs^q(U) &\simeq \left(\mathrm{Sym}(\Omega^*_c(U) \otimes \mathfrak{g}[2])[\hbar], d + \hbar \int_M \langle - \wedge - \rangle\right) \\ \mathcal{O}bs^q(V) &\simeq \left(\mathrm{Sym}(\Omega^{0,*}_c(V) \otimes \mathfrak{g}[1])[\hbar], \bar\partial - \hbar \int \langle - \wedge \partial - \rangle\right) \end{aligned} \tag{5.29}$$

for U in the interior and V an open at the boundary. Again, this induces a factorization algebra on Σ which can be shown to correspond to the abelian Kac-Moody vertex algebra expected for the WZW model.

Remark. More generally, one should be able to show that Chern-Simons and WZW theory form an absolute constructible factorization algebra on smooth oriented 3-manifolds with boundary (so we choose as tangential structure the full subcategory of conically smooth basics on $\mathbb{R}^3$ and $\mathbb{R}^2 \times \mathbb{R}_{\geq 0} \to [1]$). The only obstruction to formally proving this is 2.2.13, as always.

Remark. Again, one notices a sort of bulk-boundary correspondence: Observables at the boundary can be interpreted both as forming the Hilbert space of states for Chern-Simons theory, and as elements of the induced locally constant factorization algebra on Σ describing the WZW field theory.

5.5. Hamiltonian Approach to the Scalar Field

In this section, we study the free scalar field $\phi \in C^\infty(M)$ on a compact Riemannian manifold (M, g) with boundary, using an approach motivated by [CMR14] and [Cap]. We need to be careful about the precise way we write down its action: Unlike in the previous chapters, we set

$$S[\phi] := \int_M \frac{1}{2}\left(-\partial_i\phi\partial^i\phi + m^2\phi^2\right)\mathrm{vol}_g = \int_M \frac{1}{2}\left(-d\phi \wedge \star d\phi + m^2\phi \star \phi\right) \tag{5.30}$$

which differs from our previous action containing $\phi(\Delta_g + m^2)\phi$ by a boundary term. Variation of this action yields

$$\begin{aligned} \delta S &= -\int_{\partial M} \delta\phi \wedge \star d\phi + \int_M \delta\phi \wedge \left(d \star d\phi + m^2 \star \phi\right) = \\ &= -\int_{\partial M} \delta\phi \cdot d\phi\, \nu_{\partial M} + \int_M \delta\phi \wedge \left(\Delta_g\phi + m^2\phi\right)\mathrm{vol}_M \end{aligned}$$

where $\star$ always denotes the Hodge star in M and $\nu_{\partial M}$ is the differential normal vector along the boundary, so $d\phi\nu_{\partial M} = \partial_n\phi\,\mathrm{vol}_{\partial M}$ is the normal derivative of ϕ. We obtain the Klein-Gordon equation, provided that the boundary term $-\int_{\partial M}\delta\phi\partial_n\phi\nu_{\partial M} =: \langle\delta\phi, d\phi\rangle_\partial$ vanishes. This allows for an educated guess for the boundary BFV complex: For an open $U \subseteq \partial M$,

$$\mathcal{E}_\partial(U) = (C^\infty(U) \oplus C^\infty(U))[0] \cong (\Omega^0(U) \oplus \Omega^{n-1}(U))[0] \tag{5.31}$$

so $\mathcal{E}_\partial$ is the sheaf of sections of the differential complex over ∂M given by the direct sum of two trivial vector bundles $\epsilon_\phi \oplus \epsilon_n$ in degree 0 on ∂M. One is determining the values of ϕ on ∂M, and the other as fixing the normal derivative $\partial_n\phi$. In other words, initial conditions consist of values of ϕ and its first derivative in the normal direction.

Proposition 5.5.1. The pairing $\langle(\alpha_1,\beta_1),(\alpha_2,\beta_2)\rangle_\partial := \int_{\partial M}(\alpha_1\beta_2-\alpha_2\beta_1)\,\mathrm{vol}_{\partial M}$ determines a 0-shifted symplectic structure on $\mathcal{E}_\partial$.

Proof. This is clearly antisymmetric, and since ∂M is still orientable our identification $\mathcal{E}_\partial = \Omega^0(\partial M) \cong \Omega^{n-1}(\partial M)$ exhibits above it as the integration pairing on ∂M that we know is symplectic. □

Next, we would have to check that the restriction map $r : \mathcal{E}(M) \to \mathcal{E}_\partial(\partial M)$ is Lagrangian. Spelling this out requires existence and uniqueness results from the PDE theory of the Klein-Gordon equation, and relies heavily on the functional analytic details we suppress, we refer to [Cap] for more information.

Proposition 5.5.2. The following subbundles of $\mathcal{E}_\partial = \epsilon_\phi \oplus \epsilon_n$ are Lagrangian and therefore determine boundary conditions:

- *Dirichlet boundary condition:* Sections that lie in $0 \oplus \epsilon_n$ come from fields that vanish at the boundary
- *Neumann boundary condition:* Sections that lie in $\epsilon_\phi \oplus 0$ come from fields that are constant in the direction normal to the boundary
- *Robin boundary condition:* We can combine these concepts by looking at the subbundle where $c_0\phi + c_1\partial_n\phi = 0$, for $\phi, \partial_n\phi$ the coordinates along ϵ_ϕ and ϵ_n respectively and $(c_0, c_1) \in \mathbb{R}^2 - \{0\}$.

Proof. Since everything is concentrated in degree 0, we only have to check that the symplectic structure vanishes restricted to the bundles and that they are of half rank. The latter part is clear since the above subbundles are all of rank 1, and the first claim follows since any antisymmetric form restricted to a one-dimensional subspace vanishes and the symplectic pairing only depends on the fiber coordinates in the bundles. Explicitly for Robin boundary conditions, sections (α_1, β_1) and (α_2, β_2) as above and $\lambda := -\frac{c_0}{c_1}$, we impose $\beta_i = \lambda\alpha_i$ so $\alpha_1\beta_2 - \alpha_2\beta_1 = \alpha_1\alpha_2(\lambda - \lambda) = 0$. □

Remark. Note that setting $\phi|_{\partial M} \overset{!}{=} \phi_0$ for $0 \neq \phi_0 \in C^\infty(\partial M)$ does *not* determine a boundary condition, since it does not yield a sub-vector bundle. This is to be expected since we are dealing with perturbative field theory around $\phi = 0$, while fixing such a condition would be strictly non-perturbative. The correct way to implement non-vanishing Dirichlet boundary conditions would be by fixing a background solution that satisfies them, and performing perturbation theory around it.

As in the previous sections, the above results allow us to define constructible factorization algebras $\mathcal{O}bs^{cl}$ and $\mathcal{O}bs^q$ on any manifold with boundary that describe classical or quantum operators on the scalar field. One can even define a corresponding absolute factorization algebra on all Riemannian manifolds with boundary, compare [Cap].

Let us also talk about periodic boundary conditions. Suppose $\partial M = N \sqcup -N$ for a closed oriented $(n-1)$-manifold N, where we fix a diffeomorphism between the two boundary components in other to identify $\mathcal{E}_\partial(N)$ and $\mathcal{E}_\partial(-N)$. As before, one can show that the restrictions $\mathcal{E}_\partial \overset{r}{\leftarrow} \mathcal{E} \overset{r'}{\to} \mathcal{E}_\partial$ to both boundary components constitute a Lagrangian correspondence, so we apply self-gluing 5.2.17 to see that

$$\mathcal{E}_{\text{period}} := \text{fib}(r-r') = \left(C^\infty(M) \overset{(\Delta+m^2)\oplus(r-r')}{\longrightarrow} C^\infty(M)[-1] \oplus \left(C^\infty(\partial M) \oplus C^\infty(\partial M)\right)[-1] \right)$$

obtains a canonical (-1)-shifted symplectic structure. Notice that with all of $C^\infty(M)$ as source, $r - r'$ is surjective, so instead of the fiber we can take the kernel $C^\infty_{\text{period}}(M) := \{\alpha \in C^\infty(M) \mid \alpha|_N = \alpha|_{-N} \text{ and } \partial_n\alpha|_N = \partial_n\alpha|_{-N}\}$ via the fixed diffeomorphism between N and $-N$. This means

$$\mathcal{E}_{\text{period}} = \left(C^\infty_{\text{period}}(M) \overset{\Delta+m^2}{\longrightarrow} C^\infty(M) \right) \tag{5.32}$$

and a similar argument to 3.3.2 using results from PDE theory (uniqueness and existence of solutions, and Green's function) identifies this with the solutions of the field equations on the self-gluing of M along N. In particular, it is enough to require continuity of the field and its normal derivative at the boundary since by uniqueness, we still eventually glue to smooth solutions.

Now, let us apply this to understand the Hamiltonian formalism and canonical quantization. For this purpose, choose as underlying Riemannian manifold a product $(M, g) = (N, h) \times [0, T]$ with $T \in \mathbb{R}_{>0}$, $\pi : M \to [0, T]$ the canonical restriction map and (N, h) a closed oriented Riemannian $(n-1)$-manifold. Remember from 3.3.11 that for $(\lambda_i)_{i\in\Lambda}$ the spectrum of $\Delta_h + m^2$ with eigenfunctions e_i, the subspace $\bigoplus_i \mathbb{R}e_i \subseteq C^\infty(N)$ is dense, a fact that allowed us to describe a dense subspace the pushforward factorization algebra $\pi_* \mathcal{O}bs^q$ as a product of individual Weyl algebras for every energy eigenvalue. Let us generalize this to the boundary case:

Theorem 5.5.3. Denote by $\mathcal{O}bs^q_{TQM}$ the constructible factorization algebra of topological quantum mechanics on $[0, T]$ from 5.3, with underlying symplectic vector space $V =$

$\bigoplus_{i\in\Lambda}(\mathbb{R}a_i+\mathbb{R}a_i^\dagger)$ and its canonical symplectic form. Then, the pushforward factorization algebra $\pi_* \mathcal{O}bs^q$ agrees, up to some functional analytic completion and the fact that Λ is usually not finite, with $\mathcal{O}bs^q_{TQM}$; and similarly for the classical case.

Remark. From a physicists point of view, our functional analytic/ finiteness problems can be discussed away since physical particles should only carry a finite amount of energy. Then, this result essentially says that Topological Quantum Mechanics with values in V is the *compactification* of scalar field theory on $N \times [0,T]$ along N.

Proof. We have already seen that the statement is true in the interior of $[0,T]$ as $\mathcal{O}bs^q_{TQM}$ corresponds to the associative algebra $W(V) = \bigotimes_i W\langle a_i, a_i^\dagger\rangle$ there, where we take the tensor product over $\mathbb{R}[\hbar]$, agreeing with the result from 3.3.11.

On the boundary, $\mathcal{O}bs^q_{TQM}$ is described by the Fock space $F(L') = \bigotimes_i \mathbb{R}[\hbar][a_i^\dagger]$ equipped with a module structure over $W(V)$. For the scalar field, we need to choose a complement $\mathcal{L}^\perp$ to the Lagrangian subbundle $\mathcal{L}$ of $\epsilon_\phi \oplus \epsilon_n$ determined by the respective boundary condition, which is always a trivial bundle of rank 1 so that

$$\pi_* \mathcal{O}bs^q([0,b)) = \mathcal{O}bs^q(N \times [0,b)) \cong \operatorname{Sym}\Gamma(N, \mathcal{L}^\perp)[\hbar] \cong$$
$$\cong \operatorname{Sym} C^\infty(N)[\hbar] \supseteq \operatorname{Sym}\bigoplus_i \mathbb{R}[\hbar]e_i \cong \bigotimes_i \mathbb{R}[\hbar][e_i]$$

contains this Fock space as a dense subspace. We have, similarly to the last sections, used the main result of [GRW20] to retract the quantum observables to $\operatorname{Sym}(\mathcal{L}^\perp)$. $\square$

In the above calculation, the precise manner in which we identify the two factorization algebras depends on $\mathcal{L}^\perp$, and therefore on the boundary condition $\mathcal{L}$. Eventually, this ambiguity is irrelevant since it is captured by our convention on how to identify a_i and $a_i^\dagger$ with wave functions. Sticking close to the notation from Chapter 3, regard a_i as behaving like e^{mx} and $a_i^\dagger$ as behaving like e^{-mx} normal to the boundary. Then, Dirac boundary conditions correspond on the TQM side to $L' = \bigoplus_i \mathbb{R}(a_i - a_i^\dagger)$ as the involved perpendicular Lagrangian subspace; Neumann boundary conditions to $L' = \bigoplus_i \mathbb{R}(a_i + a_i^\dagger)$ and Robin conditions to

$$L' = \bigoplus_i \mathbb{R}\left((c_1 + c_0)a_i + (c_1 - c_0)a_i^\dagger\right) .$$

Notice that every 1-dimensional subspace L_0 of $\mathbb{R}\langle a_i, a_i^\dagger\rangle$ has to be Lagrangian since it is spanned by a single basis vector l_0 with $\langle l_0, l_0\rangle_\partial = 0$ by antisymmetry.

Eventually, comparing with 3.5.1, we have thereby proven that the real scalar field on a closed manifold can be described as a collection of infinitely many Harmonic Oscillators, and how Dirichlet and Neumann boundary conditions are encoded in this picture. Our result agrees with what is expected by physical intuition, and comparing both sides tells us a lot about the rôle our formalism plays for general Hamiltonian Field Theories. Let us speculate on how our methods could be extended, using the following theorem from sheaf theory:

Theorem 5.5.4 (Recollement, [HA, A.8.16] and [PT22, 5.18])**.** For X a topological space, $j : U \hookrightarrow X$ an open subset and $i : Z = X - U \hookrightarrow X$ its closed complement, the ∞-category $\mathcal{S}h(X;\mathcal{V})$ of ∞-sheaves with values in a presentable stable (or compactly generated) ∞-category $\mathcal{V}$ is a *recollement* of $\mathcal{S}h(Z;\mathcal{V})$ and $\mathcal{S}h(U;\mathcal{V})$. This means in particular that $\mathcal{S}h(X;\mathcal{V})$ is equivalent to the ∞-category of triples of

- A sheaf $\mathcal{F}_Z \in \mathcal{S}h(Z;\mathcal{V})$,
- A sheaf $\mathcal{F}_U \in \mathcal{S}h(U;\mathcal{V})$,
- A morphism $\mathcal{F}_Z \to i^* j_* F_U$.

Under this equivalence, any sheaf $F \in \mathcal{S}h(X;\mathcal{V})$ can we recovered as the pullback

$$\begin{tikzcd} F \arrow[r] \arrow[d] & i_* F_Z \arrow[d] \\ j_* F_U \arrow[r] & i_* i^* j_* F_U \end{tikzcd}$$

where $F_U = j^* F$ and $F_Z = i^* F$.

Example 5.5.5. Any sheaf F on a manifold with boundary can be recovered from its restriction $F_\circ$ to the interior, its restriction F_∂ to the boundary, and a morphism $F_\partial \to i^* j_* F_\circ$.

Corollary 5.5.6. For any stratified space $f : X \to P$ and $\mathcal{V}$ as above, let (P_-, P_+) be a *slicing* of P, which means that P_- is downwards closed, P_+ is upwards closed, and $P_- \cup P_+ = P$ while $P_- \cap P_+ = \emptyset$. The subspaces $X_- := f^{-1}(P_-)$ and $X_+ := f^{-1}(P_+)$ then decompose X into a closed and an open subset, and the ∞-category $\mathcal{S}h^{cbl}(X;\mathcal{V})$ is a recollement of $\mathcal{S}h^{cbl}(X_+;\mathcal{V})$ and $\mathcal{S}h^{cbl}(X_-;\mathcal{V})$.

Proof. The statement about the decomposition of X follows by continuity of f and the definition of the Alexandrov-Topology on P. The statement about sheaves follows from 5.5.4 and the observation that a sheaf F on X is constructible iff its restrictions to X_- and X_+ are constructible. See [Zet, Section 5] for a refined statement. $\square$

Construction 5.5.7. We now sketch how the Hamilton formalism for an arbitrary topological, conformal or free field theory may be introduced from the point of view of constructible sheaves. Given a manifold M with boundary, the BV-complex of the field theory we consider should define a locally constant sheaf $\mathcal{E} \in \mathcal{S}h^{lc}(\mathring{M};\mathcal{V})$ that is Verdier self-dual up to a shift by 3. The pairing inducing this Verdier self-duality is the (-1)-shifted symplectic structure $\mathcal{E}_c^! \otimes \mathcal{E} \to \mathrm{Dens}_{\mathring{M}}$ as $\mathcal{E}^\vee \otimes \mathrm{Dens}_{\mathring{M}}$ is the Verdier dual sheaf.

Now, there is a canonical boundary BFV complex $\mathcal{E}_\partial \in \mathcal{S}h^{lc}(\partial M;\mathcal{V})$ associated to $\mathcal{E}$, defined by taking the stalks of $\mathcal{E}$ at the boundary. The discussion in [Ban07, Section

8.2.2] shows that $\mathcal{E}_\partial := i^* j_* \mathcal{E}[-1]$ acts as a universal boundary sheaf to $\mathcal{E}$, and this should be the general way the unpolarized Hilbert Space (or, in the classical case, the Hamiltonian Phase Space) is defined.

If the space of off-shell fields is given by the sections of a vector bundle $\mathcal{F} = \Gamma(-, E)$, the boundary bracket may only depend on the values of the fields and their derivatives at the boundary, or even just on the fields themselves. One should then be able to reduce from this universal $\mathcal{E}_\partial$ to a smaller subsheaf, namely the jet bundle $J^\infty E|_{\partial M}$ or $E|_M$.

5.6. Examples of Corner Theories

After our lengthy discussion of boundary conditions, let us now look at field theories on manifolds with corners. As an example, fix the compact oriented 2-dimensional $M = \Delta^2$. A field theory on it should contain a bulk BV complex $\mathcal{E}$, a boundary BFV complex $\mathcal{E}_\partial$ and a corner BFFV complex $\mathcal{E}_{\partial\partial}$, where we follow the convention of [CMR14] to add one factor of "Fradkin" inside the term "Batalin-Vilkovisky" for each codimension.

It is a classical fact that on any smooth manifold with corners, we may smooth out all corners to obtain a homotopy equivalent manifold with boundary. We therefore expect that the introduction of corners should involve less physical content than the introduction of boundaries, in particular we do not need polarizations at corners. Instead, the corner complex often imposes continuity or smoothness conditions on the sections of $\mathcal{E}_\partial$, making the resulting theory very similar to the smoothed out theory.

On Δ^2, locality tells us that the contributions from different boundary components or different corners should a priori be independent, so set $\mathcal{E}_\partial = \mathcal{E}_{01} \oplus \mathcal{E}_{02} \oplus \mathcal{E}_{12}$ and $\mathcal{E}_{\partial\partial} = \mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \mathcal{E}_2$, where we suppress pushforward along the inclusions of the respective components. As an example, let us choose a finite-dimensional vector space V equipped with a (possibly degenerate) antisymmetric form $V \otimes V \to \mathbb{R}$ inducing a map $\Pi : V^* \to V$.

- For each vertex i let $\mathcal{E}_i := (\Omega^*(\{i\}) \otimes V^*[1] \oplus \Omega^*(\{i\}) \otimes V, d \otimes \mathrm{id}_V + \mathrm{id} \otimes \Pi) = V^*[1] \xrightarrow{\Pi} V$ regarded as a sheaf on $\{i\}$,
- For each edge $[i,j]$ and $U \subseteq [i,j]$ the sheaf $\mathcal{E}_{ij}(U) := (\Omega^*(U) \otimes V^*[1] \oplus \Omega^*(U) \otimes V, d \otimes \mathrm{id}_v + \mathrm{id} \otimes \Pi)$
- And $\mathcal{E} := \Omega^*_M \otimes V^*[1] \oplus \Omega^*_M \otimes V$ on all of M, with differential again induced by d and Π.

We obtain restriction maps fitting into a diagram

$$\begin{tikzcd}
 & & \mathcal{E}_1 & & \\
 & \mathcal{E}_{01} \arrow[ur] \arrow[ddl] & & \mathcal{E}_{12} \arrow[ul] \arrow[ddr] & \\
 & & \mathcal{E} \arrow[ul] \arrow[ur] \arrow[d] & & \\
\mathcal{E}_0 & & \mathcal{E}_{02} \arrow[ll] \arrow[rr] & & \mathcal{E}_2
\end{tikzcd}$$

Physically, this *extended BFV-theory* describes the free Poisson σ-model with target V. Be aware that we use the term *BFV-theory* only due to the conceptual similarity with [CMR14]; the precise setting differs quite a bit.

Remark. One can write down similar data for non-linear Poisson σ-model – this works along the same lines of the compactification of B-twisted Kapustin-Witten-theory in [BY16, Section 4.5]. Generally, the examples in this paper fit into our context as well.

There is a large class of field theories that can be extended to arbitrary codimensions in a fairly trivial way, namely the class of *AKSZ-theories.* We give no precise definition, but essentially, these are theories where the covariant phase space on an arbitrary manifold M is constructed as a mapping stack out of a derived stack associated to M. For example, the covariant phase space of Chern-Simons theory was claimed in 1.56 to be

$$X(M) = \underline{\mathrm{Map}}(\flat M, \mathbb{B}G) \ . \tag{5.33}$$

Now, we can define boundary and corner complexes using the exact same formula; one can show [Cal14] that the functor

$$\underline{\mathrm{Map}}(\flat(-), \mathbb{B}G) : \mathbf{Bord_3} \to \mathbf{Lagr_3} \tag{5.34}$$

is an extended topological field theory, where the right side is an (∞, n)-category with derived stacks as objects, Lagrangian correspondences of stacks as morphisms, and higher correspondences as higher morphisms. Taking the tangent space around a background solution on a fixed manifold with corners M intuitively should yield the global BV-complex (not a sheaf) on the interior of M, a BFV-complex on each boundary component, and so on – again forming a network of (higher) Lagrangians.

For example, on the tetraeder Δ^3, we obtain an extended BFV theory corresponding to abelian Chern-Simons theory with

- BFFFV complex $\mathcal{E}_{\partial\partial\partial} = \bigoplus_{i=1}^4 (\Omega^*(\{i\})[1], d) \cong \mathbb{R}[1]$ where i goes through all the vertices,
- BFFV complex $\mathcal{E}_{\partial\partial} = \bigoplus_{e=1}^6 (\Omega^*(e)[1], d)$ living on the edges of Δ^3,
- BFV complex $\mathcal{E}_\partial := \bigoplus_{f=1}^4 (\Omega^*(f)[1], d)$ on the faces,
- BV complex $\mathcal{E} := (\Omega^*(\Delta^2)[1], d)$ on the whole simplex.

Similarly for non-abelian Chern-Simons theory, where we tensor a factor of $\mathfrak{g}$ to each sheaf, giving us non-vanishing ℓ_2-brackets via wedge product and Lie bracket. We have restriction maps from the bottom to the top, and

- $\mathcal{E}_{\partial\partial\partial}$ possesses a 2-shifted symplectic structure, given by multiplication in $\mathbb{R}$ as $(\mathbb{R}[1])^\vee[2] \cong \mathbb{R}$.
- $\mathcal{E}_{\partial\partial} \to \mathcal{E}_{\partial\partial\partial}$ induces Lagrangian correspondences between the components for vertices connected by an edge, so that combining 5.2.15 and 5.2.17 the fiber $\mathcal{E}'_{\partial\partial}$ of this map becomes 1-shifted Lagrangian, describing forms on the edges that are compatible at the corners.
- $\mathcal{E}_\partial \to \mathcal{E}_{\partial\partial}$ factors through $\mathcal{E}'_{\partial\partial}$ since forms that live on the faces, when restricted to the edges, are automatically compatible at the corners. This again, essentially by 5.2.14, yields Lagrangian correspondences between the components, equipping the fiber $\mathcal{E}'_\partial$ of this map with a 0-shifted symplectic structure.
- $\mathcal{E} \to \mathcal{E}_\partial$ factors through $\mathcal{E}'_\partial$ for the same reason, and this map is Lagrangian.

We could obtain a physical BV-complex with a (-1)-shifted symplectic structure by producing another Lagrangian $L \to \mathcal{E}'_\partial$ acting as a boundary condition.

There is one big problem with this description: What we have obtained are global complexes on our manifold, its boundary and so on, but to construct factorization algebras, we would need to start with a constructible sheaf as in the previous examples. Also, instead of the global shifted symplectic structures we have just described, those should also be defined in a local way. This resembles surgery theory, where we are interested in CW complexes that not only satisfy global, but local Poincaré duality. We discuss two ways to incorporate this:

- We can make $\mathcal{E}$ and the boundary and corner complexes into sheaves that are Verdier self-dual up to a shift inside the interiors of the components they live on, compare 5.8.14. This involves (higher) bordism theory of Verdier-self dual sheaves, compare [Ban07, 8.2] and [Ban01].
- By working on simplicial complexes instead of manifolds with corners, we do not have to use sheaves any more (as long as we are working with topological field theories), since simplices are contractible and locally constant ∞-sheaves on them are therefore constant. Simplicial BV theories thus have very efficient descriptions as we will see in the next section.

For more examples of extended BV-BFV theories including BF-theory and Yang-Mills theory, see [CMR14, Chapter 5]; examples for the boundary case in [Rab21] and [BY16] can also often be extended.

5.7. Simplicial Topological Field Theories

In this last section, we will tease how our considerations about field theories on manifolds with corners can be used to glue together field theories from triangulations on compact manifolds. Generally, we introduce field theories on finite simplicial complexes, generating an abstract framework for cellular topological field theories including those discussed in [Mne06] and [CMR20].

Definition 5.7.1. A *simplicial complex* K consists of a set of vertices K_0, together with a partially ordered set $\mathcal{I}_K$ of faces that is a collection of nonempty finite subsets of K_0, such that

- for every $v \in K_0$, the set $\{v\}$ is in $\mathcal{I}_K$,
- if $\sigma \in \mathcal{I}_K$ and $\tau \subseteq \sigma$, then $\tau \in \mathcal{I}_K$,
- and the partial order relation is given by inclusion.

The *dimension* of a face $\sigma \in \mathcal{I}_K$ is defined as its cardinality minus 1. The dimension of K is the maximum over the dimensions of all its faces.

Definition 5.7.2. A *map of simplicial complexes* $f : K \to L$ is a map of underlying sets $f : K_0 \to L_0$ such that the image of a face is again a face. One obtains a category of simplicial complexes.

A simplicial complex K should be regarded as special case of a simplicial set, where

- no ordering is fixed on the faces of an n-simplex,
- the gluing of simplices is *regular*, i.e. all n-simplices of K are isomorphic to the standard simplex Δ^n, and
- the intersection of two simplices is again a *single* simplex.

In particular, we can associate a simplicial set to any simplicial complex K (which is unique after fixing an order on the vertices), and use this to define e.g. the geometric realization $|K|$. This agrees with its usual definition.

In the following, let us fix a *finite* simplicial complex K; i.e. the set of vertices K_0 is finite. Also, let $\mathcal{V}$ be a stable ∞-category with duality functor $(-)^\vee$, for example $D^{\mathrm{perf}}(\mathbb{R})$ or the category $\mathcal{D}$ of topological vector spaces we have fixed by assumption.

Definition 5.7.3. If we regard the poset $\mathcal{I}_K$ as an ∞-category, functors in $\mathrm{Fun}(\mathcal{I}_K, \mathcal{V})$ will be called *$\mathcal{V}$-valued constructible sheaves on K* and functors in $\mathrm{Fun}(\mathcal{I}_K^{op}, \mathcal{V})$ will be called *gluing data* on K.

Proposition 5.7.4. If we assume $\mathcal{V}$ is presentable, this coincides with the usual definition of a constructible ∞-sheaf on the geometric realization $|K|$ if we stratify it by the poset of simplices $\mathcal{I}_K$, namely

$$\mathcal{S}h^{cbl}(|K|, \mathcal{V}) \simeq \mathrm{Fun}(\mathcal{I}_K, \mathcal{V}) . \tag{5.35}$$

We are however interested in non-presentable $\mathcal{V}$ like $D^{\mathrm{perf}}(\mathbb{R})$ or our functional-analytic category $\mathcal{D}$, that can be embedding in a larger presentable category $\hat{\mathcal{V}}$ like $D(\mathbb{R})$ or $\hat{\mathcal{D}}$. In this case, $\mathrm{Fun}(\mathcal{I}_K, \mathcal{V})$ is equivalent to the full subcategory of $\mathcal{S}h^{cbl}(|K|, \hat{\mathcal{V}})$ on constructible sheaves whose stalks lie in $\mathcal{V} \subseteq \hat{\mathcal{V}}$.

Proof. Since $\mathrm{Sing}^{\mathcal{I}_K} |K| \simeq \mathcal{I}_K$ by B.2.19, the presentable case follows immediately from exodromy B.5.11. In the non-presentable case, use the exodromy correspondence for $\hat{\mathcal{V}}$ and notice that the value of the functor $\mathcal{I}_K \to \hat{\mathcal{V}}$ on a simplex σ is the stalk of the associated constructible sheaf at a point in the interior of σ. Compare [PT22, 6.32] for this fact. □

Definition 5.7.5. To every constructible sheaf $S : \mathcal{I}_K \to \mathcal{V}$, we can associate a gluing datum $\mathbb{G}S : \mathcal{I}_K^{op} \to \mathcal{V}$ defined by

$$\mathbb{G}S(\sigma) := \lim_{(\tau\subseteq\sigma)\in(\mathcal{I}_K)_{/\sigma}} S(\tau) . \tag{5.36}$$

Since $\mathcal{V}$ is stable and K finite, this limit exists. Intuitively, S gives the stalks of a constructible sheaf at points in the interiors of the respective simplices of K, and $\mathbb{G}S$ gives the sections of the respective sheaf on the closures of the simplices.

Definition 5.7.6 ([CDH+20a, 6.3.2])**.** Given a gluing datum $F \in \mathrm{Fun}(\mathcal{I}_K^{op}, \mathcal{V})$, we define its *dual gluing datum DF* as the functor $\mathcal{I}_K^{op} \to \mathcal{V}$ defined by

$$DF(\sigma) := \lim_{(\tau\subseteq\sigma)\in(\mathcal{I}_K)_{/\sigma}} F(\tau)^\vee . \tag{5.37}$$

Note that $DF = \mathbb{G}(F^\vee)$ where $F^\vee := F \circ (-)^\vee : \mathcal{I}_K \to \mathcal{V}$.

Example 5.7.7. Let $K = \Delta^2$ and fix $F \in \mathrm{Fun}(\mathcal{I}_K^{op}, \mathcal{V})$. Then,

$$\begin{aligned} DF(\{0\}) &= \lim_{\tau\subseteq\{0\}} F(\tau)^\vee = F(\{0\})^\vee , \\ DF(\{0,1\}) &= \lim_{\tau\subseteq\{0,1\}} F(\tau)^\vee = F(\{0\})^\vee \times_{F(\{0,1\})^\vee} F(\{1\})^\vee , \\ DF(\{0,1,2\}) &= \lim_{\tau\subseteq\{0,1,2\}} F(\tau)^\vee = \mathrm{fib}\left(\lim_{\tau\in\partial\Delta^2} F(\tau)^\vee \to F(\{0,1,2\})^\vee \right) . \end{aligned}$$

One can of course also write out the last line; we write it this way to make the connection with polarizations easier to see in a moment.

Proposition 5.7.8 ([CDH⁺20a, 6.6.1])**.** For every gluing datum $F \in \mathrm{Fun}(\mathcal{I}_K^{op}; \mathcal{V})$, there is a canonical biduality isomorphism $F \cong DDF$. In fact, $(\mathrm{Fun}(\mathcal{I}_K^{op}, \mathcal{V}), D)$ is again a stable ∞-category with duality functor.

Technical Remark. This is a special case of the so-called *cotensor Poincaré ∞-category* $(\mathcal{V}, (-)^\vee)^{\mathcal{I}_K}$. See the above reference for more, we warn the reader that this duality does *not* agree with Verdier duality on constructible sheaves, which is on compact oriented manifolds described by the tensor Poincaré ∞-category on $\mathrm{Fun}(\mathcal{I}_K, \mathcal{V})$. To describe it, we essentially only replaces the limit over $(\mathcal{I}_K)_{/\sigma}$ in our formula for $DF(\sigma)$ by a colimit over $(\mathcal{I}_K)_{\sigma/}$.

Corollary 5.7.9. We can recover a constructible sheaf $S : \mathcal{I}_K \to \mathcal{V}$ from its associated gluing datum $\mathbb{G}F$ by taking the dual gluing datum and pointwise applying $(-)^\vee$:

$$S \cong (D\mathbb{G}S)^\vee \tag{5.38}$$

Proof. Since we have seen $D(S^\vee) = \mathbb{G}S$, write out $(D\mathbb{G}S)^\vee \cong (DDS^\vee)^\vee \cong S^{\vee\vee} \cong S$. □

Proposition 5.7.10 ([CDH⁺20a, 6.6.2])**.** Refinement of the triangulation, and pullback along maps of simplicial complexes, commute with the duality functor – see the reference for a precise statement.

Definition 5.7.11. For $S \in \mathrm{Fun}(\mathcal{I}_K, \mathcal{V})$ a constructible sheaf, its *simplicial cochain complex with values in the local system F*, or *global sections*, are defined as

$$C^*S := \lim_{\sigma \in \mathcal{I}_K} S(\sigma) \in \mathcal{V} \; . \tag{5.39}$$

which is again well-defined since K is finite. We also define the *simplicial chain complex* as $C_*S := \operatorname*{colim}_{\sigma \in \mathcal{I}_K} S(\sigma)$.

Remark. Homotopy colimits in the derived category of a Grothendieck abelian category can be calculated using the *bar construction* (see [CG16, C.5.11]). Using this fact, we see that for $\mathcal{V} = D(R)$ with R a commutative ring or $D(\mathrm{DVS})$, this simplicial chain complex $C_*(S)$ agrees with the one well-known from topology, and similarly for $C^*(S)$. In particular, if $S = \underline{R}$ is constant,

$$C^*(\underline{R}) \cong C^*_{\mathrm{cell}}(K; R), \quad C_*(\underline{R}) \cong C_*^{\mathrm{cell}}(K; R) \; . \tag{5.40}$$

One can also define cellular and singular (co-)chains with values in a constructible sheaf in this way.

Definition 5.7.12. Similarly, for $F : \mathcal{I}_K^{op} \to \mathcal{V}$ a gluing datum, we define C^*F and C_*F by taking limits and colimits over $\mathcal{I}_K^{op}$.

Proposition 5.7.13. For S a constructible sheaf, the global sections $C^*S \cong C^*\mathbb{G}S$ agree. This is however generally not true for C_*.

Proof. We need to show that

$$C^*S = \lim_{\sigma\in\mathcal{I}_K} S(\sigma) \overset{!}{\cong} C^*\mathbb{G}S = \lim_{\sigma\in\mathcal{I}_K^{op}} \lim_{\tau\subseteq\sigma} S(\tau) \cong \lim_{(\tau\subseteq\sigma)\in\mathrm{TW}(\mathcal{I}_K)} S(\tau) = \int_{\tau\in\mathcal{I}_K} S(\tau)$$

where the last expression is an ∞-end, and $\mathrm{TW}(\mathcal{I}_K)$ denotes the twisted arrow category. Since this end is mute in one entry, morphisms in the twisted arrow category that only involve changing σ are sent to identities, so it calculates the limit on the left. One can of course also see this more explicitly. □

Theorem 5.7.14 (Poincaré Duality)**.** If K is a triangulation of a closed oriented smooth n-manifold, then for any gluing datum $F \in \mathrm{Fun}(\mathcal{I}_K^{op}, \mathcal{V})$,

$$C^*(DF) = \lim_{\sigma\in\mathcal{I}_K^{op}} \lim_{\tau\subseteq\sigma} F(\tau)^\vee \cong (\lim_{\sigma\in\mathcal{I}_K^{op}} F(\sigma))^\vee[-n] = C^*(F)^\vee[-n] \tag{5.41}$$

Proof. We again combine the limits on the left side into a single limit over the poset $\{\sigma, \tau \in \mathcal{I}_K | \tau \subseteq \sigma\}$ which is ordered by inclusion in τ and containment in σ. Again, this yields a coend that is mute in one variable, so we simplify to

$$C^*(DF) = \lim_{\tau\in\mathcal{I}_K} F(\tau)^\vee \overset{!}{\cong} \operatorname*{colim}_{\sigma\in\mathcal{I}_K} F(\sigma)^\vee[-n] = C^*(F)^\vee[-n] \; . \tag{5.42}$$

If we rewrite $F^\vee =: S$ as a constructible sheaf, then this can be written as

$$C^*S \overset{!}{\cong} C_*S[-n] \; , \tag{5.43}$$

which is just Poincaré duality for PL manifolds as in [Lur11, Lecture 26]. □

Warning. By triangulation of a smooth manifold, we mean a Whitehead triangulation; what we actually want to talk about are triangulations of PL manifolds. Our argument breaks down for topological manifolds, as their triangulations can have links that are not spheres.

Technical Remark. The functor C^* is a special case of the exceptional pushforward introduced in [CDH$^+$20a, 6.5.14], for the terminal map of simplicial complexes $! : K \to *$. One should be able to generalize above result to more general fibrations with manifold fibers. Also, there should be generalizations to triangulations of PL pseudomanifolds if we take perversity into account.

Example 5.7.15. For $K = \partial\Delta^2$, equation 5.42 says that limit and colimit over the diagram

$$
\begin{tikzcd}
F(\{0\})^\vee \arrow[dr] \arrow[drr] & & F(\{1\})^\vee \arrow[dl] \arrow[dr] & & F(\{2\})^\vee \arrow[dll] \arrow[dl] \\
& F(\{0,1\})^\vee & F(\{0,2\})^\vee & F(\{1,2\})^\vee &
\end{tikzcd}
$$

agree. Let us check this explicitly, without using Poincaré Duality, by filling in

$$
\begin{tikzcd}
& & C^*(DF) \arrow[dl] \arrow[d] \arrow[dr] & & \\
& DF(\{0,1\}) \arrow[dl] \arrow[dr] & DF(\{0,2\}) \arrow[dll] \arrow[drr] & DF(\{1,2\}) \arrow[dl] \arrow[dr] & \\
F(\{0\})^\vee \arrow[dr] \arrow[drr] & & F(\{1\})^\vee \arrow[dl] \arrow[dr] & & F(\{2\})^\vee \arrow[dll] \arrow[dl] \\
& F(\{0,1\})^\vee \arrow[dr] & F(\{0,2\})^\vee \arrow[d] & F(\{1,2\})^\vee \arrow[dl] & \\
& & C^*(F)^\vee & &
\end{tikzcd}
$$

where the three middle squares are pullbacks. We can make pullback squares into fiber squares using the formula

$$
\lim\left(A \to B \leftarrow A'\right) = \operatorname{fib}\left(A \oplus A' \to B\right) \ , \tag{5.44}
$$

obtaining a giant diagram

$$
\begin{tikzcd}
& & & C^*(DF) \arrow[dl] \arrow[dr] & & & \\
& & {} \arrow[dl] \arrow[dr] & & {} \arrow[dl] \arrow[dr] & & \\
& DF(\{0,1\}) \arrow[dl] \arrow[dr] & & DF(\{1,2\}) \oplus F(\{0\})^\vee \arrow[dl] \arrow[dr] & & DF(\{0,2\}) \arrow[dl] \arrow[dr] & \\
0 \arrow[dr] & & F(\{1\})^\vee \oplus F(\{0\})^\vee \arrow[dl] \arrow[dr] & & F(\{2\})^\vee \oplus F(\{0\})^\vee \arrow[dl] \arrow[dr] & & 0 \arrow[dl] \\
& F(\{0,1\})^\vee \arrow[dr] & & F(\{1,2\})^\vee \oplus F(\{0\})^\vee \arrow[dl] \arrow[dr] & & F(\{0,2\})^\vee \arrow[dl] & \\
& & {} \arrow[dr] & & {} \arrow[dl] & & \\
& & & C^*(F)^\vee & & &
\end{tikzcd}
$$

where, due to the pasting lemma, every square is a pullback. In particular, from the big square, we read off that $C^*(DF) = 0 \times_{C^*(F)^\vee} 0 = C^*(F)^\vee[-1]$ as anticipated.

Example 5.7.16. The argument in the above example extends to the case where K is the boundary of an n-gon, for $n \geq 1$. It is particularly simple for $n = 1$, where $\mathcal{I}_K = \left(\bullet \rightrightarrows \bullet \right)$. For a gluing datum $F : \mathcal{I}_K^{op} \to \mathcal{V}$ determined by parallel morphisms $f, f' : C \to D$, we have

$$\begin{aligned} C^*(DF) &= \lim_{\mathcal{I}_K} F^\vee = \mathrm{equ}(f^\vee, f'^\vee) = \mathrm{fib}(f^\vee - f'^\vee) = \\ &= \mathrm{cofib}(f^\vee - f'^\vee)[-1] = \mathrm{coequ}(f^\vee, f'^\vee)[-1] = \operatorname*{colim}_{\mathcal{I}_K} F^\vee[-1] = C^*(F)^\vee[-1] \ , \end{aligned}$$

since it is a general property of stable ∞-categories that fiber and cofiber sequences agree. This also yields an alternative proof of 5.2.17.

Warning. In the above claim, 1-gon and 2-gon are no actual simplicial complexes in our sense, since edges are not uniquely determined by the vertices they end on. Our conjecture should still hold for a slight generalization of simplicial complexes, namely for semi-simplicial sets. It does not hold for general simplicial sets, as can be seen from the example of $\Delta^2/\partial\Delta^2$. We will see how to fix this by extending to a semi-simplicial set in the next example.

Example 5.7.17. Let K be the triangulation of S^2 consisting of two vertices v_1, v_2, two edges e_1, e_2 and two 2-dimensional faces σ_1, σ_2. A gluing datum $F : \mathcal{I}_K^{op} \to \mathcal{V}$ is then given by a diagram:

$$\begin{tikzcd} F(\sigma_1) \arrow[r] \arrow[dr] & F(e_1) \arrow[r] \arrow[dr] & F(v_1) \\ F(\sigma_2) \arrow[r] \arrow[ur] & F(e_2) \arrow[r] \arrow[ur] & F(v_2) \end{tikzcd}$$

Let $K_0 \subseteq K$ be the subcomplex spanned by vertices and edges triangulating S^1; by the example above the restriction $F|_{K_0}$ satisfies $C^*D(F|_{K_0}) \cong C^*(F|_{K_0})^\vee[-1] =: F_\partial$. Also, $D(F|_{K_0}) = (DF)|_{K_0}$ since K_0 is closed under containment, so the limits involved in calculating DF only depend on values of F in K_0. This lets us rewrite

$$\begin{aligned} (C^*F)^\vee[-2] &\cong \lim \left(F(\sigma_1) \to C^*(F|_{K_0}) \leftarrow F(\sigma_2)\right)^\vee [-2] \cong \\ &\cong \mathrm{colim}\left(F(\sigma_1)^\vee[-2] \leftarrow F_\partial[-1] \to F(\sigma_2)^\vee[-2]\right) \cong \\ &\cong \mathrm{cofib}\left(F_\partial \to F(\sigma_1)^\vee[-1] \oplus F(\sigma_2)^\vee[-1]\right)[-1] \cong \\ &\cong \mathrm{fib}\left(F_\partial \to (F(\sigma_1) \oplus F(\sigma_2))^\vee[-1]\right) \ , \end{aligned}$$

and our goal is to explicitly show this agrees with $C^*(DF) = \lim_{\mathcal{I}_K} F^\vee$, as 5.7.14 states. Using $F_\partial = C^*(DF|_{K_0})$, this becomes a mere manipulation of limits in stable ∞-categories.

Now, let us finally apply all of this to field theory.

Definition 5.7.18. An *m-dimensional Poincaré object* (F, ω) in the stable ∞-category of gluing data $\mathrm{Fun}(\mathcal{I}_K^{op}, \mathcal{V})$ is an object F equipped with an isomorphism $\omega : F \xrightarrow{\cong} DF[-m]$ that is induced by a symmetric pairing.

Technical Remark. There are two ways to make sense of what we mean by a symmetric pairing. First, notice that $D\omega[-m] : DF[-m] \to F$ is still an isomorphism, and we want to require it to be an inverse to ω. Further, we require the 2-morphisms $\omega \circ D\omega[-m] \to \mathrm{Id}$ and $D\omega[-m] \circ \omega$ to satisfy a tower of higher coherence conditions.

It turns out (see [CDH+20a, 6.3.2]) that for $\mathcal{V} = D^{\mathrm{perf}}(\mathbb{R})$, we may equivalently require ω to be induced by a pairing β that is on object in

$$\lim_{\sigma \in \mathcal{I}_K^{op}} \mathrm{Map}_{D^{\mathrm{perf}}(\mathbb{R})}(F(\sigma) \otimes F(\sigma), \mathbb{R}[-m])^{hS_2} \,, \tag{5.45}$$

a compatible symmetric (up to coherent homotopy) pairing on $F(\sigma)$ for each σ. Properly explaining this would require introducing methods connecting quadratic forms, symmetric bilinear forms and self-dualities in Poincaré ∞-categories which we do not want to do so that we ignore this pairing in the following.

Theorem 5.7.19. For K a triangulation of a compact oriented smooth n-manifold and (F, ω) an m-dimensional Poincaré object in $\mathrm{Fun}(\mathcal{I}_K, \mathcal{V})$, the pair $(C^*F, C^*\omega)$ is an $(n+m)$-dimensional Poincaré object in $\mathcal{V}$.

Proof. We apply the isomorphism $C^*\omega$ and 5.7.14 to calculate

$$C^*F \cong C^*(DF[-m]) \cong (C^*F)^\vee[-n][-m] \cong (C^*F)^\vee[-n-m] \,. \tag{5.46}$$

Abstractly, C^* is a duality-preserving functor and those preserve Poincaré objects. $\square$

Definition 5.7.20. A (free topological) *simplicial BV theory* on a finite simplicial complex K of dimension n is a $(3-n)$-dimensional Poincaré object in $\mathrm{Fun}(\mathcal{I}_K^{op}, \mathcal{V})$.

Remark. By 5.7.9, we could equivalently have defined a simplicial BV theory as a constructible sheaf S equipped with an isomorphism $\mathbb{G}S \to D\mathbb{G}S[3-n] \cong S^\vee[3-n]$. The self-duality this imposes is different from Verdier self-duality, since $\mathbb{G}S(\sigma)^\vee$ takes a colimit over faces of σ, while Verdier duality takes a colimit over simplices containing σ.

Corollary 5.7.21. Given a simplicial BV theory $F : \mathcal{I}_K^{op} \to \mathcal{V}$ on a triangulation of a smooth oriented manifold, its global sections C^*F are a 3-dimensional Poincaré object in $\mathcal{V}$. In other words, they possess a (-1)-shifted symplectic structure as $2-3=-1$.

Remark. Similarly, one can show that for a simplicial BV theory F on a triangulation K of a smooth oriented manifold with boundary, with K_∂ the subcomplex that triangulates the boundary, the canonical map $C^*(F) \to C^*(F|_{K_\partial})$ is a Lagrangian of the 2-dimensional Poincaré object $C^*(F|_{K_\partial})$. We obtain a BV-BFV theory on our manifold! See 5.8.12 for a way to introduce boundary conditions.

Generally, our definition of a simplicial BV-theory is chosen in a way that

- On every individual top-dimensional simplex, we obtain something resembling an extended BV-BFV theory, and
- For a PL triangulation of a manifold with corners, we should obtain a BV-BFV theory on it by taking sections of the restricted gluing data.

One can view it as a middle ground between Lagrangian extended topological field theories and extended BV-BFV theories, see 5.8.13.

Theorem 5.7.22. Any simplicial BV-theory $F : \mathcal{I}_K^{op} \to \mathcal{V}$ on an n-dimensional finite simplicial complex K defines a constructible factorization algebra $\mathcal{O}bs^{cl} := \operatorname{Sym} \mathcal{E}^\vee$ of classical observables on $|K| \to \mathcal{I}_K$, where $\mathcal{E}$ is the constructible sheaf on $|K|$ associated to F.

Proof. We have seen that F can be identified with a constructible sheaf on $|K| \to \mathcal{I}_K$ in 5.7.4 and 5.7.9, which induces a constructible factorization algebra by 4.3.8. □

Let us finally develop several example theories on a fixed compact oriented manifold triangulation K, to show that our considerations are not purely academic:

Definition 5.7.23. For a given gluing datum $F : \mathcal{I}_K^{op} \to \mathcal{V}$, the datum $\operatorname{hyp}(F) := F \oplus DF$ is a Poincaré-object in a canonical way, since $D(F \oplus DF) = DF \oplus D^2F \cong F \oplus DF$. We call it the *hyperbolic Poincaré-object* associated to F. Similarly, we define the *n-dimensional Poincaré-object* associated to F as

$$\operatorname{hyp}^{[n]}(F) := F \oplus DF[-n] \; . \tag{5.47}$$

This n-dimensional Poincaré object admits two canonical Lagrangians $F \to \operatorname{hyp}^{[n]}(F)$ and $DF[-n] \to \operatorname{hyp}^{[n]}(F)$.

Construction 5.7.24. For G a group with associated one-object groupoid BG, we define a *G-local system* on an n-dimensional simplicial complex K to be a functor $\omega : \mathcal{I}_K \to BG$. For a fixed representation $\rho : BG \to \mathcal{V}$ of G (this contains the case of ordinary representations on $\mathbb{R}$-vector spaces if we restrict to functors with values in $\mathrm{Vect}_\mathbb{R} \subseteq D^{\mathrm{perf}}(\mathbb{R})$), we obtain an associated constructible sheaf $A := \rho \circ \omega : \mathcal{I}_K \to \mathcal{V}$. We define *abelian BF-theory* on K as the simplicial field theory defined by the $3-n$-dimensional Poincaré object

$$F_{BF} := \operatorname{hyp}^{[3-n]}(\mathbb{G}A) = \mathbb{G}A \oplus D\mathbb{G}A[-3+n] \; . \tag{5.48}$$

More generally, we could have chosen A to be an arbitrary gluing datum. The global section BV-complex is given by

$$\begin{aligned} C^* F_{BF} &= C^* \mathbb{G} A \,\oplus\, C^* D \mathbb{G} A[-3+n] \cong C^* A \,\oplus\, C^* A^\vee[-3+n] \cong \\ &\cong C^*_{\text{simp}}(K, A) \,\oplus\, C^{\text{simp}}_{*+n-3}(K, A)^\vee \end{aligned} \tag{5.49}$$

because $C^* A^\vee = \lim A^\vee = (\operatorname{colim} A)^\vee = (C_* A)^\vee$.

Remark. On closed, oriented manifolds, this is essentially equivalent to the field theory studied in Section 5 of [CMR20], if we identify C^* of constructible sheaves on the dual cell complex with C_* on the ordinary complex. We explain in the next chapter how our constructions generalize to regular cell complexes, but note at this point that by construction, BF theory does not even require our simplicial complex to be a manifold triangulation. This paper also explains how to quantize such theories, obtaining a partition function that for abelian BF theory depends on Reidemeister torsion and Betti numbers of the manifold.

Remark. Constructing hyperbolic objects might be seen as a perturbative analogue of the *universal bulk theory* in [BY16], as we can locally split the coordinates on the cotangent bundle used for its construction into "positions and momenta" as in $F \oplus DF$.

Construction 5.7.25. For $\mathfrak{g}$ an $\mathbb{R}$-vector space with non-degenerate inner product exhibiting $\mathfrak{g} \cong \mathfrak{g}^\vee$, and assuming K triangulates a compact oriented 3-dimensional manifold, we define *abelian Chern-Simons theory* with values in $\mathfrak{g}$ as the simplicial field theory associated to the constant constructible sheaf $S = \underline{\mathfrak{g}[0]} : \mathcal{I}_K \to D^{\text{perf}}(\mathbb{R})$ sending $\sigma \mapsto \mathfrak{g}[0]$. To see that this is a simplicial BV-theory, evaluate

$$\mathbb{G}\underline{\mathfrak{g}[0]}(\sigma) = \lim_{\tau \subseteq \sigma} \mathfrak{g}[0] \cong C^*_{\text{simp}}(\sigma, \mathfrak{g}) \cong \mathfrak{g}[0] \tag{5.50}$$

where since σ is contractible, its simplicial cochain complex is quasi-isomorphic to $\mathfrak{g}[0]$. Composing with the inner product on $\mathfrak{g}$ yields an isomorphism

$$\mathbb{G}\underline{\mathfrak{g}[0]} \cong \underline{\mathfrak{g}[0]} \cong \underline{\mathfrak{g}[0]}^\vee \tag{5.51}$$

making $\mathbb{G}\underline{\mathfrak{g}[0]}$ into a $(3-3)$-dimensional self-dual gluing datum. The global section BV theory induced by 5.7.21 is

$$C^*\underline{\mathfrak{g}[0]} \cong C^*_{\text{simp}}(K, \mathfrak{g}) \tag{5.52}$$

as expected for Chern-Simons theory (replacing simplicial chains by differential forms), with (-1)-shifted symplectic structure given by the integration pairing since it can be integrated up from contractible subspaces.

Construction 5.7.26. We can also define higher-dimensional abelian Chern-Simons theory. Choose an r-shifted Poincaré object $\mathfrak{g} \cong \mathfrak{g}^\vee[-r]$ in $\mathcal{V}$, where the above corresponds to $r = 0$. If K is n-dimensional with $n+r$ odd, define the constructible sheaf $S : \mathcal{I}_K \to \mathcal{V}$ sending $\sigma \mapsto \mathfrak{g}[s]$ with $s := \frac{n+r-3}{2}$. Then,

$$\mathbb{G}S \cong \underline{\mathfrak{g}[s]} \cong \underline{\mathfrak{g}}^\vee[s-r] \cong S^\vee[2s-r] \tag{5.53}$$

so that $\mathbb{G}S$ is a Poincaré object of dimension $-(2s-r) = 3-n$ and hence defines a simplicial BV theory. This contains as special cases simplicial variations of

- *Topological Quantum Mechanics* if $n = 1, r = 2, s = 0$ and $\mathfrak{g} = V[-1]$ with V a symplectic vector space,
- the *free Poisson σ-model* if $n = 2, r = 1, s = 0$ and we set $\mathfrak{g} = (V^* \to V[-1])$ with V a finite-dimensional vector space and the differential $\Pi : V^* \to V$ induced by an (anti-)symmetric pairing on V, so that $\mathfrak{g}^\vee[-r] = \left(V^*[1] \xrightarrow{\Pi^*} V\right)[-1] \cong \mathfrak{g}$,
- *abelian BF-theory* valued in a finite-dimensional vector space $\mathfrak{h}$ for n arbitrary, $r = 3-n$ and $s = 0$, choosing $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}[3-n]$. This is also a special case of the example above.

Finally, we sketch how interactions can (classically) be incorporated:

Definition 5.7.27. An *interacting simplicial BV-theory* on a finite n-dimensional simplicial complex K is a gluing datum $F : \mathcal{I}_K^{op} \to D^{\mathrm{perf}}(\mathbb{R})$ where

- For each $\sigma \in \mathcal{I}_K$, the complex $F(\sigma)$ is equipped with the structure of an L_∞-algebra, such that for $\sigma \subseteq \tau$ the map $F(\tau) \to F(\sigma)$ is a morphism of L_∞-algebras,
- There is a symmetric pairing $\beta = (\beta_\sigma)_{\sigma \in \mathcal{I}_K}$ inducing $F \cong DF[n-3]$ as in the free case,
- The symmetric pairing is compatible with the L_∞-structure in the sense that for each $\sigma \in \mathcal{I}_K$ and $k \in \mathbb{N}_{>1}$, the map

$$\beta_\sigma(-, \ell_k(-, \dots, -)) : F(\sigma)^{\otimes k} \to \mathbb{R}[n-3] \tag{5.54}$$

  is totally graded antisymmetric.

Construction 5.7.28. If $\mathfrak{g}$ is a Lie-algebra with ad-invariant non-degenerate inner product $\kappa : \mathfrak{g} \cong \mathfrak{g}^\vee$, we can define non-abelian Chern-Simons theory as an interacting simplicial BV-theory on the constructible sheaf $S = \underline{\mathfrak{g}[0]}$. Unlike in the abelian case, $\mathbb{G}S \cong \underline{\mathfrak{g}}[0]$ obtains a non-trivial Lie bracket that is part of a simplex-wise L_∞-structure on this gluing datum. The self-duality isomorphism is compatible with the Lie bracket in the sense that $\beta_\sigma(-, [-, -])$ is indeed totally antisymmetric: Since β_σ is, up to quasi-isomorphism, just given by κ, this is immediate from its ad-invariance.

Remark. This also works for higher-dimensional Chern-Simons theory, yielding in particular a simplicial analog of classical non-abelian BF theory if $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\vee[n-3]$ for $\mathfrak{h}$ a Lie algebra with an ad-invariant symmetric pairing, inducing a graded Lie bracket on $\mathfrak{g}$. Further interesting examples to consider would be the Poisson σ-model with a non-linear target, or quantizations of the above theories.

Remark. In the limit where the triangulation on the respective manifold gets finer and finer, these simplicial theories should approximate the respective physical theories. For the example of BF-theory, this is explained in [CMR20].

5.8. Epilogue: Further Developments

We have defined free topological field theories on manifolds with boundaries, and on finite simplicial complexes. Let us conclude with an outlook on how we would proceed on more general classes of spaces. The most straightforward generalization is passing from a fixed triangulation on a space to a whole class of triangulations:

Definition 5.8.1. A *piecewise linear space*, or *PL space* in short, is a topological space X together with a set $\mathcal{T}$ of locally finite triangulations such that

- if $T \in \mathcal{T}$, then the barycentric subdivision $\mathrm{sd}(T)$ is also in $\mathcal{T}$, and
- any $T, T' \in \mathcal{T}$ possess a common refinement T''.

We refer to the standard literature (e.g. [Lur11, Lecture 17]) for more information, and different characterizations. In particular, one can define a notion of *PL maps* between PL spaces that preserve the PL structure, making PL spaces into a category.

Definition 5.8.2. An *n-dimensional PL manifold* is a PL space X that is locally isomorphic to $\mathbb{R}^n$ with its canonical PL structure, in the sense that any point $x \in X$ has an open neighborhood that, together with its restricted PL structure, possesses mutually inverse PL maps to and from $\mathbb{R}^n$.

Definition 5.8.3. If K is a simplicial complex and $\sigma \in \mathcal{I}_K$ a simplex, the *star* of σ is the set of simplices in K that have a non-empty intersection with σ, closed under the operation of taking faces (so that it becomes a simplicial complex itself). The *link* of σ consists of those simplices in the star that do not intersect σ. We identify star and link with the respective subspaces of $|K|$ spanned by their simplices.

Example 5.8.4. Usually, we are interested in the case where $\sigma = \{v\}$, so the star of $\{v\}$ is the closure of all simplices containing v as a vertex. If we for example let $v = 0 \in \Delta^n$, then its star is all of Δ^n, and its link consists of those simplices in Δ^n that do not contain v, making up $\Delta^{1<\cdots<n} \cong \Delta^{n-1}$.

Proposition 5.8.5. A PL space X is an n-dimensional PL manifold iff it possesses a triangulation K such that for every vertex $v \in K_0$, (the geometric realization of) its link is homeomorphic to S^{n-1}. This then automatically holds for every triangulation of X. In particular, (Whitehead) triangulations of smooth manifolds yield PL manifolds, but not triangulations of topological manifolds.

Definition 5.8.6. For X a PL space with set of triangulations $\mathcal{T}$ ordered by refinement, we define the ∞-category of *combinatorial sheaves* on M as the direct limit

$$\mathcal{S}h^{comb}(X, \mathcal{V}) := \lim_{T \in \mathcal{T}} \mathrm{Fun}(\mathcal{I}_T, \mathcal{V}) \tag{5.55}$$

where for S a refinement of T, the transition maps in this limit are given by right Kan extension along the induced map $\mathcal{I}_S \to \mathcal{I}_T$ since every simplex of S is contained in a corresponding simplex of T. By [CDH+20a, 6.6.2], the duality D carried out componentwise is still well-defined on this limit, making $\mathcal{S}h^{comb}(X, \mathcal{V})$ into a stable ∞-category with duality functor.

Definition 5.8.7. A *BV field theory* on an n-dimensional PL space X is a $(3-n)$-dimensional Poincaré object S in $\mathcal{S}h^{comb}(X, \mathcal{V})$; spelling this out, for every triangulation T, we must specify a simplicial BV theory on T determined by a functor $S_T : \mathcal{I}_T \to \mathcal{V}$, together with a symmetric pairing exhibiting $\mathbb{G}S_T \cong S_T^\vee[3-n]$ and compatible with pushing forward (via right Kan extension) along refinements of triangulation.

Example 5.8.8. One can check the our constructions of cellular field theories are indeed compatible with refinements, thus defining field theories on compact oriented PL manifolds. In fact, most results of the last section carry over to the PL case. We still expect relatively good properties in the slightly more general case of PL pseudomanifolds.

Proposition 5.8.9. By the discussion in the last section, the global sections of a BV-theory on PL manifolds (defined as the global sections of the induced constructible sheaf on any triangulation, since those agree) yield a (-1)-shifted symplectic object in $\mathcal{V}$.

Next, let us discuss the case of regular CW complexes and cellular field theories.

Construction 5.8.10. Let X be a finite, regular CW complex (also called finite ball complex), where by regular we mean that for every cell in X, the respective gluing map $D^n \to X$ is injective, i.e. the boundary of the cell is not collapsed in any way. Compare this to the distinction between simplicial complexes and simplicial sets. We show in B.2.18 that if we stratify X by its poset of cells $\mathcal{I}_X$, then $\mathrm{Sing}^{\mathcal{I}_X}(X) \simeq \mathcal{I}_X$ so that constructible sheaves can be identified with functors $\mathcal{I}_X \to \mathcal{V}$. The discussion in the last section carries through without any changes, if we replace triangulations of compact oriented smooth manifolds by cell decompositions. Even Poincaré Duality can still be applied following [CMR20, Equation 8].

Definition 5.8.11. A *cellular BV theory* on a finite regular CW complex X of dimension n is a $(3-n)$-dimensional Poincaré object in $\mathrm{Fun}(\mathcal{I}_X^{op}, \mathcal{V})$.

Definition 5.8.12. Let X be a finite, regular CW complex that makes up a cell decomposition of an n-dimensional compact oriented manifold M with boundary, and let $F : \mathcal{I}_X^{op} \to \mathcal{V}$ be a cellular BV-theory, i.e. $F \cong DF[n-3]$. Denote by Y the subcomplex that decomposes the boundary ∂M. A *polarization* or *boundary condition* is an element $P \in \mathcal{V}$, together with a natural transformation $\rho : \underline{P} \to F|_{\mathcal{I}_Y^{op}}$ of functors $\mathcal{I}_Y^{op} \to \mathcal{V}$, such that the adjoint map $P \to C^*(F|_{\mathcal{I}_Y^{op}})$ is Lagrangian (note that the target is a $(2-n)$-dimensional Poincaré object by 5.7.14). Using 5.2.15 and the remark after 5.7.21, we can glue the Lagrangians

$$C^*(F) \to C^*(F|_{\mathcal{I}_Y^{op}}) \leftarrow P \tag{5.56}$$

to obtain a $(3-n)$-dimensional Poincaré object $C_P^*(F) := C^*(F) \times_{C^*(F|_{\mathcal{I}_Y^{op}})} P \in \mathcal{V}$ of fields satisfying the boundary condition P.

Finally, let us discuss how to systematically define simplicial field theories. We start with an n-dimensional *Lagrangian extended field theory* as discussed in [CHS21] and [Cal14]. This is a symmetric monoidal functor of (∞, d)-categories

$$Z : \mathbf{Bord}_{\boldsymbol{n}}{}^{\sqcup} \to \mathbf{Lagr}_{\boldsymbol{n}}{}^{\times}\,, \tag{5.57}$$

i.e. a fully extended topological field theory as in 5.1.2 with values in the (∞, n)-category of *Lagrangian correspondences*, with

- Objects derived Artin stacks equipped with a $(3-n)$-shifted symplectic structure,
- Morphisms Lagrangian correspondences $X \leftarrow L \to Y$ between shifted symplectic stacks,
- Higher Morphisms up to degree n given by higher Lagrangian correspondences,
- $(n+1)$-morphisms given by homotopies of higher Lagrangian correspondences,
- Higher Morphisms given by higher homotopies.

While we do not give a precise definition, the notion of shifted symplectic stacks and Lagrangian correspondences yields on tangent spaces precisely our notions of Poincaré complexes and Lagrangian correspondences in 5.2, and higher Lagrangian correspondences are on the level of tangent spaces the same thing as Poincaré objects on Δ^n in 5.7. In particular, this should allow us to do the following construction:

Construction 5.8.13. Given a Lagrangian extended field theory (for example, an AKSZ theory) Z and an n-manifold with borders M, we can

- Triangulate M by a simplicial complex K,

- Regard the simplices in K as n-manifolds with borders, i.e. higher bordisms, whose composition as n-morphisms in $\mathbf{Bord}_{\boldsymbol{n}}$ is M regarded as a higher bordism,
- Apply Z to this composition, obtaining a system of derived Artin stacks and higher Lagrangian correspondences,
- Choose a common geometric point ϕ of this system as in the diagram below,
- Take the tangent complex at ϕ for every involved stack, obtaining a Poincaré object in $\mathbb{T}_\phi X_- : \mathrm{Fun}(\mathcal{I}_K^{op}, \mathcal{D})$
- and associate to this a constructible sheaf $\mathcal{E}$ on $|K| \simeq M \to \mathcal{I}_K$.

Or course, this involves that our chosen category $\mathcal{D}$ is adapted to our notion of derived stack as in [Wal16], and it would require some more work to show that all of these definitions fit together. In particular, one should hope that this does not depend on how we understand M as a higher bordism (i.e. which is the incoming, which the going side etc.), and that the sheaf $\mathcal{E}$ is not only constructible over $\mathcal{I}_K$, but also when we stratify M by its boundary and corner components. Below, we have sketched how our diagram of Lagrangian correspondences of derived stacks looks like for $K = \Delta^2$; we regard the value X on the interior as the covariant phase space of our theory.

$$\begin{tikzcd}
 & & & X \arrow[dl] \arrow[d] \arrow[dr] & & \\
 & & X_{01} \arrow[dl] \arrow[dr] & X_{02} \arrow[dll] \arrow[drr] & X_{12} \arrow[dl] \arrow[dr] & \\
\mathrm{Spec}(\mathbb{R}) \arrow[uurrr, dashed, bend left, "\phi"] & X_0 & & X_1 & & X_2
\end{tikzcd}$$

Note how ϕ induces compatible geometric points on all involved stacks.

Upshot: Extended field theories possess an intrinsic notion of locality, since we can always triangulate spaces and glue the field theory together from this composition of higher bordisms. This cutting-and-gluing notion of locality is at first glance in stark contrast with the locality that BV data possess by being ∞-sheaves. The construction above would show that in the regarded special case, triangulating and transitioning to perturbation theory allows us to go from the former to the latter! This should make simplicial (and cellular) BV theories a very useful tool in the study of perturbative properties of extended field theories, particularly since it is extremely easy to write down examples as we had seen.

Finally, let us sketch how in our language, extended BFV theories can be defined on arbitrary manifolds with corners.

Definition 5.8.14. A BV-theory on an n-dimensional topological manifold with corners $M \to [n]$ is determined by a constructible sheaf $\mathcal{E} \in \mathcal{S}h^{cbl}(X; \hat{\mathcal{D}})$ such that

- The stalk $\mathcal{E}_x$ at every point $x \in M$ lies in $\mathcal{D} \subseteq \hat{\mathcal{D}}$,
- The restriction $\mathcal{E}|_{M_{n-k}}$ to the stratum M_{n-k} of codimension k is a $(3-k)$-dimensional Verdier self-dual sheaf on M_{n-k}. By this, we mean that there is an isomorphism $\mathcal{E} \cong \mathcal{E}^\vee \otimes \mathrm{Dens}_{M_{n-k}}[k-3]$ induced by a symmetric pairing.
- If we denote by $i_k : M_k \hookrightarrow M$ the inclusions of strata, then the sheaves $(i_k)_*(i_k)^*\mathcal{E}$ form a system of Poincaré objects and Lagrangians in $\mathcal{S}h^{cbl}(X; \hat{\mathcal{D}})$ with maps induced by the recollement decomposition 5.5.4 extended to stratifications.

Remark. This definition even makes sense on a topological pseudomanifold with corners, if we replace $\mathrm{Dens}_{M_{n-k}}$ by the Verdier dualizing complex $\omega_{M_{n-k}}$.

In other words, the pair of M and $\mathcal{E}$ should be a higher bordism of self dual sheaves, generalizing [Ban01, Chapter 4] or [Lur11, Lecture 26] in the case of ordinary bordisms. A fist step at getting a better understanding of this is our study of the algebraic L-theory of $\mathcal{S}h^{cbl}(M, \hat{\mathcal{D}})$ in [Zet], which is helpful to precisely formulate the last part of the definition above.

A. Higher Categories and Higher Algebra

For a complete introduction to higher category theory, we refer the reader to [KER], since we definitely can not do this subject justice here. We try to give a working intuition for the definitions we need in the main text without delving into too many technicalities, which unfortunately means that we often have to be a bit imprecise or refer to the literature.

A.1. ∞-categories

We assume the reader is familiar with basic category theory (e.g. limits, adjunctions, slice categories), enriched categories and Kan extensions. Let us still, for comparison, repeat the definition of an ordinary category:

Definition A.1.1. A (small) category $\mathcal{C}$ consist of

- A set of objects
- For any two objects $X, Y \in \mathcal{C}$ a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms between X and Y
- For all $X, Y, Z \in \mathcal{C}$ an associative composition map

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z) \tag{A.1}$$

- For any $X \in \mathcal{C}$ an identity morphism id_X that does not change morphisms under composition

Remark. Being small refers to the fact that objects and morphisms are sets (in a fixed universe), we will also encounter many cases where this is not the case. Still, let us avoid set-theoretic problems unless they are actually important.

Definition A.1.2. A category if called a *groupoid* if every morphism $f : C \to D$ in it is *invertible*, i.e. there exists a $g : D \to C$ such that $f \circ g = \mathrm{id}_D$ and $g \circ f = \mathrm{id}_C$.

Example A.1.3.
- Examples of categories can be found all over mathematics, e.g. the category Set of sets and maps between them, the category Top of topological spaces and continuous maps, the category Ab of abelian groups and homomorphisms or the category Cat categories and functors.
- There are also important examples of groupoids: For each group G, we can construct a groupoid BG with one object $*$ and $\mathrm{Hom}_{BG}(*,*) = G$, where composition is given by the group operation and inverses exist because G has inverses.
- For X a topological space, we may also introduce the *fundamental groupoid* $\pi_{\leq 1}X$ with objects the points of X, and morphisms the homotopy classes of paths between the respective points. Composition is given by concatenation of paths, and inverses exist since paths can be followed in the inverse direction.

Some of these examples seem to possess further information that we were not able to capture:

- Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, the category Cat allows for a space of natural transformations $\mathrm{Nat}(F, G)$. In other words, there are morphisms between morphisms, and these can also be composed.
- It is a bit strange that $\pi_{\leq 1}X$ contains a large amount of objects (uncountably many for almost all manifolds), but the morphisms only consist of homotopy classes of paths, instead of actual paths. Why is that? Note that concatenation of paths is, in itself, not associative, but only so up to a reparametrization (i.e. a homotopy)! If we want to keep information about individual paths, we therefore need to add information about homotopies into the mix.

Both of these problems can be resolved by 2-categories, also called *bicategories*. They should consist of a set of objects, together with a set of morphisms between any two objects and a set of 2-morphisms between any two morphisms that have a common source and target. Also, they feature composition operations for morphisms and 2-morphisms, as well as associativity constraints and identity (2-)morphisms. Composition of 1-morphisms should only be associative up to an invertible 2-morphisms (the associator), and identity 1-morphisms should only act as identities up to invertible 2-morphisms as well; we see this in the second example since concatenation of paths is not strictly associative. We therefore always speak about *weak* 2-categories, instead of *strict* 2-categories where associativity and identity conditions hold on the nose. Finally, the invertible 2-morphisms in the above definition should be considered as extra data in a 2-category, and they must themselves satisfy higher coherence relations, like the pentagon identity (see [KER, Tag 007Q] for a precise definition).

- The (strict) 2-category ***Cat*** consists of (small) categories as objects, functors as 1-morphisms and natural transformations as 2-morphisms.

- The (weak) 2-category $\pi_{\leq 2}X$ of a topological space X consists of points of X as objects, paths in X as morphisms, and homotopy classes of homotopies as 2-morphisms (we need to think about homotopy classes again to satisfy the strict associativity for 2-morphisms). This is even a *2-groupoid*, since morphisms and 2-morphisms are invertible.

But now, the second example suffers from a similar issue concerning homotopy classes as before. This points us toward a straightforward idea: Why do we not define 3-categories, 4-categories etc., as well as fundamental n-groupoids $\pi_{\leq 3}X, \pi_{\leq 4}X, \ldots$ consisting of objects, morphisms, 2-morphisms, 3-morphisms and so on? The reasons why this is not a priori a good idea:

- We need to add composition operations for each type of morphism, that can interact with each other (horizontal composition, whiskering) and satisfy associativity and identity constraints up to higher isomorphisms – that also need to be part of our data! Also, these higher isomorphisms must satisfy their own coherence relations up to even higher isomorphisms, which are subject to even higher coherence relations and so on. Even the definition of a (weak) 3-category is so complicated that it is extremely hard to work with – of course, things are a lot simpler for strict n-categories.
- Even if we could define an n-category (even n-groupoid) $\pi_{\leq n}X$ for each $n \in \mathbb{N}$, this still would not resolve our problem since the n-morphisms are still given by homotopy classes of maps.

Surprisingly, it is possible to resolve both problems at once by figuratively going two steps forward and one step back: Things surprisingly become a lot simpler when we do not look at n-categories, but at n-groupoids, where m-morphisms for all $1 \leq m \leq n$ are invertible. Letting n go towards ∞, there should be for each topological space X an ∞-groupoid $\pi_{\leq\infty}X$ that knows about points, paths, homotopies, homotopies of homotopies etc. in X. Since homotopies from the constant path to itself are just embedding of S^2 into X, and similarly for the other levels, this means that $\pi_{\leq\infty}X$ knows about all homotopy groups and hence, at least if X is a CW complex (by Whitehead), the full homotopy type of X. Thus, ∞-groupoids, which contain n-groupoids as special cases, are intimately related to (CW) topology and homotopy theory – but their definition should still be "algebraic", which can be achieved by working with simplicial complexes as models. A bit of ordinary category theory is necessary to understand it:

Definition A.1.4. We define the *presheaf category* of a given small category $\mathcal{C}$ as $\mathrm{PSh}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$, note that it is never small unless $\mathcal{C} = \emptyset$. There is always a fully faithful, limit-preserving functor $h : \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$, the *Yoneda embedding*, which sends $C \mapsto \mathrm{Hom}_{\mathcal{C}}(-, C)$.

Theorem A.1.5 (coYoneda Lemma)**.** For $\mathcal{C}$ a (small) category, any presheaf $F \in \mathrm{PSh}(\mathcal{C})$ may be written as a colimit of representable ones (i.e. those that lie in the image of the

Yoneda embedding):

$$\forall C' \in \mathcal{C}: \; F(C') = \operatorname*{colim}_{C \in \mathcal{C}_{/F}} \mathrm{Hom}_{\mathcal{C}}(C', C) = \int^{C \in \mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(C', C) \times F(C) \tag{A.2}$$

Remark. $\mathcal{C}_{/F}$ denotes the generalized slice category (also called comma category) $\mathcal{C} \times_{\mathrm{PSh}(\mathcal{C})} \mathrm{PSh}(\mathcal{C})_{/F}$, which by the Yoneda lemma agrees with the category of elements $\int F$ (i.e. the category of pairs (C, a) with $C \in \mathcal{C}$ and $a \in F(C)$). The latter coend expression is also called the *Ninja Yoneda Lemma* [Lor15], it tells us that the Hom-functor acts as a delta distribution in the coend. We will not use it further, but it is helpful in the proof.

Proof. We start by recalling the usual Yoneda Lemma:

$$F(C') = \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(-, C'), F) \tag{A.3}$$

For $G \in \mathrm{PSh}(\mathcal{C})$ any other presheaf, above colimit is characterized by

$$\mathrm{Nat}(\operatorname*{colim}_{C \in \mathcal{C}_{/F}} \mathrm{Hom}_{\mathcal{C}}(-, C), G) \cong \lim_{C \in \mathcal{C}_{/F}} \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(-, C), G) = \lim_{C \in \int F} G(C) \tag{A.4}$$

As the last limit is taken in Set, we may describe it as the set of families $(b_{C,a} \in G(C))_{C \in \mathcal{C}, a \in F(C)}$ such that for any morphism $f : C \to C'$ in $\mathcal{C}$, we have the compatibility $b_{F(C'),F(f)(a)} = G(f)(b_{C,a})$. Rewriting this, we see that $\eta_C : F(C) \to G(C)$ sending $a \mapsto b_{C,a}$ assemble into a natural transformation $F \Rightarrow G$. In other words, the limit agrees with $\mathrm{Nat}(F, G)$, as claimed. □

Technical Remark. While this was quite cumbersome, proving the coend expression is a lot easier (and the colimit expression can ultimately be derived from it, using how weighted colimits/ Kan extensions can be written as coends). Let $S \in \mathrm{Set}$, then:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Set}}\left(\int^{C \in \mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(C', C) \times F(C), S\right) &\cong \int_{C \in \mathcal{C}} \mathrm{Hom}_{\mathrm{Set}}\left(\mathrm{Hom}_{\mathcal{C}}(C', C) \times F(C), S\right) \cong \\ &\cong \int_{C \in \mathcal{C}} \mathrm{Hom}_{\mathrm{Set}}\left(\mathrm{Hom}_{\mathcal{C}}(C', C), \mathrm{Hom}_{\mathrm{Set}}(F(C), S)\right) \cong \\ &\cong \mathrm{Nat}\left(\mathrm{Hom}_{\mathcal{C}}(C', -), \mathrm{Hom}_{\mathrm{Set}}(F(-), S)\right) \cong \mathrm{Hom}_{\mathrm{Set}}(F(C'), S) \end{aligned}$$

Corollary A.1.6. Let $\mathcal{C}, \mathcal{D}$ be small categories, and let $\mathcal{D}$ contain all small colimits. Then, precomposing with the Yoneda embedding h induces an isomorphism

$$\mathrm{Fun}^{colim}(\mathrm{PSh}(\mathcal{C}), \mathcal{D}) \cong \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \quad , \tag{A.5}$$

where Fun^{colim} denotes the colimit-preserving functors and the inverse is given by *Yoneda extension*, i.e. Left Kan Extension Lan_h. In other words, any colimit-preserving functor on the presheaf-category of $\mathcal{C}$ is determined by its action on $\mathcal{C}$.

Theorem A.1.7 (Nerve-Realization Paradigm)**.** Let $r : \mathcal{C} \to \mathcal{D}$ be a functor, $\mathcal{C}$ be small, and $\mathcal{D}$ admit all (small) colimits. This functor induces an adjunction

$$\begin{tikzcd} \mathrm{PSh}(\mathcal{C}) \arrow[r, shift right, "|-|"'] & \mathcal{D} \arrow[l, shift right, "N"'] \end{tikzcd}$$

where the Yoneda extension $|-| := \mathrm{Lan}_h\, r$ is called the associated *realization* functor, and $N(D) := \mathrm{Hom}_{\mathcal{D}}(r(-), D)$ the associated *nerve*. In fact, any adjunction containing a presheaf category arises in this way.

Proof. The left Kan extension exists because $\mathcal{D}$ has all colimits, we show that $|-| \dashv N$. For $D \in \mathcal{D}, F \in \mathrm{PSh}(\mathcal{C})$, we must construct a natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(\mathrm{Lan}_h\, r(F), D) \cong \mathrm{Nat}(F, \mathrm{Hom}_{\mathcal{D}}(r(-), D)) \ . \tag{A.6}$$

Both sides send colimits in the argument F to limits, and by the coYoneda lemma above the presheaf F is a colimit of representable presheaves. Without loss of generality, we may therefore assume that $F = \mathrm{Hom}_{\mathcal{C}}(-, C)$ is representable. But then

$$\mathrm{Hom}_{\mathcal{D}}(\mathrm{Lan}_h\, r(F), D) \cong \mathrm{Hom}_{\mathcal{D}}(r(C), D) \cong \mathrm{Nat}(\mathrm{Hom}(,C), \mathrm{Hom}_{\mathcal{D}}(r(-), D)) \ , \tag{A.7}$$

where we use A.1.6 in the first equality (the universal property of presheaf category and Yoneda extension), and the Yoneda lemma in the second. $\square$

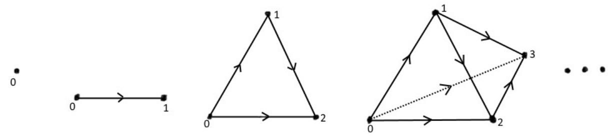
Now, let us apply this to construct models for ∞-groupoids and ∞-categories:

Definition A.1.8. The *simplex category* Δ consists of the nonempty finite totally ordered sets $[n] = \{0 < 1 < 2 < \cdots < n\}$, for $n \in \mathbb{N}_0$, as objects; and order-preserving maps as morphisms.

Definition A.1.9. A *simplicial set* is a functor $X : \Delta^{op} \to \mathrm{Set}$. Let us write $\mathrm{sSet} := \mathrm{PSh}(\Delta)$ for their category. We denote $X_n := X([n])$, and the Yoneda embedding $h([n]) = \mathrm{Hom}_{\Delta}(-, [n]) =: \Delta^n$.

By the coYoneda-Lemma, elements of sSet are colimits of representable presheaves Δ^n, and the presheaf category is in some sense freely generated by such colimits. Morphisms between the Δ^n are, since the Yoneda embedding is fully faithful, the same thing as morphisms in Δ, i.e. order-preserving maps. These can be written as compositions of *face maps* that leave out one number, like $[1] \to [2]$ via $0 \mapsto 0, 1 \mapsto 2$; and *degeneracy maps* that double one number, like $[2] \to [1]$ via $0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 1$. Geometrically, we should imagine $[n]$ and Δ^n as n-simplices, i.e. n-dimensional triangles/pyramids with vertices labeled by the numbers 0 to n, so that these maps can be identified with face inclusions, and regarding an n-Simplex as a degenerate $(n+1)$-simplex (e.g. regarding a line as a triangle with an angle of 0°).

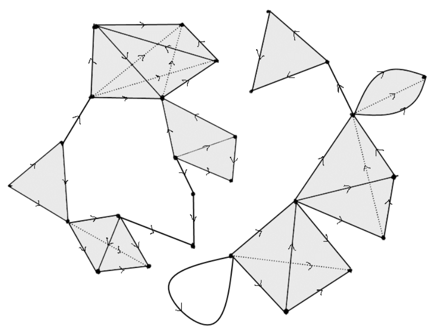
Figure A.1.: Objects of Δ regarded as topological simplices



Due to the fact that presheaves are free gluings of these representables, we expect that simplicial sets are abstract gluings of simplices along their faces, in other words a slightly generalized version of simplicial complexes:

Example A.1.10.
- Δ^n for any $n \geq 0$ are simplicial sets.
- The boundary $\partial\Delta^n$ is the sub-simplicial set of Δ^n that is obtained when erasing the interior. For example, $\partial\Delta^1 : \Delta^{op} \to \mathrm{Set}$ sends each $[n] \in \Delta$ to the order-preserving maps $[n] \to [1]$ that are not surjective – there are precisely two of those, corresponding to $\partial\Delta^1 \cong \Delta^0 \amalg \Delta^0$.
- The *horn* Λ^n_i, for $0 \leq i \leq n$, is the sub-simplicial set of Δ^n that is obtained when erasing both the interior and the face opposite to the vertex i.

Figure A.2.: Example of a simplicial set



Example A.1.11. Let us define a functor $r_{top} : \Delta \to \mathrm{Top}$ sending $[n]$ to the topological n-Simplex $|\Delta^n| := \{(x_0, \dots, x_n) \in [0,1]^{n+1} \,|\, x_1 + \cdots + x_n = 1\}$, with the action on morphisms that we geometrically expect. Since Top has all colimits, we may employ the nerve-realization paradigm to obtain an adjunction

$$\begin{tikzcd} \mathrm{sSet} \arrow[r, shift right, "|-|"'] & \mathrm{Top} \arrow[l, shift right, "\mathrm{Sing}"'] \end{tikzcd}$$

where $|-|$ is called *geometric realization* and $\mathrm{Sing}(X) = \mathrm{Hom}_{\mathrm{Top}}(|\Delta^\bullet|, X)$ is the *singular simplicial set* of a topological space X.

Definition A.1.12. A *Kan complex* is a simplicial set K that satisfies the horn filler property: Any map of simplicial sets $\Lambda_i^n \to K$ can be filled, i.e. extended, to a map $\Delta^n \to K$ such that the following diagram commutes:

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d, hook] & K \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Theorem A.1.13 (Homotopy hypothesis, [KER, Tag 012Y])**.** For any topological space X, the singular simplicial set $\mathrm{Sing}(X)$ is a Kan complex. The adjunction $|-| \dashv \mathrm{Sing}$ induces an equivalence of categories between CW-complexes and Kan complexes. In fact, it even induces a Quillen equivalence between sSet (with the Quillen model structure) and Top that induces above equivalence on homotopy categories.

Kan complexes *are* homotopy types!

This resolves our first problem: Kan complexes should be the same thing as ∞-groupoids, if we regard their vertices as objects, edges as morphisms, n-simplices as n-morphisms. We say that a morphism h in a Kan complex K is a composition of morphisms f, g if there is a 2-simplex $\sigma \in K_2$ such that, identifying σ with a map $\Delta^2 \to K$ via the Yoneda lemma, restriction of this map to the boundary component $\{0 < 2\}$ agrees with h, while the restrictions to $\{0 < 1\}$ and $\{1 < 2\}$ agree with f and g, respectively. We say that σ *witnesses* h as a composition $g \circ f$.

Such a composition exists for any morphisms $f : X \to Y$ and $g : Y \to Z$ as can be seen by filling up Λ_1^2; but since this filling is not required to be unique, the composition of morphisms is not uniquely defined. However, using higher horn fillers, one can show it is unique up to a contractible space of choices. Composition of higher morphisms, as well as associativity, are witnessed by higher horn fillers; and identity n-morphisms are induced by the degeneracy maps. In particular, by filling up the other horns Λ_0^2 and Λ_2^2 where one edge is degenerate, we see that every morphism in a Kan complex has an inverse.

We have also solved our problem concerning fundamental ∞-groupoids if we set $\pi_{\leq\infty}(X) := \mathrm{Sing}(X)$. Because of the homotopy hypothesis, our wish that this should know about the entire homotopy type of a CW-complex comes true. But what about ∞-categories?

Example A.1.14. Let $r_{cat} : \Delta \to \mathrm{Cat}$ be the functor that sends the partially ordered set $[n]$ to the corresponding thin category with objects $0, \ldots, n$. Again, we can apply the nerve-realization paradigm to obtain an adjunction

$$\begin{tikzcd} \mathrm{sSet} \arrow[r, shift right, "h"'] & \mathrm{Cat} \arrow[l, shift right, "\mathrm{N}"'] \end{tikzcd}$$

where hX is called the *homotopy category* of X. The nerve functor N is fully faithful, so categories are a special case of simplicial sets; but $N\mathcal{C}$ is a Kan complex iff $\mathcal{C}$ is a groupoid.

The problem is that if $\mathcal{C}$ is not a groupoid, then the horns Λ_0^2 and Λ_2^2 will not always have fillers, since these would require the existence of inverse morphisms (in the case of degenerate simplices). We therefore must relax the horn filler condition:

Definition A.1.15. A simplicial set X is called *quasi-category* if it satisfies the weak horn filler condition: Any *inner horn* $\Lambda_n^i \to X$ with $0 < i < n$ can be extended to Δ^n.

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d, hook] & K \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

We will interchangeably also use the terms ∞-category or $(\infty, 1)$-category for this construction; the difference in terminology is useful to distinguish the explicit simplicial model we have constructed from the abstract, ontological concept of a higher category that we tried to motivate in the beginning. In particular, morphisms in an $(\infty, 1)$-category can be non-invertible, while one can show that all n-vertices in a quasi-category, for $n > 1$, are invertible in some sense – this is what the 1 in the name refers to. Clearly, every Kan complex is a quasi-category; also the nerve of an ordinary category is one (in fact, ordinary categories are precisely those quasi-categories where the choice of an inner horn filler is always unique).

But what about $(\infty, 2)$-categories? To find a common generalization of $(\infty, 1)$-categories and 2-categories, we should as a fist step find a fully faithful functor from 2-categories into simplicial sets, as we did for 1-categories.

Definition A.1.16 ([KER, Tag 009T])**.** The *Duskin nerve* of a 2-category is defined via the nerve-realization paradigm; applied to the functor $r_{2-cat} : \Delta \to \mathrm{Cat}_2$, which is given by composing r_{cat} with the inclusion of categories into 2-categories. It is fully faithful, and the Duskin nerve of a $(2, 1)$-category is a quasi-category.

However, the Duskin nerve of a 2-category with non-invertible 2-morphisms can never be a quasi-category. One can however proceed as above, and define a class of simplicial sets that contains Duskin nerves to model $(\infty, 2)$-categories. This is a lot more complicated than for quasi-categories, see [KER, Tag 01W6].

There are also notions for (∞, k)-categories, with $k \in \mathbb{N}_0$, but these generally follow a slightly different philosophy in their definition – see [Lur09] for more. Also, there are currently two different notions of (∞, ∞)-categories, via a projective or an inductive limit in k; and both are still poorly developed. We only need $k \leq 2$ in this text.

k \ n	-2	-1	0	1	2	...	∞
0	point	boolean	set	groupoid	2-groupoid		∞-groupoid
1	"	"	poset	category	$(2,1)$-category		$(\infty,1)$-category
2	"	"	"	2-poset	2-category	...	$(\infty,2)$-category
...					...		
∞	"	"	"	"			(∞,∞)-category ?

The inclusion functors in vertical and horizontal direction in this chart have adjoint functors that we will make use of regularly. We already know that the nerve functor from categories to $(\infty, 1)$-categories, and the Duskin nerve from 2-categories to $(\infty, 2)$-categories, have left adjoints (the homotopy category and the homotopy 2-category).

Definition A.1.17. Given an $(\infty, 1)$-category $\mathcal{C}$, we can forget all non-invertible 1-morphisms, obtaining its underlying ∞-groupoid $\mathcal{C}^{\simeq}$. Similarly, given an $(\infty, 2)$-category $\boldsymbol{\mathcal{C}}$, we can forget all non-invertible 2-morphisms, yielding an $(\infty, 1)$-category $\mathrm{Pith}(\boldsymbol{\mathcal{C}})$ called its *pith*. These functors are right adjoint to the respective inclusions.

Left adjoints are harder to construct, but also of interest.

- By localizing (as defined in A.2.3) an ∞-category at all 1-morphisms, one obtains a Kan complex, a process called *Quillen fibrant replacement* (equivalently one can apply Kan's Ex^{∞}-functor).
- Similarly, a sort of localization of an $(\infty, 2)$-category at all 2-morphisms is called *Joyal fibrant replacement*.

In fact, both of these constructions can be applied to arbitrary simplicial sets; they are fibrant replacements in the Quillen and Joyal model structures on sSet, respectively.

A.2. Higher Category Theory

Since ∞-categories and ordinary categories both contain objects and (possibly non-invertible) morphisms, the only difference between them is the existence of invertible higher morphisms, i.e. homotopies, homotopies of homotopies and so on, that act as coherence data for composition, associativity and identity constraints for the 1-morphisms in an ∞-category. It seems reasonable to assume that, as long as work in a homotopy coherent manner, most concepts from ordinary category theory should translate to ∞-categories without much change (similarly, concepts from 2-categories should translate to $(\infty, 2)$-categories). Let $\mathcal{C}, \mathcal{D}$ be ∞-categories, then we can define:

Definition A.2.1. The ∞-category of functors $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is the internal Hom between them in sSet. In other words, $\mathrm{Fun}(\mathcal{C}, \mathcal{D})_n = \mathrm{Hom}_{\mathrm{sSet}}(\mathcal{C} \times \Delta^n, \mathcal{D})$. Morphisms in this functor ∞-category are called natural transformations, and invertible morphisms are called natural isomorphisms.

Definition A.2.2. Functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ define an equivalence of ∞-categories if their compositions $F \circ G$ and $G \circ F$ are both naturally isomorphic to the respective identity functors.

Warning. A functor of $(\infty, 2)$-categories, regarded as simplicial sets, is a map of simplicial sets that *additionally* sends invertible 2-vertices to invertible 2-vertices.

Proposition A.2.3. For $\mathcal{C}$ and ∞-category and W a set of morphisms in it, there is another ∞-category $\mathcal{C}[W^{-1}]$, called the *localization* of $\mathcal{C}$ at W, equipped with a functor $\mathcal{C} \to \mathcal{C}[W^-1]$ such that for any ∞-category $\mathcal{D}$, precomposing with it induces a fully faithful functor

$$\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \tag{A.8}$$

with essential image spanned by those functors $F : \mathcal{C} \to \mathcal{D}$ that send each morphism in W to an isomorphism. By this universal property, $\mathcal{C}[W^{-1}]$ is uniquely determined up to equivalence (unlike for ordinary categories, where it can be made unique up to isomorphism).

Definition A.2.4. For $C, D \in \mathcal{C}$, the *morphism space*

$$\mathrm{Map}(C, D) := \{C\} \times_{\mathcal{C}} \mathrm{Fun}(\Delta^1, \mathcal{C}) \times_{\mathcal{C}} \{D\} \tag{A.9}$$

is always a Kan complex. It is homotopy equivalent to the left and right pinched morphism spaces $\{C\} \times_{\mathcal{C}} \mathcal{C}_{/D}$ and $\mathcal{C}_{C/} \times_{\mathcal{C}} \{D\}$.

Warning. For $(\infty, 2)$-categories they are different; one has to work with the left pinched morphism space (which is an $(\infty, 1)$-category).

Proving the statements we make (e.g. that the functor category is again an ∞-category) uses a lot of simplicial combinatorics that we will not discuss; see [KER] for more. In particular, we freely use:

- Join and slice constructions for simplicial sets, like $\mathcal{C}_{/C}$ above. Note that there are two simplicial models for those, that are equivalent as ∞-categories.
- The opposite simplicial set $\mathcal{C}^{op}$.

- Special kinds of morphisms between simplicial sets, for example trivial fibrations, Kan fibrations, left and right fibrations, Cartesian and coCartesian fibrations, and many more. The last four will be motivated in A.2.12.

Example A.2.5.

- If $\mathcal{C}$ is the nerve of an ordinary category, then $\mathrm{Map}_{\mathcal{C}}(C, D)$ is a discrete space.
- For X a topological space and $x, y \in X$, the mapping space $\mathrm{Map}_{\mathrm{Sing}(X)}(x, y)$ is the space of paths from x to y in X.

We have learned that morphism spaces of ∞-categories are Kan complexes. Are ∞-categories the same thing as categories enriched over Kan complexes? This can not literally be true, since enriched categories have strict composition maps, while composition in an ∞-category is, as we have seen, only defined up to a contractible space of choices. But it is essentially true:

Definition A.2.6. Denote by sSet-Cat the ordinary category of sSet-enriched categories, and by $r_{cube} : \Delta \to \mathrm{sSet}\text{-}\mathrm{Cat}$ the functor that sends $[n]$ to a simplicially enriched category with objects $0, \ldots, n$ and morphisms between $i, j \in [n]$ given by

$$\underline{\mathrm{Hom}}_{r_{cube}([n])}(i, j) := N(P(\{i, i+1 \ldots, j\}), \subseteq) \in \mathrm{sSet} \ . \tag{A.10}$$

Putting this into words, we take the nerve of the ordinary category associated to the poset of subsets of the set $\{i, i+1 \ldots, j\}$, ordered by inclusion. For intuition: This simplicial set is just a $(i - j + 1)$-dimensional cube.

Theorem A.2.7. Applying the nerve-realization paradigm A.1.7 to r_{cube} yields an adjunction

$$\begin{tikzcd} \mathrm{sSet} \arrow[r, shift right, "\mathrm{Path}"'] & \mathrm{sSet}\text{-}\mathrm{Cat} \arrow[l, shift right, "\mathrm{N}_{hc}"'] \end{tikzcd}$$

where $\mathrm{N}_{hc}(\mathcal{C})_n = \mathrm{Fun}(r_{cube}([n]), \mathcal{C})$ is called the *homotopy coherent nerve* of the simplicially enriched category $\mathcal{C}$. This is a Quillen equivalence with respect to certain model structures on both sides, yielding an equivalence of the homotopy categories: Quasi-categories are the same thing as Kan-enriched categories!

Remark. Since Kan complexes are the same thing as (good) topological spaces, one could via a change of enrichment also say that quasi-categories are the same thing as topologically enriched categories.

Proposition A.2.8 ([KER, Tag 01YL])**.** Similarly, if $\mathcal{C}$ is a simplicially enriched category where all morphism spaces are quasi-categories, then its homotopy coherent nerve is an $(\infty, 2)$-category. Every $(\infty, 2)$-category can be obtained this way up to equivalence of $(\infty, 2)$-categories. However, one can not proceed like this to obtain all $(\infty, 3)$-categories.

Proposition A.2.9 ([KER, Tag 01LG])**.** For $\mathcal{C}$ a category enriched over quasi-categories and $X, Y \in \mathcal{C}$, there is an equivalence of the internal Hom with the left pinched mapping spaces in the $(\infty, 2)$-category $N_{hc}(\mathcal{C})$:

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \simeq \mathrm{Hom}^{L}_{N_{hc}(\mathcal{C})}(X, Y) \simeq \mathrm{Hom}^{R}_{N_{hc}(\mathcal{C})}(X, Y)^{op} \tag{A.11}$$

In particular, if $\mathcal{C}$ is even enriched over Kan complexes, this is a homotopy equivalence

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \simeq \mathrm{Map}_{N_{hc}(\mathcal{C})}(X, Y)\ . \tag{A.12}$$

Example A.2.10.

- Let Kan be the Kan-enriched category with objects Kan complexes, and morphisms spaces $\underline{\mathrm{Hom}}_{\mathrm{Kan}}(K, L) := \mathrm{Fun}(K, L)$, which is indeed a Kan complex. The homotopy coherent nerve $\mathcal{S} := N_{hc}\,\mathrm{Kan}$ is the ∞-*category of spaces.* Its rôle in higher category theory is the same as the rôle of Set in ordinary category theory; one might argue it is the most important ∞-category.
- $\mathcal{S}_* := \mathcal{S}_{/\Delta^0}$ is the ∞-*category of pointed spaces.* Equivalently, it is the homotopy coherent nerve of the Kan-enriched slice category $\mathrm{Kan}_{/\Delta^0}$.
- Denote by $\mathcal{S}^{fin}$ the full subcategory of $\mathcal{S}$ on Kan complexes with finitely many non-degenerate simplices, and similarly $\mathcal{S}^{fin}_*$.
- Let QC be the category enriched over quasi-categories, where objects are quasi-categories and $\underline{\mathrm{Hom}}_{\mathrm{QC}}(\mathcal{C}, \mathcal{D}) := \mathrm{Fun}(\mathcal{C}, \mathcal{D})$. The homotopy coherent nerve $\boldsymbol{\mathcal{C}at}_\infty := N_{hc}(\mathrm{QC})$ is the $(\infty, 2)$-*category of all* ∞-*categories.*
- If we denote by $\mathrm{QC}^{\simeq}$ the Kan-enriched category with objects quasi-categories, and morphisms given by the Kan complexes $\underline{\mathrm{Hom}}_{\mathrm{QC}^{\simeq}}(\mathcal{C}, \mathcal{D}) := \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}$, then the homotopy coherent nerve is $\mathcal{C}at_\infty := N_{hc}\,\mathrm{QC}^{\simeq} \simeq \mathrm{Pith}(\boldsymbol{\mathcal{C}at}_\infty)$, the ∞-*category of all* ∞-*categories.*
- The homotopy coherent nerve of the quasi-category-enriched slice category $\boldsymbol{\mathcal{C}at}_{\infty,\mathrm{obj}} := N_{hc}\,\mathrm{QC}_{/\Delta^0}$ is the $(\infty, 2)$-*category of lax-pointed* ∞-*categories.* We will denote its pith ∞-category by $\mathcal{C}at_{\infty,\mathrm{obj}}$.

After these very foundational definitions, let us introduce some universal constructions for ∞-categories:

Definition A.2.11. Let $K \in \mathrm{sSet}$, and $p : K \to \mathcal{C}$ be a morphism of simplicial sets, that we interpret as a diagram in the ∞-category $\mathcal{C}$. Denote by $K^{\triangleleft}$ the left cone on K, formed by adding an initial object to it (i.e. taking the join $\Delta^0 \star K$). The *limit cone* of this diagram, if it exists, is a morphism $\overline{p} : K^{\triangleleft} \to \mathcal{C}$ with $p(-\infty) := \lim(p)$, that induces for all $C \in \mathcal{C}$ a homotopy equivalence

$$\mathrm{Map}(C, \lim(p)) \simeq \mathrm{Nat}(\mathrm{const}_{\mathcal{C}}, p) \tag{A.13}$$

where $\mathrm{const}_C : K \to \mathcal{C}$ it the constant diagram on C. Note how this agrees with the ordinary limit if $\mathcal{C}$ is a 1-category. Oppositely, we can define a colim(p) by extending p to $K^{\triangleright}$ such that

$$\mathrm{Map}(\mathrm{colim}(p), C) \simeq \mathrm{Nat}(p, \mathrm{const}_{\mathcal{C}}) \ . \tag{A.14}$$

- Special cases of this construction yield (as in ordinary category theory) products, coproducts; pullbacks, pushouts; final, initial and zero objects; kernels, cokernels; and filtered (co-)limits (see A.8.5).
- While coproducts and products can be treated with similar intuition as in ordinary categories, pullbacks and pushouts behave like homotopy pullbacks and pushouts. For example, (co-)limits in $\mathcal{S}$ are precisely homotopy (co-)limits of topological spaces; and kernels in chain complexes are mapping cones, see A.3.4.
- Just like every set is a colimit (coproduct) over its elements regarded as one-element sets, every Kan complex K is the colimit over the functor $\mathrm{const}_* : K \to \mathcal{S}$.
- Using the mapping space construction in a similar way, one can define adjunctions, Kan extensions, and so on.
- Almost all the usual formulae for limits and colimits still hold. Generally, almost all theorems from category theory still hold, like the Yoneda lemma, colimits commuting with colimits, uniqueness of colimits and adjoints, Quillen's theorem A for cofinal (see A.8.1) functors, and so on.
- There are notions of accessible, presentable (sometimes also called locally presentable), and compactly generated ∞-categories mimicking the ordinary notions, and the Adjoint Functor Theorem holds.
- Presentable ∞-categories are "the same thing" as combinatorial model categories! Higher category theory therefore trivializes many cumbersome model category calculations.

See [HTT, Chapter 4 and 5] for more. Finally, let us give a short comment on why higher category is so technically difficult (and why [HTT] is almost 1000 pages long). Giving a functor between two ∞-categories, and checking that it is indeed a functor, can be extremely difficult to do explicitly, for example it is very hard to see why the mapping space construction we gave above is functorial in its arguments. However, to even define a Yoneda embedding, see that limits are functorial and so on, we need to understand this. There is a very elegant, roundabout way to define the mapping space functor:

Theorem A.2.12 (Grothendieck construction, [HTT, Section 3.2])**.** For a fixed ∞-category $\mathcal{C}$, functors $F : \mathcal{C} \to \mathcal{C}at_\infty$ are essentially the same thing as simplicial sets $\mathcal{M}$ equipped with a simplicial map $p : \mathcal{M} \to \mathcal{C}$ that is a so-called *coCartesian fibration.* More explicitly, the fiber of p over an object $C \in \mathcal{C}$ is equivalent to the ∞-category $F(C)$, and the action of F on morphisms in $\mathcal{C}$ is encoded via a version of parallel transport along a certain class of morphisms in $\mathcal{M}$, called *p-coCartesian morphisms*. Conversely, one can obtain p from F as the pullback

$$\mathcal{M} := \mathcal{C}at_{\infty,\mathrm{obj}} \times_{\mathcal{C}at_\infty} \mathcal{C} \to \mathcal{C} \ . \tag{A.15}$$

Similarly, functors $F : \mathcal{C}^{op} \to \mathcal{C}at_\infty$ are essentially the same thing as *Cartesian fibrations* over $\mathcal{C}$, i.e. simplicial maps $p : \mathcal{M} \to \mathcal{C}$ such that $p^{op} : \mathcal{M}^{op} \to \mathcal{C}^{op}$ is a coCartesian fibration. To make these statements precise, one could write down a Quillen equivalence between certain model structures on (marked) simplicial sets.

Remark. We do not define coCartesian fibrations, since that is a bit cumbersome and it is enough for us to know that such a notion exists. However, we could interpret this result as saying that the category $\mathcal{C}at_\infty$ acts as a classifying space for them; see B.5 for more.

Corollary A.2.13. For a fixed ∞-category $\mathcal{C}$, functors $\mathcal{C} \to \mathcal{S}$ are essentially the same thing as *left fibrations* over $\mathcal{C}$, with a similar explicit description of this correspondence as above. Oppositely, functors $\mathcal{C}^{op} \to \mathcal{S}$ are the same thing as *right fibrations.*

Remark. This is immediately clear if we define a left fibration to be a coCartesian fibrations where all fibers are Kan complexes, and similarly for right fibrations. There is however a much simpler characterization of when a map of simplicial sets is a left or right fibration via a horn lifting property. We recommend reading [KER, Tag 01J2] for more on the Grothendieck construction.

Remark. One can deduce that for an ∞-groupoid K, the ∞-categories $\mathrm{Fun}(K, \mathcal{S}) \simeq \mathcal{S}_{/K}$ are equivalent. This relies on a model category argument; given an arbitrary map of Kan complexes $M \to K$, we can always replace it by a left fibration that is weakly homotopy equivalent to M. This induces an equivalence between $\mathcal{S}_{/K}$ and its full subcategory spanned by the left fibrations, so we can apply A.2.13. Note that this argument would break down for K an arbitrary ∞-category, where we would have to work with this subcategory.

Example A.2.14. One can show that for $\mathcal{C}$ an ∞-category and $C \in \mathcal{C}$, the projection $\mathcal{C}_{C/} \to \mathcal{C}$ out of the slice category is a left fibration [KER, Tag 018F]. The associated functor $\mathcal{C} \to \mathcal{S}$ sends D to the fiber $\mathcal{C}_{C/} \times_{\mathcal{C}} \{C\} \simeq \mathrm{Map}_{\mathcal{C}}(C, D)$, so it can be used together with the analogous observation for the right fibration $\mathcal{C}_{/C} \to \mathcal{C}$ to define the mapping space functor.

A.3. Stable ∞-categories

Definition A.3.1. A *zero object* 0 in an ∞-category $\mathcal{C}$ is an object that is both initial and final; in other words for any $C \in \mathcal{C}$,

$$\mathrm{Map}_{\mathcal{C}}(C, 0) \simeq \mathrm{Map}_{\mathcal{C}}(0, C) \simeq \Delta^0 \tag{A.16}$$

are contractible. Since this is a universal property, a zero object is (if it exists) unique up to a contractible space of choices. Also, for $C, D \in \mathcal{C}$, the composition of the essentially unique maps $C \to 0 \to D$ specifies a zero-morphism in every mapping space of $\mathcal{C}$, so they become pointed spaces.

Definition A.3.2. If $\mathcal{C}$ has a zero object 0, and $f : C \to D$ is a morphism in $\mathcal{C}$, then its *fiber* $\mathrm{fib}(f)$ is the equalizer of f and the zero morphism 0 from C to D (just like the kernel in ordinary category theory). Similarly, its *cofiber* $\mathrm{cofib}(f)$ is the coequalizer of f and 0. Sequences of the form

$$\mathrm{fib}(f) \to \mathcal{C} \xrightarrow{f} \mathcal{D} \tag{A.17}$$

$$\mathcal{C} \xrightarrow{f} \mathcal{D} \to \mathrm{cofib}(f) \tag{A.18}$$

(up to isomorphism) are called *fiber sequences* and *cofiber sequences*, respectively.

Definition A.3.3. An ∞-category $\mathcal{C}$ is called *stable* if:

- It has a zero object 0,
- Every morphism in $\mathcal{C}$ has a fiber and a cofiber,
- Any cofiber sequence is also a fiber sequence.

One can show, using these axioms, that:

- Fiber and cofiber sequences agree
- All finite limits and colimits exist in $\mathcal{C}$
- A square is a pushout square iff it is a pullback square
- The *loop space* functor $\Omega : \mathcal{C} \to \mathcal{C}$ sending $C \mapsto 0 \times_C 0$ is an equivalence of categories, with inverse the *suspension* functor $\Sigma : C \mapsto 0 \amalg_C 0$.
- The homotopy category $h\mathcal{C}$ has a natural Ab-enrichment.

Theorem A.3.4 ([HA, 1.1.2.14])**.** If $\mathcal{C}$ is a stable ∞-category, then the homotopy category $h\mathcal{C}$ is a triangulated category. In particular, (co-)fiber sequences in $\mathcal{C}$ become triangles in $h\mathcal{C}$, cofibers become (functorial) mapping cones and Σ becomes the shift functor [1].

The upshot: Stable ∞-categories are equipped to take over the rôle of triangulated categories (and their dg enhancements), just like presentable ∞-categories took over the rôle of (combinatorial) model categories. This is extremely nice, since their definition is just a homotopy coherent formulation of the axioms of an abelian category, in particular very simple compared to Verdier's definition. Similarly, presentable stable ∞-categories are one analogon of Grothendieck abelian categories.

Definition A.3.5. An *exact functor* $F : \mathcal{C} \to \mathcal{D}$ between stable ∞-categories is a functor that preserves finite limits (or equivalently, finite colimits). We obtain an ∞-category $\mathcal{C}at_\infty^{ex}$ of stable ∞-categories and exact functors as a (non-full) subcategory of $\mathcal{C}at_\infty$.

Proposition A.3.6 ([HA, 1.1.3.2])**.** A full subcategory $\mathcal{D}$ of a stable ∞-category $\mathcal{C}$ that contains the zero object, and is closed under fibers and cofibers, is itself stable. We call this a *stable subcategory.*

Proof. $\mathcal{D}$ has a zero object, fibers and cofibers since they can be calculated in $\mathcal{C}$ because their universal properties restrict. For the same reason, fiber or cofiber sequences in $\mathcal{D}$ are precisely fiber or cofiber sequences in $\mathcal{C}$ where every object lies in $\mathcal{D}$, so the notions coincide. $\square$

Proposition A.3.7 ([Gro16])**.** An ∞-category $\mathcal{C}$ is stable iff it admits finite limits and colimits, and finite limits and colimits commute.

Let us construct a few examples.

Definition A.3.8. A *simplicial abelian group* is an ordinary functor in $\mathrm{sAb} := \mathrm{Fun}(\Delta^{op}, \mathrm{Ab})$. Forgetting about the group operation, it has an underlying simplicial set, which can be shown to automatically be a Kan complex. Conversely, every $X \in \mathrm{sSet}$ defines a simplicial abelian group $\mathbb{Z}X : \Delta^{op} \to \mathrm{Ab}$ by composing with the free $\mathbb{Z}$-module functor.

Definition A.3.9. For X a simplicial abelian group, let its *Moore complex* $C_*(X)$ be the chain complex (in homological convention) with $C_n(X) = X_n$ concentrated in non-negative degrees and differential induced by the face maps $\delta_i : X_n \to X_{n-1}$ in X:

$$\forall c \in X_n : \ dc := \sum_{i=0}^{n} (-1)^i \delta_i X_n \in C_*(X)_{n-1} \tag{A.19}$$

The *normalized Moore complex* $N_*(X)$ is the subcomplex of $C_*(X)$ spanned by the non-degenerate simplices, where all contributions form degenerate simplices in the differential are set to 0.

Both C_* and N_* are additive functors and preserve colimits. Therefore, N_* is the Left Kan Extension along the (Ab-enriched Yoneda embedding) of its restriction to

Δ (by A.1.6), and therefore arises by applying the nerve-realization paradigm A.1.7 to this restriction. Hence, it has a right adjoint $K : \mathrm{Ch}(\mathbb{Z})_{\geq 0} \to \mathrm{sAb}$ sending $C \mapsto \mathrm{Hom}(N_*(-), C)$. We call $K(C)$ the *Eilenberg-MacLane space* of C, in particular for A an abelian group, $K(A, n) := |K(A[n])|$ is the Eilenberg-MacLane space from topology.

Theorem A.3.10 (Dold-Kan correspondence)**.** The functors N_* and K form an equivalence of categories between non-negatively graded chain complexes and simplicial abelian groups, $\mathrm{Ch}(\mathbb{Z})_{\geq 0} \simeq \mathrm{sAb}$. This can be generalized to any abelian category, instead of Ab.

Example A.3.11. Applying C_* to $\mathbb{Z}\,\mathrm{Sing}(X)$ yields the singular chain complex of a topological space X.

Construction A.3.12. Let $\mathcal{C}$ be a differential graded (dg) category (a category enriched over $\mathrm{Ch}(\mathbb{Z})$). Truncating the morphism complexes at 0 and applying K yields a category enriched over simplicial abelian groups, and forgetting the group structure yields a Kan-enriched category because of A.3.8. Finally, applying the homotopy coherent nerve yields an ∞-category $N_{dg}\mathcal{C}$, called the dg-nerve of $\mathcal{C}$.

Remark. There is an equivalent construction of the dg-nerve that is easier to compute; the shortest way to define it is to apply the nerve-realization paradigm to a functor that realizes objects of Δ as A_∞-categories, see [Fao13]. This paper also shows that if $\mathcal{C}$ is a pretriangulated dg-category, then $N_{dg}(\mathcal{C})$ is stable.

Example A.3.13. For R any commutative ring, let $\mathrm{Ch}(R)$ be the dg-category of chain complexes of R-modules. Denote by $\mathcal{C}h(R) := N_{dg}\,\mathrm{Ch}(R)$ its dg-nerve, the stable ∞-category of chain complexes. Explicitly, its

- Objects are chain complexes of R-modules
- Morphisms are chain maps
- 2-morphisms are chain homotopies
- 3-morphisms are chain homotopies between chain homotopies, and so on.

Example A.3.14. Localization (as in A.2.3) of $\mathcal{C}h(R)$ at the quasi-isomorphisms yields the *derived ∞-category* $D(R)$ of R. Similarly for the bounded variants $D^b(R), D^+(R), D^-(R)$, all of which are stable.

More generally, one can define the derived ∞-category $D(\mathcal{A})$ of any abelian category $\mathcal{A}$ by inverting the quasi-isomorphisms in the ∞-category of chain complexes in $\mathcal{A}$. This is particularly well-behaved for Grothendieck abelian categories, where $D(\mathcal{A})$ actually is a presentable ∞-category.

Example A.3.15. Let $\mathcal{S}_*^{fin}$ be the ∞-category of finite pointed spaces from A.2.10. Denote by $\mathrm{Exc}^*(\mathcal{S}_*^{fin}, \mathcal{S})$ the full subcategory of $\mathrm{Fun}(\mathcal{S}_*^{fin}, \mathcal{S})$ on functors that are

- *reduced*, i.e. preserve the final object, and
- *excisive*, i.e. send pushout squares to pullback squares.

This is the stable ∞-category $\mathcal{S}p$ of *spectra*. Its homotopy category agrees with the triangulated category of (symmetric) spectra.

A.4. Sheaves and ∞-Topoi

Definition A.4.1. Let $\mathcal{C}$ be an ∞-category, then denote by $\mathcal{PSh}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ its *presheaf category*, and by $h : \mathcal{C} \to \mathcal{PSh}(\mathcal{C})$ the fully faithful Yoneda embedding.

As in ordinary category theory, we often want to restrict our attention to a full subcategory of $\mathcal{PSh}(\mathcal{C})$ that contains *sheaves*, which are presheaves that satisfy descent with respect to a particular notion of covering.

Definition A.4.2. A *Grothendieck pretopology* τ on $\mathcal{C}$ consists of, for every $U \in \mathcal{C}$, a set of *coverings* $\mathrm{Cov}_\tau(U)$ whose elements are families $(U_i \to U)_{i \in I}$ with $U_i \in \mathcal{C}$, such that the following hold:

- Given an isomorphism $U' \to U$, the one-element family $(U' \to U)$ is a covering.
- For any morphism $V \to U$, the pullbacks $(U_i \times_U V \to V)_i$ exist and form a covering again.
- If for any i, the family $(U_{ij} \to U_i)_j$ is a covering, then the composition $(U_{ij} \to U)_{ij}$ is a covering.

Technical Remark. While every *Grothendieck topology*, as in [HTT, 6.2.2.1], is a Grothendieck pretopology, the latter or usually much smaller. However, every pretopology specifies a unique topology by defining the covering sieves as those that contain a whole covering family, see [Pst18, A.5]. We will therefore work with this simpler notion.

Definition A.4.3. An ∞-*site* $\mathcal{C}_\tau$ is an ∞-category $\mathcal{C}$ equipped with a Grothendieck pretopology τ. Since covering families are invariant under isomorphisms (combining the first and third axiom), it is enough to specify the pretopology on the homotopy category.

Definition A.4.4. Given a covering $(U_i \to U)$, we define its *Čech nerve* $C(U_i \to U) \in \mathrm{Fun}(\Delta^{op}, \mathcal{PSh}(\mathcal{C}))$ as the simplicial diagram

$$\cdots \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; \coprod_{i,j,k} h(U_i) \underset{h(U)}{\times} h(U_j) \underset{h(U)}{\times} h(U_k) \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; \coprod_{i,j} h(U_i) \underset{h(U)}{\times} h(U_j) \longrightarrow \coprod_i h(U_i)$$

which by functoriality of h possesses a canonical morphism to $h(U)$.

Definition A.4.5. A *sheaf* on an ∞-site $\mathcal{C}$ is a presheaf $F : \mathcal{C}^{op} \to \mathcal{S}$ that is local with respect to these morphisms; that is for every covering $(U_i \to U)$,

$$\lim_{\Delta^{op}} \mathrm{Map}_{\mathcal{PSh}(\mathcal{C})}(C(U_i \to U), F) \overset{!}{=} \mathrm{Map}_{\mathcal{PSh}(\mathcal{C})}(h_U, F) = F(U)\ . \tag{A.20}$$

In other words, we require

$$F(U) = \lim_{\Delta^{op}} \left(\prod_i F(U_i) \longrightarrow \prod_{i,j} F(U_i \times_U U_j) \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \cdots \right)\ . \tag{A.21}$$

Technical Remark. We denote the full subcategory on them by $\mathcal{Sh}(\mathcal{C}_\tau)$, leaving out the topology if it is clear. This is equivalent to the definition in [HTT] by [Pst18, A.8, A.9].

Theorem A.4.6 ([HTT, 6.2.2.7])**.** For any ∞-site $\mathcal{C}$, there is a *sheafification* functor $(-)^{sh}$, which can be constructed as a transfinite composition of a *plus construction* (mimicking the classical double-plus-construction) is left adjoint to the canonical inclusion

$$\begin{tikzcd} \mathcal{Sh}(\mathcal{C}) \arrow[r, hook, shift right, "i"'] & \mathcal{PSh}(\mathcal{C}) \arrow[l, shift right, "(-)^{sh}"'] \end{tikzcd}\ .$$

This leads to a general axiom for ∞-categories that "look like" categories of sheaves:

Definition A.4.7. An ∞-*topos* $\mathcal{X}$ is an ∞-category that can be written as a left exact accessible localization of a presheaf category. In other words, there must exist a (small) ∞-category $\mathcal{C}$ and an adjunction

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook, shift right, "i"'] & \mathcal{PSh}(\mathcal{C}) \arrow[l, shift right, "L"'] \end{tikzcd}$$

such that i is fully faithful and preserves κ-filtered colimits for some regular cardinal κ, and L preserves finite limits.

Technical Remark. The accessibility condition (preserving κ-filtered colimits) is equivalent to ensuring $\mathcal{X}$ is again presentable. It is currently not known whether it is automatic (as it is in the case of n-topoi).

Remark. This definition is extrinsic, since it tells us how to construct ∞-topoi, but not how to check if a specific ∞-category is one. There are also several intrinsic definitions, for example the *Giraud-Rezk-Lurie axioms.*

Warning. Not every left exact accessible reflective localization of a presheaf category arises as sheaves with respect to a Grothendieck category! It is not even known whether any ∞-topos can be written as sheaves on an ∞-site at all.

Example A.4.8. Since identity functors are always left exact accessible localizations, presheaf categories are always ∞-topoi. In particular, $\mathcal{S} = \mathcal{PSh}(*)$ is an ∞-topos. Also, for any ∞-site $\mathcal{C}_\tau$, the sheaves $\mathcal{Sh}(\mathcal{C}_\tau)$ form an ∞-topos using the adjunction A.4.6.

Example A.4.9. For $\mathcal{X}$ an ∞-topos and C an object in it, the *slice topos* $\mathcal{X}_{/C}$ is again an ∞-topos.

Definition A.4.10. A *geometric morphism* between ∞-topoi is an adjunction

$$\begin{tikzcd} \mathcal{X} \arrow[r, shift right, "f_*"'] & \mathcal{Y} \arrow[l, shift right, "f^*"'] \end{tikzcd}$$

where f^* preserves finite limits. Let us denote the subcategory of $\mathcal{C}at_\infty$ on ∞-topoi and geometric morphisms by $\mathcal{L}$Top.

Proposition A.4.11. $\mathcal{S}$ is the terminal object of $\mathcal{L}$Top. This means that every ∞-topos $\mathcal{X}$ is equipped with an essentially unique adjunction

$$\begin{tikzcd} \mathcal{X} \arrow[r, shift right, "\Gamma_*"'] & \mathcal{Y} \arrow[l, shift right, "\Gamma^*"'] \end{tikzcd}$$

In particular, for $*$ the terminal object, $\Gamma_* = \mathrm{Map}_{\mathcal{X}}(*, -)$ and if $\mathcal{X} = \mathcal{Sh}(\mathcal{C})$ over a ∞-site, $\Gamma^*(K) = (C \mapsto K)^{sh}$. Also, note that since Γ^* preserves colimits and every Kan-complex is the colimit over its points, Γ^* can be understood via its value on Δ^0.

Definition A.4.12 ([Pst18, A.10 and A.12])**.** Given ∞-sites $\mathcal{C}$ and $\mathcal{D}$, a *morphism of sites* is a functor $F : \mathcal{C} \to \mathcal{D}$ that sends coverings to coverings.

Further, F has the *covering lifting property* if for any $U \in \mathcal{C}$ and $(V_i \to F(U))$ covering family in $\mathcal{D}$, there is a covering $(U_j \to U)$ in $\mathcal{C}$ such that for every j one can find an i such that one can factor $F(U_j) \to V_i \to F(U)$.

Proposition A.4.13 ([Pst18, A.11 and A.13])**.** For any morphism of sites $F : \mathcal{C} \to \mathcal{D}$, precomposition $F_* := - \circ F$ preserves sheaves and, together with sheafification of the Left Kan Extension along it $F^* = (-)^{sh} \circ \mathrm{Lan}_F$, induces an adjunction

$$\begin{tikzcd} \mathcal{Sh}(\mathcal{D}) \arrow[r, shift right, "F_*"'] & \mathcal{Sh}(\mathcal{D}) \arrow[l, shift right, "F^*"'] \end{tikzcd} .$$

If F has the covering lifting property, then F_* commutes with sheafification, in particular it preserves colimits and admits another left adjoint $F^- : \mathcal{S}h(\mathcal{C}) \to \mathcal{S}h(\mathcal{D})$.

Let $\mathcal{D}$ be an arbitrary ∞-category.

Definition A.4.14. A functor $F : \mathcal{C}^{op} \to \mathcal{D}$ is a *$\mathcal{D}$-valued sheaf on $\mathcal{C}$* if, for any $D \in \mathcal{D}$, the composition $\mathrm{Map}_{\mathcal{D}}(D, F(-)) : \mathcal{C}^{op} \to \mathcal{S}$ is a sheaf on $\mathcal{C}$. We denote the subcategory on them by $\mathcal{S}h(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{D})$.

Remark. Again, explicitly we impose that for any cover $(U_i \to U)$, we have Čech descent:

$$F(U) = \lim_{\Delta^{op}} \Big(\prod\nolimits_i F(U_i) \longrightarrow \prod\nolimits_{i,j} F(U_i \times_U U_j) \rightrightarrows \cdots \Big) . \tag{A.22}$$

In particular, this limit should exist in $\mathcal{D}$.

Proposition A.4.15 ([SAG, 1.3.1.7])**.** If $\mathcal{D}$ has all limits, there is an equivalence

$$\mathcal{S}h(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}^{lim}(\mathcal{S}h(\mathcal{C})^{op}, \mathcal{D}) , \tag{A.23}$$

where Fun^{lim} denotes the subcategory of Fun on the limit-preserving functors.

This description can be further refined when we restrict to the class of *presentable* ∞-categories, which generalizes the class of (locally) presentable ordinary categories. To put it loosely, an ∞-category is presentable if is *accessible*, that is, generated under colimits by a small subcategory of compact objects; and it has all colimits (and automatically all limits).

Theorem A.4.16 ([HA, 4.8.1.17])**.** For $\mathcal{C}$ and $\mathcal{D}$ any presentable ∞-categories, one can define their tensor product $\mathcal{C} \otimes \mathcal{D} := \mathrm{Fun}^{lim}(\mathcal{C}^{op}, \mathcal{D})$ that is again a presentable ∞-category, and a natural functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ such that for any presentable $\mathcal{E}$,

$$\mathrm{Fun}^{colim}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}^{colim,colim}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) . \tag{A.24}$$

Here, $\mathrm{Fun}^{colim,colim}$ denotes functors that preserve colimits in both variables. In particular, $\mathcal{C} \otimes \mathcal{D}$ is symmetric in $\mathcal{C}$ and $\mathcal{D}$.

Corollary A.4.17. For $\mathcal{D}$ a presentable ∞-category, $\mathcal{S}h(\mathcal{C}, \mathcal{D}) \simeq \mathcal{S}h(\mathcal{C}) \otimes \mathcal{D}$.

Corollary A.4.18. If $\mathcal{D}$ is a presentable (and/ or stable) ∞-category, then $\mathcal{S}h(\mathcal{C}, \mathcal{D})$ is presentable (and/ or stable) as well.

Proof. For $\mathcal{D}$ presentable, $\mathcal{S}h(\mathcal{C}, \mathcal{D}) = \mathcal{S}h(\mathcal{C}) \otimes \mathcal{D}$ is presentable by A.4.16.

If $\mathcal{D}$ is stable, then $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{D})$ is stable because limits and colimits in a functor category are computed pointwise, so we need to show that the sheaves form a stable subcategory in the sense of A.3.6. This follows since the sheafification functor is left exact, so that the category of sheaves is in particular closed under fibers and contains the zero object. $\square$

A.5. Sheaves on Topological Spaces

Let us apply this machinery to the probably most interesting case:

Definition A.5.1. For X a topological space and $\mathcal{D}$ a complete ∞-category, equip the thin category of open subsets $\mathrm{Open}(X)$ with the Grothendieck pretopology τ where covering families are open coverings. We denote

$$\mathcal{S}h(X) := \mathcal{S}h(\mathrm{Open}(X)_\tau)\,, \quad \mathcal{S}h(X;\mathcal{D}) := \mathcal{S}h(\mathrm{Open}(X)_\tau;\mathcal{D})\,. \tag{A.25}$$

Remark. In particular, a functor $F : \mathrm{Open}^{op}(X) \to \mathcal{D}$ is an ∞-sheaf if for any open cover $(U_i \subseteq U)$,

$$F(U) = \lim_{\Delta^{op}} \left(\prod_i F(U_i) \longrightarrow \prod_{i,j} F(U_i \cap U_j) \mathrel{\substack{\longrightarrow\\[-0.6ex]\longrightarrow\\[-0.6ex]\longrightarrow}} \cdots \right)\,. \tag{A.26}$$

There are several different ways to intuitively make sense of this descent condition. First of all, note the similarity with the Čech complex which also involves comparing sections at higher intersections of the U_i. One can show that for every ordinary sheaf $F_0 \in \mathrm{Sh}(X;\mathbb{Z})$, the derived sections $R\Gamma(-,F_0) \in \mathcal{S}h(X;D(\mathbb{Z}))$ form an ∞-sheaf; the descent condition in this case is equivalent to the statement that sheaf cohomology of F_0 agrees with the Čech hypercohomology of $R\Gamma(-,F_0)$ on any cover (this follows from the Čech-to-sheaf-cohomology spectral sequence).

As a second example, suppose we are given a collection of topological spaces $(X_i)_{i\in I}$ and open subsets $U_j^{(i)} \subseteq X_i$ for $i,j \in I$, together with homeomorphisms $\phi_{ij} : U_j^{(i)} \cong U_i^{(j)}$ such that $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ on the respective intersections. Then, we can glue the spaces X_i together along the gluing maps ϕ_{ij}. Note how this involves triple intersections, unlike the ordinary sheaf condition which only compares sections on intersections of two open sets in a covering. A similar descent via triple intersections holds for the functors of points of stacks in algebraic geometry. Since descent for ∞-sheaves involves intersections of arbitrary order, they are sometimes called *higher stacks.*

Proposition A.5.2. A continuous map of topological spaces $f : X \to Y$ induces a geometric morphism

$$(f_* \dashv f^*) : \mathcal{S}h(X) \to \mathcal{S}h(Y)\,. \tag{A.27}$$

In particular, for $F \in \mathcal{S}h(X)$ and $x \in X$ with inclusion $x : \{x\} \to X$ we can define the *stalk* x^*F of F at x.

Proof. This follows immediately from A.4.13, since the inverse image $f^{-1} : \mathrm{Open}(Y) \to \mathrm{Open}(X)$ is a morphism of sites by definition of continuity. $\square$

An ∞-topos $\mathcal{X}$ can be understood as an exotic world to do topology in; in particular the terminal ∞-topos $\mathcal{S}$ describes usual topology (of CW complexes), and the topoi $\mathcal{S}h(X)$ describe (if X is paracompact and Hausdorff) topology relative to X. In particular, one can define homotopy groups of objects in every ∞-topos. However, not all results from usual topology still hold – in particular, the theorem of Whitehead can break down:

Definition A.5.3. A morphism $f : X \to Y$ in an ∞-topos $\mathcal{X}$ is called ∞-*connected* if it induces an isomorphism on all homotopy groups internal to $\mathcal{X}$ (we do not define what this means).

Proposition A.5.4 ([HA, A.3.9])**.** A morphism $f : F \to G$ in $\mathcal{S}h(X)$ is ∞-connected iff for any $x \in X$, it induces an isomorphism on stalks $x^*f : x^*F \xrightarrow{\cong} x^*G$.

Definition A.5.5. Let $\mathcal{X}$ be an ∞-topos, then an object $X \in \mathcal{X}$ is *hypercomplete* if it is local with respect to ∞-connected morphisms, meaning that for any ∞-connected $f : C \to D$, precomposing with f induces a homotopy equivalence

$$- \circ f : \mathrm{Map}_{\mathcal{X}}(D, X) \xrightarrow{\simeq} \mathrm{Map}_{\mathcal{X}}(C, X) \ . \tag{A.28}$$

Theorem A.5.6. The full subcategory on the hypercomplete objects $\mathcal{X}^{hyp}$ is again an ∞-topos, the *hypercompletion* of $\mathcal{X}$. The functor $(-)^{hyp}$ is a reflection on the subcategory of $\mathcal{L}\mathrm{Top}$ on the hypercomplete ∞-topoi.

Definition A.5.7. For $\mathcal{C}$ an ∞-site, we call a $\mathcal{D}$-valued sheaf $F \in \mathcal{S}h(\mathcal{C}, \mathcal{D})$ *hypercomplete* if for any $D \in \mathcal{D}$, the composition $\mathrm{Map}_{\mathcal{D}}(D, F) \in \mathcal{S}h(\mathcal{C})$ is hypercomplete. Denote their full subcategory by $\mathcal{S}h^{hyp}(\mathcal{C}, \mathcal{D})$.

Definition A.5.8. We call hypercomplete $\mathcal{D}$-valued sheaves on X *hypersheaves*, and denote their category by $\mathcal{S}h^{hyp}(X, \mathcal{D})$

Proposition A.5.9. If X is paracompact Hausdorff and of finite covering dimension, every sheaf on X is hypercomplete.

Proof. This is very technical and only added for lack of reference. [HTT, 7.1.1.1] assures that we can find a basis U_i for the topology of X, such that every U_i is itself open, paracompact Hausdorff and of finite covering dimension; therefore [HTT, 7.2.3.6] tells us that $\mathcal{S}h(U_i)$ has finite homotopy dimension. Since the Yoneda embeddings $h_{U_i} \in \mathcal{S}h(X)$ generate $\mathcal{S}h(X)$ under colimits and the slice topoi $\mathcal{S}h(X)_{/U_i} \simeq \mathcal{S}h(U_i)$, we even know that $\mathcal{S}h(X)$ is locally of finite homotopy dimension. Because of [HTT, 7.2.1.12], every ∞-topos that is locally of finite homotopy dimension is hypercomplete. $\square$

A.6. ∞-Operads

Let us first review the classical notion of a colored operad, also called multicategory or simply *operad*. Albeit looking complicated, it just generalizes the definition of an ordinary category by allowing for morphisms with multiple sources (and a single target).

Definition A.6.1. An *operad* $\mathcal{O}^{\otimes}$ consists of a set of objects, and for every objects $C_1, C_2, \dots, C_n, D \in \mathcal{O}$ with $n \in \mathbb{N}_0$ a set of multimorphisms, or *n-ary morphisms*,

$$\mathrm{Mul}_{\mathcal{O}}(C_1, \dots, C_n; D)$$

together with composition operations or the form

$$\mathrm{Mul}_{\mathcal{O}}(C_1, \dots, C_n; D_1) \times \mathrm{Mul}_{\mathcal{O}}(D_1, \dots, D_m; E) \to \mathrm{Mul}_{\mathcal{O}}(C_1, \dots, C_n, D_2, \dots, D_m; E)$$

that satisfy evident associativity conditions. Finally, there should be identity multimorphisms $\mathrm{id}_C \in \mathrm{Mul}_{\mathcal{O}}(C; C)$ for all $C \in \mathcal{O}$, acting as identities for all kinds of compositions.

Remark. We use the superscript $\otimes$ to indicate that $\mathcal{O}^{\otimes}$ is an operad, and in particular cases to keep track of how the multimorphisms are defined. It is however left out if it clutters the notation and is obvious.

Example A.6.2.

- The operad $\mathrm{Vec}_{\mathbb{R}}^{\otimes}$ of contains real vector spaces as objects, and $\mathrm{Mul}_{\mathrm{Vec}}(V_1, \dots, V_n; W)$ consists of multilinear maps $V_1 \otimes \dots \otimes V_n \to W$.
- Similarly to the first example, any monoidal category $\mathcal{C}$ can be interpreted as an operad by setting $\mathrm{Mul}_{\mathcal{C}}(C_1, \dots, C_n; D) := \mathrm{Hom}_{\mathcal{C}}(C_1 \otimes \dots \otimes C_n, D)$.
- A partially monoidal category is defined similarly to a monoidal category, but the tensor product does not have to be well-defined for all pairs of objects. We can still capture the data of it inside an operad, by setting $\mathrm{Mul}_{\mathcal{C}}(C_1, \dots, C_n; D) = \emptyset$ if the product $C_1 \otimes \dots \otimes C_n$ is not well-defined. An example would be the category of open subsets in a topological space, with inclusions as morphisms and disjoint union as partial monoidal product.
- $\mathrm{Comm}^{\otimes} = \mathbb{E}_{\infty}^{\otimes}$ is the operad with a single object $*$, and one multimorphism of every degree $\mathrm{Mul}_{\mathbb{E}_1}(*, \dots, *; *) = *$.
- $\mathrm{Triv}^{\otimes}$ is the operad with a single object $*$ and only the identity multimorphism.
- $\mathbb{E}_0^{\otimes}$ is the operad with a single object $*$ and only two multimorphisms $\mathrm{id}_* \in \mathrm{Mul}_{\mathbb{E}_0}(*; *)$ and $1_* \in \mathrm{Mul}_{\mathbb{E}_0}(\emptyset; *)$.
- $\mathrm{Assoc}^{\otimes} = \mathbb{E}_1^{\otimes}$ is the operad with a single object $*$ and n-ary multimorphisms corresponding to orderings of the set $\{1, \dots, n\}$. Composition is given by inserting one ordering at the position of the respective element into the other ordering.

In principle, a generalization to ∞-operads seems evident: We should allow for the multimorphism spaces to be Kan complexes instead of discrete sets, and compositions should be simplicial maps – this yields *Kan-enriched operads*, analogously to Kan-enriched categories. Again, they are technically difficult to work with, and a definition as a simplicial set is usually preferred. We do not compile it here since it is extremely technical; however we can give many interesting examples.

Definition A.6.3. A *functor* $F : \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ of Kan-enriched operads consists of a map on the set of objects, and for all $C_1, \ldots, C_n, C \in \mathcal{O}$ a simplicial map

$$\mathrm{Mul}_{\mathcal{O}}(C_1, \ldots, C_n; C) \to \mathrm{Mul}_{\mathcal{O}'}(F(C_1), \ldots, F(C_n); F(C)) \tag{A.29}$$

that is compatible with the composition maps. The functors $\mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ are themselves the objects of a Kan-enriched category, i.e. via the homotopy-coherent nerve of an ∞-category. This is called the *∞-category of $\mathcal{O}$-algebras in $\mathcal{O}'$* and denoted $\mathrm{Alg}_{\mathcal{O}}(\mathcal{O}')$ (in fact, it is in some cases an ∞-operad itself). This equips (small) ∞-operads with the structure of a category enriched over ∞-categories, so that taking the homotopy coherent nerve yields the *$(\infty, 2)$-category of ∞-operads* $\boldsymbol{\mathcal{O}p}_{\infty}$. Replacing the morphism spaces by the Kan complexes $\mathrm{Alg}_{\mathcal{O}}(\mathcal{O}')^{\simeq}$ yields its pith, the *∞-category of ∞-operads* $\mathcal{O}p_{\infty}$.

Example A.6.4. Ever ∞-category is an ∞-operad with only 1-ary multimorphisms. Conversely, every ∞-operad has an underlying ∞-category that is obtained by forgetting all multimorphisms that are not 1-ary.

Example A.6.5 (Little Cubes Operads)**.** Let $[0,1]^n \subseteq \mathbb{R}^n$ be the n-dimensional unit cube. An embedding $[0,1]^n \hookrightarrow [0,1]^n$ is called *rectilinear* if it is of the form

$$(x_1, \ldots, x_n) \mapsto (a_1 x_1 + b_1, \ldots, a_n x_n + b_n) \tag{A.30}$$

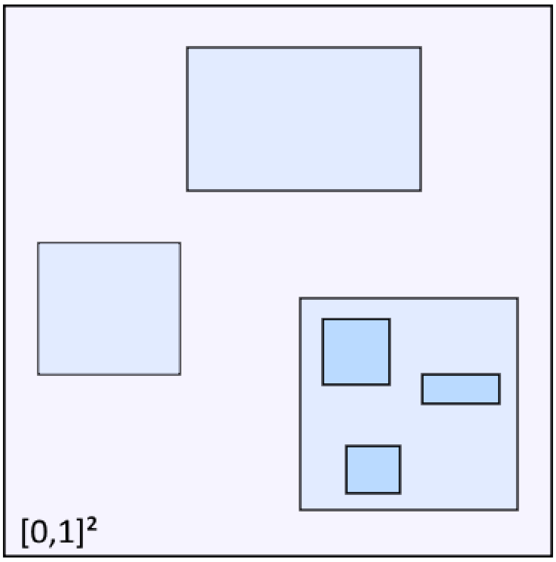
for $a_n \in \mathbb{R}_{>0}, b_n \in \mathbb{R}_{\geq 0}$ chosen in a way that its image lies inside $[0,1]^n$. Similarly, an embedding $[0,1]^n \sqcup \cdots \sqcup [0,1]^n = [0,1]^n \times \{1, \ldots, k\} \hookrightarrow [0,1]^n$ is called rectilinear if its restriction to each copy of $[0,1]^n$ is, and their images are disjoint. Equip the set $\mathrm{Rect}([0,1]^n \times \{1, \ldots, k\}, [0,1]^n)$ of such rectilinear embeddings with the topology induced from its natural embedding into $\mathbb{R}^{2 \cdot n \cdot k}$ via the coordinates a_n, b_n.

The ∞-operad $\mathbb{E}_n^{\otimes}$ has a single object $[0,1]^n$, and the Kan complex of k-ary multimorphisms of is defined as the Kan complex

$$\mathrm{Mul}_{\mathbb{E}_k^{\otimes}}([0,1]^n, \ldots, [0,1]^n; [0,1]^n) := \mathrm{Sing}\,\mathrm{Rect}([0,1]^n \times \{1, \ldots, k\}, [0,1]^n) \tag{A.31}$$

where composition is induced by inserting a cube with embedded cubes at the place of one embedded cube, see the figure.

Figure A.3.: Composition of 3-ary multimorphisms in the $\mathbb{E}_2$-operad



Remark. Instead of embedding n-cubes into n-cubes in a rectilinear way, we could have embedded n-disks into n-disks in a framing-preserving way, or points into $\mathbb{R}^n$. The reason these constructions yield the same ∞-operad is that the respective spaces of multimorphisms are homotopy equivalent (see also the proof of 2.4.4):

$$\begin{aligned}&\mathrm{Rect}([0,1]^n \times \{1,\dots,k\},[0,1]^n) \simeq \mathrm{Emb}^{fr}(\mathbb{R}^n \times \{1,\dots,k\},\mathbb{R}^n) \simeq \\ &\quad \simeq \mathrm{Conf}_k(\mathbb{R}^n) := \mathrm{Emb}(\{1,\dots,k\},\mathbb{R}^n)\end{aligned}$$

Construction A.6.6. Every rectilinear embedding $[0,1]^n \times \{1,\dots,k\} \hookrightarrow [0,1]^n$ can be extended to a rectilinear embedding $[0,1]^{n+1} \times \{1,\dots,k\} \hookrightarrow [0,1]^{n+1}$ applying the identity on the $(n+1)$-component. This induces a continuous map $\mathrm{Rect}([0,1]^n \times \{1,\dots,k\},[0,1]^n) \to \mathrm{Rect}([0,1]^{n+1} \times \{1,\dots,k\},[0,1]^{n+1})$ that is compatible with composition, hence one obtains maps of ∞-operads

$$\mathbb{E}_0^{\otimes} \to \mathbb{E}_1^{\otimes} \to \mathbb{E}_2^{\otimes} \to \dots \tag{A.32}$$

Proposition A.6.7 ([HA, 5.1.1.5])**.** This recovers the ordinary operads $\mathbb{E}_0^{\otimes}, \mathbb{E}_1^{\otimes} = \mathrm{Assoc}^{\otimes}$ as special cases (the first claim is clear; for a proof of the latter, see 2.4.4). Also, taking a colimit in $\mathcal{O}p_\infty$ over the sequential diagram constructed in A.6.6 yields

$$\mathrm{Comm}^{\otimes} = \mathbb{E}_\infty^{\otimes} = \operatorname*{colim}_{n\in\mathbb{N}} \mathbb{E}_n^{\otimes}\,. \tag{A.33}$$

A.7. Symmetric Monoidal ∞-categories and Algebras

Definition A.7.1. Let Fin_* be the ordinary category (or its nerve) of finite pointed sets $\langle n\rangle = \{*, 1, 2, \ldots, n\}$ with pointed maps. In particular, denote by $\rho_i : \langle n\rangle \to \langle 1\rangle$ the map that sends $i \mapsto 1$ and all other elements to $*$, for $i = 1, \ldots, n$.

Definition A.7.2. A *symmetric monoidal ∞-category* $\mathcal{V}^{\otimes}$ with underlying ∞-category $\mathcal{V}$ is a functor $v : \mathrm{Fin}_* \to \mathcal{C}at_\infty$ such that for each n, the functors $v(\rho_i)$ exhibit $v(\langle n\rangle)$ as the product $\mathcal{V}^{\times n}$.

Remark ([HA, 4.1.2.5]). Similarly, *monoidal ∞-categories* can be defined as functors $\Delta^{op} \to \mathcal{C}at_\infty$ satisfying similar properties, but we will need them.

Definition A.7.3. A *partial symmetric monoidal ∞-category* $\mathcal{V}^{\otimes}$ with underlying ∞-category $\mathcal{V}$ is a functor $v : \mathrm{Fin}_* \to \mathcal{C}at_\infty$ such that for each n, the functors $v(\rho_i)$ exhibit $v(\langle n\rangle)$ as a full subcategory of the product $\mathcal{V}^{\times n}$.

Construction A.7.4. For $\mathcal{V}^{\otimes}$ a symmetric monoidal ∞-category, the unique morphism $u : \langle 0\rangle \to \langle 1\rangle$ in Fin_*, and the morphism $t : \langle 2\rangle \to \langle 1\rangle$ sending $* \mapsto *$ and everything else to 1, induce morphisms

$$1_{\mathcal{V}} : * \to \mathcal{V}\,, \quad \otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V} \tag{A.34}$$

called the *unit object* and *tensor product* of $\mathcal{V}$. Other morphisms in Fin_* induce higher coherence relations on them, in particular the morphism $\langle 2\rangle \to \langle 2\rangle$ interchanging 1 and 2 induces a symmetric braiding $V \otimes W \cong W \otimes V$. Similarly for partial symmetric monoidal ∞-categories, but here the tensor product of two objects is not always defined.

Example A.7.5.

- For R a ring, the ∞-category of chain complexes $\mathcal{C}h(R)$ is symmetric monoidal with respect to the tensor product of chain complexes. Similarly for the derived ∞-category $D(R)$ and the derived tensor product.
- $\mathcal{C}h(R)^{op}$ is also symmetric monoidal with respect to $\otimes$, similarly for any symmetric monoidal ∞-category.
- The ∞-category os spectra $\mathcal{S}p$ is symmetric monoidal with respect to the smash product $\wedge$.
- Any ∞-category with finite products is symmetric monoidal with respect to the product, similarly for coproducts.
- For X a topological space, the partially ordered set $\mathrm{Open}(X)$ is (after regarding it as a thin category and taking the nerve) partial symmetric monoidal with respect to disjoint union $\sqcup$.

Definition A.7.6. Any (partial) symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$ can be interpreted as an ∞-operad, with the same objects as $\mathcal{V}$ and multimorphism spaces

$$\mathrm{Mul}_{\mathcal{V}^{\otimes}}(C_1, \ldots, C_n; C) := \mathrm{Map}_{\mathcal{V}}(C_1 \otimes \ldots C_n, C) . \tag{A.35}$$

In the partial case, we set $\mathrm{Mul}_{\mathcal{V}^{\otimes}}(C_1, \ldots, C_n; C) = \emptyset$ if $C_1 \otimes \ldots C_n$ does not exist. In other words, symmetric monoidal ∞-categories are precisely those ∞-operads where the multimorphism spaces are representable by objects in $\mathcal{V}$.

Example A.7.7. Given a symmetric monoidal ∞-category $\mathcal{V}$, we will often look at the ∞-categories of algebras $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{V})$.

- An algebra $A : \mathbb{E}_0 \to \mathcal{V}$ is fully determined by the image $C := A(*)$ of the unique object $* \in \mathbb{E}_0$, and the image $F(u) : 1_{\mathcal{V}} \to C$ of the unique 0-ary morphism $u \in \mathrm{Mul}_{\mathbb{E}_0^{\otimes}}(\emptyset; *)$. In fact, $\mathrm{Alg}_{\mathbb{E}_0}(\mathcal{V}) \simeq \mathcal{V}_{1_{\mathcal{V}}/}$ so we say that $\mathbb{E}_0^{\otimes}$-algebras are *pointed objects* in $\mathcal{V}$.
- An algebra over $\mathbb{E}_{\infty}^{\otimes} = \mathrm{Comm}^{\otimes}$ is determined by the image $C := A(*)$ of the unique object, and the images of the unique n-ary multimorphism for each $n \in \mathbb{N}$. In other words, C is equipped with maps
$$u : 1_{\mathcal{V}} \to C,\ \mathrm{id}_C : C \to C,\ m : C \otimes C \to C,\ C \otimes C \otimes C \to C, \ldots \tag{A.36}$$
These equip the object C with a unit, a multiplication and higher associativity relations on it. In particular, since there is only a single n-ary multimorphism in $\mathbb{E}_{\infty}^{\otimes}$ for each n, the multiplication can not depend on the ordering of the arguments and is automatically commutative (for any reordering $\sigma : \langle n \rangle \to \langle n \rangle$ of the n arguments, and $m_0 : \langle n \rangle \to \langle 1 \rangle$ the map that sends every element except for the point to 1, the multimorphisms $m_0 \circ \sigma = m_0$ in $\mathbb{E}_{\infty}^{\otimes}$ agree). We call $\mathbb{E}_{\infty}$-algebras *commutative algebras.*
- Algebras $A : \mathbb{E}_1^{\otimes} \to \mathcal{V}$ are determined by the image $C := A(*)$ and the induced unit map $u : 1_{\mathcal{V}} \to C$, as well as precisely one n-ary multiplication map $C^{\times n} \to C$ for each ordering on the set $\{1, \ldots, n\}$, together with higher coherence relations. This allows the tensor product to depend on the ordering in a very free way, so that what we obtain is an *associative algebra* without any commutativity.
- $\mathbb{E}_2^{\otimes}$-algebras are again given by a fixed object $C \in \mathcal{V}$ together with a multiplication, but since the space of rectilinear embeddings of 2-dimensional cubes is connected, all orderings on the arguments of this multiplication give isomorphic results. These isomorphisms however are part of the data, and can lead to a sort of braiding. This is explicitly discussed in 2.4.5, it leads to a *braided commutative algebra* that is commutative up to isomorphisms, but has weaker coherence relations on these isomorphisms as an $\mathbb{E}_{\infty}$-algebra.
- $\mathbb{E}_n$-algebras for even higher n become more and more commutative.

In particular,

- Precomposition with the maps of ∞-operads from A.6.6 yields restriction maps from $\mathbb{E}_n$-algebras to $\mathbb{E}_k$-algebras for $k < n$, as expected (e.g. every symmetric algebra is associative).
- Conversely, to any $\mathbb{E}_k$-algebra, we can associate a *center* $\mathbb{E}_n$*-algebra* as we discuss in 4.5.
- If $\mathcal{C}$ is an ordinary symmetric monoidal category, $\mathbb{E}_1$-algebras in $\mathcal{C}$ are the same thing as ordinary algebra objects in $\mathcal{C}$, and $\mathbb{E}_2, \mathbb{E}_3, \dots, \mathbb{E}_\infty$-algebras all agree with commutative algebra objects (by an Eckmann-Hilton-type argument, see below).
- For R a commutative ring, $\mathbb{E}_1$-algebras in $\mathcal{C}h(R)$ are differential graded algebras, and $\mathbb{E}_\infty$-algebras are commutative graded algebras (at least, the respective ∞-categories are equivalent; some straightening is necessary). $\mathbb{E}_n$-algebras for $1 < n < \infty$ interpolate between them, increasing the amount of symmetry. For example, Tannaka-Duality associates to an $\mathbb{E}_1$-algebra a Hopf algebra, and to an $\mathbb{E}_2$-algebra a quasitriangular Hopf algebra.
- For **Cat** the $(2,1)$-category of ordinary categories, with symmetric monoidal structure given by the product, $\mathbb{E}_1$-algebras are monoidal categories, $\mathbb{E}_2$-algebras are braided monoidal categories, and $\mathbb{E}_n$-algebras for $n > 2$ are symmetric monoidal categories.
- For $\mathbf{Cat_2}$ the $(3,1)$-category of $(2,1)$-categories, with symmetric monoidal structure given by the product, $\mathbb{E}_1$-algebras are monoidal, $\mathbb{E}_2$-algebras are braided monoidal, $\mathbb{E}_3$-algebras are sylleptic monoidal and $\mathbb{E}_n$-algebras for $n > 3$ are symmetric monoidal $(2,1)$-categories.
- In the symmetric monoidal $(n+1,1)$-category of $(n,1)$-categories, there are $(n+2)$ distinct notions of $\mathbb{E}_k$-algebras (known as k-tuply monoidal $(n,1)$-categories), where for $k > n+1$ the notions coincide. This is known as the *Baez-Dolan stabilization hypothesis*. It also holds for (n,m)-categories with m arbitrary [GH15]. To visualize what all these notions actually mean: One can identify k-tuply monoidal (n,m)-categories with $(n+k, m+k)$-categories that only have a single object, morphism, ..., $(k-1)$-morphism.
- Eventually, $\mathbb{E}_1$-algebras in $\mathcal{C}at_\infty$ are monoidal ∞-categories and $\mathbb{E}_\infty$-algebras are symmetric monoidal ∞-categories.

Theorem A.7.8 (Dunn additivity)**.** For $k, k' \in \mathbb{N}_0$, the maps from A.6.6 exhibit $\mathbb{E}_{k+k'}$ as the tensor product of the operads $\mathbb{E}_k$ and $\mathbb{E}_{k'}$. This means that for any ∞-operad $\mathcal{O}^\otimes$,

$$\mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Alg}_{\mathbb{E}_{k'}}(\mathcal{O})) \simeq \mathrm{Alg}_{\mathbb{E}_{k+k'}}(\mathcal{O}) \,. \tag{A.37}$$

Remark. Using this result, we may compare the increasing commutativity of $\mathbb{E}_n^\otimes$ with the classical *Eckmann-Hilton argument*: For any set A with two binary operations $\cdot, \times : A \times A \to A$ making A into a monoid with common unit $e \in A$, such that $\cdot$ and $\times$

are compatible in the sense that $(a \cdot b) \times (c \cdot d) = (a \times c) \cdot (b \times d)$ (imagine this by putting a, b, c, d into the corners of a square), both multiplications necessarily agree and are commutative:

$$a \times b = (a \cdot 1) \times (1 \cdot b) = (a \times 1) \cdot (1 \times b) = a \cdot b = (1 \times a) \cdot (b \times 1) = (1 \cdot b) \times (a \cdot 1) = b \times a$$

This leads for example to the fact that higher homotopy groups are commutative.

Definition A.7.9. For $\mathcal{V}_1^{\otimes}, \mathcal{V}_2^{\otimes}$ (partial) symmetric monoidal ∞-categories, a *symmetric monoidal functor* $F : \mathcal{V}_1^{\otimes} \to \mathcal{V}_2^{\otimes}$ is a morphism between them in the slice category $(\mathcal{C}at_\infty)_{/\mathrm{Fin}_*}$. Equivalently, it is a functor of the underlying ∞-categories that preserves unit, tensor product and its braiding up to isomorphism. Such functors are the objects of an ∞-category $\mathrm{Fun}^{\otimes}(\mathcal{V}_1, \mathcal{V}_2)$.

Similarly, a *lax monoidal functor* F is a map between the ∞-operads associated to $\mathcal{V}_1^{\otimes}$ and $\mathcal{V}_2^{\otimes}$. This yields morphisms $F(V) \otimes F(V') \to F(V \otimes V')$ that do not have to be isomorphisms.

Example A.7.10. For $\mathcal{C}$ and arbitrary ∞-category, the *coCartesian ∞-operad* $\mathcal{C}^{\amalg}$ has the same objects as $\mathcal{C}$, and

$$\mathrm{Mul}_{\mathcal{C}^{\amalg}}(C_1, \ldots, C_n; C) := \prod_{i=1}^{n} \mathrm{Map}_{\mathcal{C}}(C_i, C) \ . \tag{A.38}$$

Remark. This ∞-operad is a symmetric monoidal ∞-category iff $\mathcal{C}$ has finite coproducts.

Definition A.7.11. An ∞-operad $\mathcal{O}^{\otimes}$ is called *unital* if there are no non-trivial 0-ary multimorphisms, i.e. for all $X \in \mathcal{O}$ the space $\mathrm{Mul}(\emptyset; X)$ is contractible.

Proposition A.7.12 ([HA, 2.4.3.9])**.** For $\mathcal{O}^{\otimes}$ a unital ∞-operad and $\mathcal{C}^{\amalg}$ a coCartesian ∞-operad, the natural inclusion

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{O}, \mathcal{C}) \tag{A.39}$$

is an equivalence of ∞-categories.

Corollary A.7.13. For $\mathcal{O}^{\otimes}$ a unital ∞-operad, $\mathcal{O}^{\amalg}$ the coCartesian ∞-operad on its underlying ∞-category $\mathcal{O}$, the identity functor on $\mathcal{O}$ induces a canonical map of ∞-operads $\mathcal{O}^{\otimes} \to \mathcal{O}^{\amalg}$.

Proof Sketch. This map is the identity on objects, and the ith component of the map of multimorphism spaces

$$\mathrm{Mul}_{\mathcal{O}^{\otimes}}(C_1, \ldots, C_n; C) \to \prod_{i=1}^{n} \mathrm{Map}_{\mathcal{O}}(C_i, C) \tag{A.40}$$

is induced by simultaneously precomposing with the unique 0-ary morphism in $\mathrm{Mul}_{\mathcal{O}^{\otimes}}(\emptyset; C_j)$ for $j \neq i$. The proof of the general proposition follows the same idea. $\square$

Corollary A.7.14 ([AFT14a, 1.20])**.** Let $\mathcal{O}^\otimes$ be a unital ∞-operad, and $f : \mathcal{D} \to \mathcal{O}$ a right fibration on the underlying ∞-category (for example, the slice projection $\mathcal{O}_{/X} \to \mathcal{O}$ for $X \in \mathcal{O}$). Then, the pullback diagram in $\mathcal{O}p_\infty$ involving the canonical map of operads $\mathcal{D}^\amalg \to \mathcal{O}^\amalg$ and the one from A.7.13

$$\begin{tikzcd} \mathcal{D}^\otimes \arrow[r] \arrow[d] & \mathcal{D}^\amalg \arrow[d] \\ \mathcal{O}^\otimes \arrow[r] & \mathcal{O}^\amalg \end{tikzcd}$$

defines an ∞-operad $\mathcal{D}^\otimes$ with underlying ∞-category $\mathcal{D}$. Explicitly, the multimorphism space $\mathrm{Mul}_{\mathcal{D}^\otimes}(D_1, \ldots, D_n; D)$ is given by

$$\mathrm{Mul}_{\mathcal{D}^\otimes}(f(D_1), \ldots, f(D_n); f(D)) \times_{\prod_i \mathrm{Map}_{\mathcal{O}}(f(D_i), f(D))} \prod_i \mathrm{Map}_{\mathcal{D}}(D_i, D) \ .$$

Proof. This is an immediate corollary of how limits in $\mathcal{O}p_\infty$ are are calculated, in particular the underlying ∞-category of the pullback above is the pullback of the underlying ∞-categories. See [AFT14a, 1.20] for a proof of this fact, since this involves the actual definition of ∞-operads as simplicial sets over Fin_*. □

Definition A.7.15. If we further assume $\mathcal{O}^\otimes$ is a symmetric monoidal ∞-category, we call a multimorphism in $\mathrm{Mul}_{\mathcal{D}^\otimes}(D_1, \ldots, D_n; D)$ *pre-coCartesian* if its image

$$f(D_1) \otimes \cdots \otimes f(D_n) \to f(D)$$

in $\mathcal{O}^\otimes$ is an isomorphism. This equips $\mathcal{D}^\otimes$ with what we call a *weak symmetric monoidal structure.* In particular, a map of ∞-operads $\mathcal{D}^\otimes \to \mathcal{E}^\otimes$ into another symmetric monoidal ∞-category will be called *symmetric monoidal* if it sends pre-coCartesian morphisms to isomorphisms.

A.8. Sifted Colimits

Definition A.8.1. A morphism of simplicial sets $f : L \to K$ is called *right cofinal* if for any ∞-category $\mathcal{C}$, every diagram $p : K \to \mathcal{C}$ and any $C \in \mathcal{C}$, precomposing with f induces a homotopy equivalence of Kan complexes

$$\mathrm{Nat}(p, \underline{C}) \simeq \mathrm{Nat}(p \circ f, \underline{C}) \ , \tag{A.41}$$

where $\underline{C}$ denotes the constant functors $K \to \mathcal{C}$ or $L \to \mathcal{C}$ with value C, respectively.

Proposition A.8.2. In particular, if in the above situation p admits a colimit, then by definition A.2.11 this is equivalent to $\mathrm{Map}_{\mathcal{C}}(\mathrm{colim}\, p, C) = \mathrm{Map}_{\mathcal{C}}(\mathrm{colim}(p \circ f), C)$. In other words, right cofinal morphisms are precisely those that preserve (universal properties of) colimits! Similarly, left cofinal morphisms are those that preserve limits.

Remark. There are many equivalent characterizations of cofinality (see [KER, Tag 02NR]) that are often easier to check than ours, most prominently:

Theorem A.8.3 (Quillen's Theorem A, [KER, Tag 02NY])**.**
A morphism of simplicial sets $F : \mathcal{C} \to \mathcal{D}$ with $\mathcal{D}$ an ∞-category is

- left cofinal iff, for all $D \in \mathcal{D}$, the fiber $\mathcal{C}_{/D} := \mathcal{C} \times \mathcal{D}_{/D}$ is weakly contractible,
- right cofinal iff, for all $D \in \mathcal{D}$, the fiber $\mathcal{C}_{D/} := \mathcal{C} \times \mathcal{D}_{D/}$ is weakly contractible.

Definition A.8.4. Here, a simplicial set K is *weakly contractible* iff the geometric realization $|K|$ is contractible, or equivalently (by the adjunction $|-| \dashv \mathrm{Sing}$), the space of maps $\underline{\mathrm{Hom}}(K, X)$ into any Kan complex X is contractible. Similarly, we define *weak homotopy equivalences* as those maps of simplicial set that become homotopy equivalences after applying $|-|$, or $\underline{\mathrm{Hom}}(-, X)$ for any Kan complex. They are the weak equivalences in the Quillen model structure on sSet.

Definition A.8.5 ([KER, Tag 02PB])**.** An ∞-category $\mathcal{C}$ is called *filtered* if for each simplicial set K, any map $K \to \mathcal{C}$ can be extended to a map $K^{\triangleright} \to \mathcal{C}$.

Proposition A.8.6. An ∞-category $\mathcal{C}$ is filtered iff for any simplicial set K, the diagonal map $\mathcal{C} \to \mathrm{Fun}(K, \mathcal{C})$ that sends C to the constant functor $\underline{C} : K \to \mathcal{C}$ is right cofinal.

This generalizes filtered diagrams in ordinary categories; and colimits parametrized by filtered simplicial sets have similarly nice properties as filtered colimits in ordinary categories. We introduce a slightly larger class of nice colimit diagrams:

Definition A.8.7. A simplicial set K is called *sifted* if it is nonempty and the diagonal morphism $\delta : K \to K \times K$ is right cofinal. A colimit over a diagram $p : K \to \mathcal{C}$ is called *sifted colimit* if K is sifted (just like for filtered colimits).

This notion behaves well with respect to algebraic (in particular, operadic) structures because of the following observation:

Proposition A.8.8. Given ∞-categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$, a functor $\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ preserves sifted colimits separately in both arguments if and only if it preserves sifted colimits in $\mathcal{C} \times \mathcal{D}$.

Proof. For a colimit diagram $p : K \to \mathcal{C} \times \mathcal{D}$ with K sifted, we can decompose it into its projections $p = (p_1 \times p_2) \circ \delta$ with $p_1 = \pi_{\mathcal{C}} \circ p : K \to \mathcal{C}$ and $p_2 = \pi_{\mathcal{D}} \circ p$. We say that $\otimes$ preserves sifted colimits if for any such K,

$$\operatorname*{colim}_K(\otimes p) = \operatorname*{colim}_K(p_1 \otimes p_2) \circ \delta \overset{!}{=} \otimes(\operatorname*{colim}_K p) = \otimes \left(\operatorname*{colim}_K \left((p_1 \times p_2) \circ \delta \right) \right) \ ;$$

and $\otimes$ preserves sifted colimits in both arguments separately iff for any $p_1 : K \to \mathcal{C}$ and $p_2 : L \to \mathcal{D}$,

$$\operatorname*{colim}_{K\times L}(p_1 \otimes p_2) = \operatorname*{colim}_{K} \operatorname*{colim}_{L}(p_1 \otimes p_2) \overset{!}{=} (\operatorname*{colim}_{K} p_1) \otimes (\operatorname*{colim}_{L} p_2) = \otimes \left(\operatorname*{colim}_{K\times L}(p_1 \times p_2) \right) .$$

We are immediately finished if we set $L = K$ and use right cofinality of δ. □

Remark. This generalizes to n-ary multiplications using [KER, Tag 02QM]. From this, one can follow (see [HA, 3.2.3.1]) that sifted colimits of commutative algebra objects and (lax) monoidal functors can be computed pointwise, while more general colimits must be reduced to this case via free resolutions (see [HA, 3.2.3.3] and remark A.8 below).

Definition A.8.9. A symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$ is called *sifted complete* if its underlying ∞-category admits sifted colimits, and the tensor product $\otimes$ preserves sifted colimits in both arguments separately (or equivalently, in $\mathcal{V} \times \mathcal{V}$ by A.8.8).

Example A.8.10 ([AFT14a, 1.18])**.** The symmetric monoidal ∞-categories $\mathcal{C}h(R)^{\otimes}$, $D(R)^{\otimes^L}$, $\mathcal{S}p^{\wedge}$, $\mathcal{S}^{\times}$, $\mathcal{C}at_{\infty}^{\times}$ and the derived category of differentiable vector spaces $D(\mathrm{DVS})^{\hat{\otimes}}$ are sifted complete. Generally, if $\mathcal{C}$ admits finite coproducts, then the co-Cartesian symmetric monoidal structure is sifted complete. The ∞-category $\mathcal{C}h(\mathbb{R})^{op}$ with $\otimes$ is not sifted, since the tensor product does not commute with limits over Δ (called totalizations).

Proposition A.8.11 ([KER, Tag 02QP])**.** The opposite simplex category Δ^{op} is sifted.

Proposition A.8.12. Every filtered simplicial set is sifted.

Proof. Immediate from A.8.6, setting $K = \Delta^0 \amalg \Delta^0$. □

In fact, these two types of diagrams generate all sifted colimits:

Theorem A.8.13 ([HTT, 5.5.8.16])**.** For $\mathcal{C}$ any ∞-category, the following subcategories of the presheaf category $\mathcal{PSh}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ are the same:

- The full subcategory spanned by presheaves that preserve finite products.
- The smallest full subcategory that contains representable presheaves and is closed under sifted colimits
- The smallest full subcategory that contains representable presheaves and is closed under filtered colimits and geometric realizations (i.e. colimits over Δ^{op})

We denote this category by $\mathrm{Fun}^{\pi}(\mathcal{C}^{op}, \mathcal{S})$ and call it the *sifted cocompletion* of $\mathcal{C}$.

Remark. Compare this to the analogous statement for the Ind-completion (or idempotent completion) of $\mathcal{C}$, that consists of presheaves preserving finite limits and is generated by filtered colimits.

Remark. If $\mathcal{C}$ is a 1-category with finite products, then this is also known as the *animation*, or *non-abelian derived category* of the cocompletion of $\mathcal{C}$ under sifted ordinary colimits (the class generated by ordinary filtered diagrams and the full subcategory of Δ spanned by [0] and [1]). For example, if $\mathcal{C}$ is the category of polynomial algebras over a ring R, then its 1-sifted cocompletion is the whole category of R-algebras, and its sifted cocompletion is the derived category of R-algebras (which therefore agrees with the animation of the category of R-algebras).

However, as this construction uses simplicial objects instead of chain complexes (equivalent via Dold-Kan), it allows for deriving non-additive functors, like the functor of taking Kähler differentials (yielding the cotangent complex we used to define the BV complex), exterior/symmetric products, divided power algebras and others. Explicitly, if $\mathcal{D}$ has sifted colimits, the derived functor of $F : \mathcal{C} \to \mathcal{D}$ is its Left Kan Extension along the (corestricted) Yoneda embedding $\mathcal{C} \hookrightarrow \mathrm{Ani}(\mathcal{C})$.

A.9. Lax Limits and Colimits

Some $(\infty, 2)$-category theory is introduced in this section to obtain intuition about lax colimits in the $(\infty, 2)$-categories $\boldsymbol{\mathcal{C}at}_\infty$ of ∞-categories and $\boldsymbol{\mathcal{O}p}_\infty$ of ∞-operads. It should be skipped on a first reading since we use it very little (mostly to build intuition for our conjecture 4.5.9 and for some proofs), it is very technical and we are extremely sketchy to the point of merely conjecturing many statements.

Definition A.9.1. Let K be a simplicial set, $\boldsymbol{\mathcal{C}}$ an $(\infty, 2)$-category, and $p : K \to \boldsymbol{\mathcal{C}}$ a map of simplicial sets. We call $\mathrm{laxlim}(p) \in \boldsymbol{\mathcal{C}}$ the *lax limit* of p and a natural transformation $\alpha : \Delta^1 \times K \to \boldsymbol{\mathcal{C}}$ with $\alpha|_{\{0\}\times K} = \mathrm{const}_{\mathrm{laxlim}(p)}$ the corresponding *lax limit cone*, if for (and naturally in) every $D \in \boldsymbol{\mathcal{C}}$, composition with α induces an equivalence

$$\mathrm{Map}^L_{\boldsymbol{\mathcal{C}}}(D, \mathrm{laxlim}(p)) \simeq \mathrm{Map}^L_{\mathrm{Fun}_{(\infty,2)}(K,\mathcal{C})}(\mathrm{const}_D, p) \, . \tag{A.42}$$

Here, Map^L denotes the left-pinched mapping space since it does not agree with how we usually define Map, and $\mathrm{Fun}_{(\infty,2)}$ is the full subcategory of the ordinary functor category on functors preserving invertible 2-morphisms. Dually, we define *lax colimits* and replacing the left by a right pinch yields *oplax limits* and *oplax colimits.*

Remark. This is relatively imprecise, since we have not even talked about compositions in $(\infty, 2)$-categories. While [KER, Tag 01W4] contains some further information, it is still difficult to give a formal definition in our setting.

These constructions are well understood in $\boldsymbol{Cat_\infty}$ – the following is a main theorem of [GHN15] and [Ber20]:

Theorem A.9.2. Let $F : \mathcal{C} \to \mathcal{C}at_\infty$ be a functor classifying the coCartesian fibration $p : \mathcal{E} \to \mathcal{C}$ in the sense of A.2.12, and let $G : \mathcal{C}^{op} \to \mathcal{C}at_\infty$ classify the Cartesian fibration $p : \mathcal{F} \to \mathcal{C}$. Then,

- $\operatorname{laxcolim} F \simeq \mathcal{E}$.
- The ordinary colimit $\operatorname{colim} F$ is the localization of $\mathcal{E}$ at the class of p-coCartesian morphisms.
- $\operatorname{laxlim} F$ is equivalent to the ∞-category of sections $\operatorname{Fun}_{\mathcal{C}}(\mathcal{C}, \mathcal{E})$.
- $\lim F$ is the full subcategory of $\operatorname{Fun}_{\mathcal{C}}(\mathcal{C}, \mathcal{E})$ spanned by sections that send all morphisms in $\mathcal{C}$ to p-coCartesian morphisms.
- $\operatorname{oplaxcolim} G \simeq \mathcal{F}$,
- $\operatorname{colim} G$ agrees with its localization at the p-Cartesian morphisms,
- $\operatorname{oplaxlim} G \simeq \operatorname{Fun}_{\mathcal{C}}(\mathcal{C}, \mathcal{F})$,
- $\lim G$ is the full subcategory on sections only hitting p-Cartesian morphisms.

Corollary A.9.3. Let $\mathcal{C}$ be an ∞-category and $\underline{\Delta^0}$ denote the constant functor $\mathcal{C} \to \boldsymbol{Cat_\infty}$ on the terminal category. We then find:

$$\mathcal{C} \simeq \underset{\mathcal{C}}{\operatorname{laxcolim}} \underline{\Delta^0} \tag{A.43}$$

Proof. By A.9.2, it suffices to notice that $\mathrm{id} : \mathcal{C} \to \mathcal{C}$ is the coCartesian fibration classified by the functor $\underline{\Delta^0}$. □

Conjecture A.9.4. Let $\boldsymbol{\mathcal{C}}$ be an $(\infty, 2)$-category, $C \in \boldsymbol{\mathcal{C}}$ and $p : K \to \boldsymbol{\mathcal{C}}$ a diagram. Then, there are natural equivalences:

$$\begin{aligned} \underset{k \in K}{\operatorname{laxlim}} \operatorname{Hom}_{\boldsymbol{\mathcal{C}}}(C, p(k)) &\simeq \operatorname{Hom}_{\boldsymbol{\mathcal{C}}}(C, \underset{K}{\operatorname{laxlim}}(p)) \\ \underset{k \in K}{\operatorname{laxlim}} \operatorname{Hom}_{\boldsymbol{\mathcal{C}}}(p(k), C) &\simeq \operatorname{Hom}_{\boldsymbol{\mathcal{C}}}(\underset{K}{\operatorname{laxcolim}}(p), C) \end{aligned} \tag{A.44}$$

Let us talk about cofinality for a moment. We want to find an analogue of the following result for (non-lax) colimits:

Proposition A.9.5 ([KER, Tag 02N5])**.** If $f : A \to B$ is a weak equivalence of simplicial sets and B is a Kan complex, then f is left (and right) cofinal.

Proof. We need to show that for any left fibration $q : B' \to B$, the induced map $\mathrm{Fun}_{/B}(B, B') \to \mathrm{Fun}_{/B}(A, B')$ is a homotopy equivalence, which is an equivalent characterization of cofinality. Since B is a Kan complex, B' is as well and q is a Kan fibration, so that we obtain a commutative diagram of Kan complexes

$$\begin{tikzcd} \mathrm{Fun}(B, B') \arrow[r, "-\circ f"] \arrow[d, "q\circ -"] & \mathrm{Fun}(A, B') \arrow[d, "q\circ -"] \\ \mathrm{Fun}(B, B) \arrow[r, "-\circ f"] & \mathrm{Fun}(A, B) \end{tikzcd}$$

where the vertical arrows are Kan fibrations since q is, and the horizontal arrows are homotopy equivalences since f is a weak equivalence (by definition of those, since Kan complexes are the fibrant objects in the Quillen model structure on sSet). We are finished when we realize that the (homotopy) fibers of the vertical arrows must therefore also be homotopy equivalent. $\square$

Conjecture A.9.6. Similarly, if $f : A \to B$ is categorical equivalence of simplicial sets and B is an ∞-category; then for any diagram $p : B \to \boldsymbol{\mathcal{C}}$ where $\boldsymbol{\mathcal{C}}$ is an $(\infty, 2)$-category, f induces an isomorphism

$$\underset{A}{\mathrm{laxcolim}}(p \circ f) \cong \underset{B}{\mathrm{laxcolim}}(p) \tag{A.45}$$

and similarly for the lax limit.

A lax colimit of a diagram in the $(\infty, 2)$-category of ∞-operads $\mathbf{Op}_\infty$ is called the *assembly* of this diagram, it is studied in [HA] Sections 2.3.3 and 2.3.4.

B. Stratified Spaces

This chapter is intended as a short introduction to the kinds of stratified spaces we want to consider in the main part of this work, with particular emphasis on their homotopy theory. We discuss exit-path categories, constructible sheaves, generalizations of the Seifert-van-Kampen theorem and (topological) exodromy. The last subject is put into the context of many other theorems throughout mathematics that study, in a broad sense, local systems on spaces via their monodromy representation.

B.1. Different Notions of Stratifications

Definition B.1.1. Let $(P, \leq)$ be a poset. We can equip it with the *Alexandrov topology*, where

- Open subsets are precisely the upwards closed subsets
- Closed subsets are precisely the downwards closed subsets
- Locally closed subsets are precisely the intervals

In particular for $p \in P$, the set $P_{\geq p} = \{q \in P | q \geq p\}$ is open, $P_{\leq p}$ is closed and $\{p\}$ is locally closed.

Definition B.1.2. A *P-stratified space*, usually called filtered space, is a topological space X equipped with a continuous map $f : X \to P$, where P carries the Alexandrov topology. The locally closed subspaces $X_p = f^{-1}(p)$ are called *strata* of X, and the closed subspaces $X_{\leq p} = f^{-1}(P_{\leq p})$ are called *closed strata*.

Example B.1.3. An $(\mathbb{N}, \leq)$-stratified space is a topological space X, together with a filtration $\bigcup_{i \in \mathbb{N}} X_i = X$ by closed subspaces X_i with $X_i \subseteq X_j$ for $i \leq j$.

Definition B.1.4. A map of stratified spaces $g : (X \to P) \to (Y \to Q)$ consists of a continuous map $X \to Y$ and an order-preserving map $P \to Q$ (equivalently, continuous with respect to the Alexandrov-topology) such that the following square commutes:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ P & \longrightarrow & Q \end{array}$$

We obtain a category $\mathrm{Top}_{/\mathrm{Alex}}$, and we will call the isomorphisms *stratified homeomorphisms.* Also, restricting to a fixed poset P and continuous morphisms that cover the identity map on P, we obtain a category $\mathrm{Top}_{/P}$.

Definition B.1.5. An *open embedding* $f : (X \to P) \hookrightarrow (Y \to Q)$ of stratified spaces is a map of stratified spaces that induces an open embedding $f : X \hookrightarrow Y$ of topological spaces, as well as open embeddings $f_p : X_p \hookrightarrow Y_{f(p)}$ for each $p \in P$.

Definition B.1.6. For $f : X \to P$ a stratified space, define its *open cone* $C(X) := \frac{X\times[0,\infty)}{X\times\{0\}}$ and equip it with its natural stratification by $P^{\triangleleft} := P \cup \{-\infty\}$ that sends $[x,t] \mapsto f(x)$ for $t > 0$, and the collapsed cone point to $-\infty$.

Definition B.1.7. A stratified space $f : X \to P$ is called *conically stratified* if for any $p \in P$ and any point $x \in X_p$, there exists a neighborhood $x \in U$ with $f(U) = P_{\geq p}$ such that the space U with its restricted stratification $U \to P_{\geq p}$ is stratified homeomorphic to a space of the form $Y \times C(L)$. Here, Y should be a (trivially stratified) topological space and L a $P_{>p}$-stratified space so that we can identity $P_{\geq p} \cong P^{\triangleleft}_{>p}$.

Being conically stratified implies many useful statements about the (stratified) homotopy type of a space, as we will see later. To capture the definition in a view words, it means that our space should locally look like a cone. There is a similar, even more refined notion we will often use in the main text, that mirrors the definition of a topological manifold:

Definition B.1.8. An *n-basic* is inductively defined to be a stratified space of the form $\mathbb{R}^i \times C(L)$, where $i \geq 0$ and its *link* Z is a compact C^0-stratified space of dimension $(n-i-1)$, inductively defined below. To start this induction, the only (-1)-dimensional C^0-stratified space is $\emptyset \to \emptyset$, and there are no basics of negative dimension.

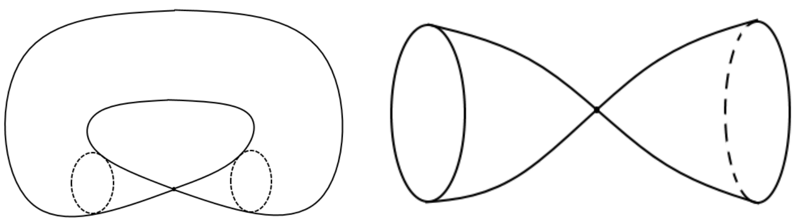
Definition B.1.9. A C^0*-stratified space* of dimension n is a paracompact Hausdorff space that is locally stratified homeomorphic an n-basic, in the sense of B.1.7. We denote the category of them, together with stratified maps, as $\mathrm{Strat}_n^{C^0}$; and if we take only stratified open embeddings of as morphisms, as $\mathrm{Sngl}_n^{C^0}$. Finally, we denote the category of n-basics with stratified open embeddings as morphisms by $\mathrm{Bsc}_n^{C^0}$.

Proposition B.1.10 ([AFT14b, 6.2.2]). Since every space in $\mathrm{Sngl}_n^{C^0}$ can be glued from n-basics, we can obtain an embedding $\mathrm{Sngl}_n^{C^0} \hookrightarrow \mathcal{PSh}(\mathrm{Bsc}_n^{C^0})$. In fact, open coverings constitute a Grothendieck pretopology on $\mathrm{Bsc}_n^{C^0}$, and the functors of points of C^0-stratified spaces are sheaves over it (but not every sheaf is of this form).

Example B.1.11. It follows that the only 0-basic is $C(\emptyset \to \emptyset) = * \to *$, and the only 1-basics are $\mathbb{R}$, $C(* \to *) = (\mathbb{R}_{\geq 0} \to \{0 < 1\})$ and generally $C(\{1, \dots, k\} \to *) = \mathbb{R}_{\geq 0} \times_{\{0\}} \cdots \times_{\{0\}} \mathbb{R}_{>=0} \to *$.

Example B.1.12.

- Since forming a cone always adds an element to the stratification poset, C^0-stratified spaces $(X \to P)$ with $P = *$ must locally look like $\mathbb{R}^i$, so they are precisely topological manifolds. Similarly, one can see that strata of C^0-stratified spaces are always topological manifolds.
- C^0 stratified spaces of dimension 0 are disjoint unions of points (with the trivial stratification), and in dimension 1 we obtain undirected graphs stratified over $\{0 < 1\}$ by sending vertices to 0 and edges to 1.
- Let $N \subseteq M$ be an embedded submanifold, and let us stratify M by $\{0 < 1\}$ by sending N to 0 and $M \backslash N$ to 1. This is a C^0-stratification; an important special case are knots $S^1 \hookrightarrow \mathbb{R}^3$.
- Irreducible complex varieties of pure dimension, with their analytic topology, have a natural C^0-stratification with only even-dimensional strata.
- The *pinched torus* $S^1 \times S^1 / \{0\} \times S^1$ and the *double cone* $S^1 \times \mathbb{R} / S^1 \times \{0\}$ are C^0-stratified of dimension 2; both consist of a singular stratum (the quotient point, with link $S^1 \times S^1$ in both cases) of dimension 0 and a regular stratum.



- The suspension $ST^2 = \frac{[0,1] \times T^2}{\{0,1\} \times T^2}$ of the torus is a C^0-stratified space of dimension 3 with two singular points.
- The topological n-simplex $|\Delta^n| = \{(x_0, \dots, x_n) \in [0,1]^{n+1} | x_0 + \cdots + x_n = 1\}$ possesses a natural $\{0 < \cdots < n\}$-stratification, sending $(x_0, \dots, x_n)$ to the maximal i with $x_i \neq 0$. For example, $|\Delta^1|$ consists of the 0-stratum $\{(1,0,0)\}$ and the 1-stratum given by the remaining half-open line. Alternatively, $|\Delta^n|$ can also be stratified differently by considering it as a manifold with corners.
- Stratifications that are not C^0 include for example most CW-complexes (stratified by their skeleta); and $\{\frac{1}{n} \,|\, n \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}$ stratified by sending everything to $1 \in \{0 < 1\}$, except for $0 \mapsto 0$.

Example B.1.13. Topological n-Manifolds with corners, i.e. spaces that are locally homeomorphic to $\mathbb{R}^{n-i} \times \mathbb{R}^i_{\geq 0}$ for $0 \leq i \leq n$, are C^0-stratified over $\{0 < \cdots < n\}$ if we send every corner to its dimension i (i.e. the interior to n, the boundary to $n-1$, ...). This follows from the fact that C^0-stratified spaces are closed under forming products.

Definition B.1.14. A C^0-stratified space is called a *topological pseudomanifold* if its top-dimensional stratum is dense, and there is no stratum of codimension 1. This allows, for example, the introduction of an orientation class.

Unfortunately, C^0-stratified spaces behave worse than topological manifolds in several aspects, making it convenient to introduce a smoothness condition. Traditionally, one uses Whitney, Thom-Mather and several other kinds of stratifications for this purposes; but we follow a different notion introduced in [AFT14b] that is both easy to work with algebraically, but in our point of view also geometrically clearer. We however only sketch its definition, since we will not use the technical details in the main text. Just like in the definition of a smooth manifold, the idea is to specify what it means for a map between basics to be smooth, and from that to build the notion of a smooth atlas.

Definition B.1.15. Let $f : \mathbb{R}^n \times C(W) \to \mathbb{R}^m \times C(Z)$ be a map of basic C^0-stratified spaces that sends the cone tip to the cone tip, in the sense that it can be restricted to $f|_{\mathbb{R}^n} : \mathbb{R}^n \times * \to \mathbb{R}^m \times *$. We say that f is C^1 *along the cone locus* if the map $\mathbb{R}_{>0} \times T\mathbb{R}^n \times C(W) \to T\mathbb{R}^m \times C(Z)$ given by

$$(t, v, p, [s, u]) \mapsto \left(t, \frac{f|_{\mathbb{R}^n}(tv + p) - f|_{\mathbb{R}^n}(p)}{t}, c_{\frac{1}{t}}(f(p, [ts, u]))\right) \tag{B.1}$$

can be continuously extended to $t = 0$ (necessarily uniquely). Here, c_λ is the map that scales the $\mathbb{R}$-component of $C(Z)$ by λ, and v is a tangent vector in $T_p\mathbb{R}^n \cong \mathbb{R}^n$. We denote this extension at $t = 0$ by $Df : T\mathbb{R}^m \times C(W) \to T\mathbb{R}^n \times C(Z)$. Further, we inductively define that f is C^k *along the cone locus* if Df is C^{k-1} along its cone locus; and f is *conically smooth along the cone locus* if this holds for every $k \in \mathbb{N}$.

Definition B.1.16. An n-dimensional C^0-stratified space $(X \to P)$ is called *conically smooth* if it is equipped with an (equivalence class of) conically smooth atlas $\mathcal{A}_X$, and a stratified open embedding $f : (X, \mathcal{A}_X) \to (Y, \mathcal{A}_Y)$ between conically smooth stratified spaces is called *smooth* if $f^*\mathcal{A}_Y$ is equivalent to $\mathcal{A}_X$. Denote the 1-category of conically smooth stratified spaces and open embeddings by Sngl_n.

Definition B.1.17. A *conically smooth atlas* on an n-dimensional C^0-stratified space $(X \to P)$ is a collection $\mathcal{A}_X = \{(U_i, \phi_i : U_i \hookrightarrow X)\}_{i \in I}$ where U_i are conically smooth n-basics and ϕ_i are stratified open embeddings, such that

- The (U_i) form an open cover of X

- and for any $(U, \phi), (V, \psi) \in \mathcal{A}_X$ and $x \in \phi(U) \cap \psi(V)$, there is a conically smooth n-basic W equipped with conically smooth open embeddings of n-basics $\alpha : W \hookrightarrow U, \beta : W \hookrightarrow V$ such that $\phi \circ \alpha = \psi \circ \beta$ and $x \in \phi \circ \alpha(W) = \psi \circ \beta(W)$.

Two atlases are equivalent if their union is an atlas. For $f : X' \to X$ a stratified open embedding of stratified spaces, we obtain a pullback atlas $f^*\mathcal{A}_X$ on X' consisting of those $\phi : U \to X'$ such that $f \circ \phi$ is compatible with $\mathcal{A}_X$. Similarly, we can form product atlases.

Definition B.1.18. An n-basic $\mathbb{R}^i \times C(Z)$ is called *conically smooth* if the link Z is a compact conically smooth stratified space of dimension $n-i-1$, equipped with an atlas $\mathcal{A}_Z$. Since the dimension decreases, this makes sense inductively. A *conically smooth open embedding of n-basics* $f : U = \mathbb{R}^i \times C(Z) \hookrightarrow V = \mathbb{R}^j \times C(Z)$ is a stratified open embedding such that

- If f preserves the cone locus, it is conically smooth along it as defined above, its differential Df is injective, and apart from the cone locus, it preserves the smooth structure induced from Z in the sense that $f^*\mathcal{A}|_{V-\mathbb{R}^j} = \mathcal{A}|_{f^{-1}(V-\mathbb{R}^j)}$.
- If f does not preserve the cone locus, since it is a stratified map, this means that it factors through $f : U \hookrightarrow \mathbb{R}^j \times \mathbb{R}_{>0} \times Z \hookrightarrow V$. We require that $U \hookrightarrow \mathbb{R}^j \times \mathbb{R}_{>0} \times Z$ is compatible with the atlas induced on the right side by $\mathcal{A}_Z$.

To start the induction, there are no conically smooth basics of negative dimension, and there is a unique (-1)-dimensional conically smooth stratified space $\emptyset$ with a unique atlas. Denote the 1-category of conically smooth n-basics and open embeddings by Bsc_n.

Definition B.1.19. A *conically smooth map* $f : X \to Y$ between conically smooth stratified spaces is a map of stratified spaces such that for all $\phi : U \hookrightarrow X$ and $\psi : V \hookrightarrow Y$ in the respective atlases with $f(\phi(U)) \subseteq \psi(V)$, the composition $\psi^{-1} \circ f \circ \phi$ is a conically smooth map of basics. A *conically smooth map of basics* $f : \mathbb{R}^i \times C(Z) \to \mathbb{R}^j \times C(W)$ is a map of stratified spaces such that

- If f sends the cone locus to the cone locus, it is conically smooth along the cone locus $\mathbb{R}^i$ as in B.1.15, and by induction, $f|_{\mathbb{R}^i \times \mathbb{R}_{>0} \times Z}$ is conically smooth;
- If f does not hit the cone locus, the factorization $f : \mathbb{R}^i \times C(Z) \to \mathbb{R}^j \times \mathbb{R}_{>0} \times W$ by induction is conically smooth.

Example B.1.20.

- A map $f : (\mathbb{R}^n \to *) \to (\mathbb{R}^m \to *)$ is conically smooth iff it is smooth.

- A map $f : (\mathbb{R}^n \times \mathbb{R}_{\geq 0} \to [1]) \to (\mathbb{R}^m \to *)$ is conically C^k iff it is C^k in the interior $\mathbb{R}_{>0} \cong \mathbb{R}$ and C^k along the cone locus. The latter means that the map $Df : T\mathbb{R}^n \times \mathbb{R}_{\geq 0} \to T\mathbb{R}^m$ given by
$$(v, p, s) \mapsto \lim_{t \to 0^+} \left(\frac{f(p + tv, 0) - f(p, 0)}{t}, f(p, ts) \right) \tag{B.2}$$
exists and is C^{k-1} along the cone locus, in particular there is a one-sided derivative along the boundary. Finally, f is conically smooth iff it is conically C^k for all $k > 0$.
- Conically smooth stratified spaces that are trivially stratified are the same thing as smooth manifolds.
- Manifolds with corners are conically smooth when equipped with their canonical stratification. More generally by [NV21], every Whitney stratified space has a canonical conically smooth atlas, in particular every complex variety of pure dimension is conically smooth.

Remark. One of the nice properties of conically smooth spaces is that basics become more rigid than in the C^0 case; for example for $(X \to P)$ conically smooth,

- The canonical map $\mathrm{Aut}(X) \to \mathrm{Aut}(C(X))$ is a homotopy equivalence,
- A map of basics is an isomorphism in $\mathcal{B}sc_n$, which we define in 4.2, iff it is an isomorphism in Bsc_n,
- For X compact, the exit-path category $\mathrm{Sing}^P(X)$ we introduce shortly is equivalent to a finite ∞-category.

The first two statements are part of [AFT14a, 4.3.1], the last one is from [Vol22].

Let us capture the most important flavors of stratified spaces in a diagram, where arrows denote an extension in generality. Note that the variants of smooth stratified spaces are always equipped with extra data like an atlas, so the respective arrows are not fully faithful.

$$\begin{tikzcd}
\text{stratified spaces} & C^0\text{-stratified} \arrow[l, hook'] & \text{topological pseudomanifolds} \arrow[l, hook'] \\
\text{Thom-Mather} \arrow[ur, hook] & \text{conically smooth} \arrow[u, hook] & \\
& \text{Whitney} \arrow[ul, hook] \arrow[u, hook] & \text{complex varieties} \arrow[l, hook'] \arrow[uu, hook]
\end{tikzcd}$$

B.2. Exit Paths

The (weak) homotopy type of a topological space X is described by its singular simplicial set, or fundamental ∞-groupoid, $\mathrm{Sing}(X)$ that we defined in A.1.11. In fact, good topological spaces and ∞-groupoids are more or less the same thing according to the homotopy hypothesis.

We want to find a similar simplicial model for the *stratified homotopy type* of a stratified space $(X \to P)$. Since a stratification equips X with a sense of ordering, or direction, we would expect that this model has non-invertible edges, ie. it should not be a Kan complex. In fact, there is a correspondence (akin to the homotopy hypothesis) between ∞-categories and so-called *directed spaces*, which we could regard stratified spaces as a special case of. We will however take a different approach.

Remember that vertices of $\mathrm{Sing}(X)$ are points of X, edges are paths, 2-simplices are homotopies and so on. What we would expect for stratified spaces is that vertices of their model $\mathrm{Sing}^P(X)$ should still be points of X, but edges should be paths that "move in the direction of the stratification". Let us formalize this:

Definition B.2.1. We introduce a functor $r_{\mathrm{strat}} : \Delta \to \mathrm{Top}_{/\,\mathrm{Alex}}$ that sends $[n]$ to $(|\Delta^n| \to [n])$ with the natural stratification of B.1.12. Using the fact that $\mathrm{Top}_{/\,\mathrm{Alex}}$ has all colimits, and the nerve-realization paradigm A.1.7, we obtain an adjunction

$$\mathrm{Top}_{/\,\mathrm{Alex}} \underset{\mathrm{Sing}^{\mathrm{strat}}}{\overset{|-|_{\mathrm{strat}}}{\rightleftarrows}} \mathrm{sSet}$$

We call the simplicial set $\mathrm{Sing}^{\mathrm{strat}}(X)$ with n-vertices given by $\mathrm{Sing}^{\mathrm{strat}}(X)_n := \mathrm{Hom}_{\mathrm{Top}_{/\,\mathrm{Alex}}}((|\Delta^n| \to [n]), (X \to P))$ the *exit path category* of X.

To be more explicit,

- Vertices of $\mathrm{Sing}^{\mathrm{strat}}(X)$ are points in X,
- For vertices $x, y \in X$, edges between them in $\mathrm{Sing}^{\mathrm{strat}}(X)$ are paths $|\Delta^1| \to X$ that cover an order-preserving map $[1] \to P$, i.e. *exit-paths* in X that start in a lower stratum and immediately exit into a higher stratum in which they stay,
- 2-simplices are homotopies between exit-paths that, according to the stratification of Δ^2, increase in strata,
- Higher simplices are higher homotopies.

In particular, for $p \leq p' \leq p''$ in P and exit-paths $\gamma : x \to y$ starting in X_p and exiting into $X_{p'}$, $\gamma' : y \to z$ starting in $X_{p'}$ and exiting into $X_{p''}$, and $\gamma'' : x \to z$ starting in X_p and exiting into $X_{p''}$, a 2-simplex starting at γ and γ' and ending at γ'' is a homotopy between the concatenation $\gamma' * \gamma$ and γ'' that, apart from beginning and end, completely lies in $X_{p''}$.

Remark. Be aware that $\mathrm{Sing}^{\mathrm{strat}}(X)$ generally does *not* have to be an ∞-category, despite the name. The reason is that paths γ' and γ as above don't necessarily need to have a composite, i.e. a third path γ'' equipped with a 2-simplex as above. The condition that the homotopy needs to lie in $X_{p''}$ may be to strong.

This resolves half of our problem – we can use $\mathrm{Sing}^{\mathrm{strat}}(X)$ as a simplicial model for X. What special properties does this simplicial set possess?

Construction B.2.2 ([DW21, 2.9])**.** For P a poset, regard it as a thin category and denote by $N(P) \in \mathrm{sSet}$ its nerve, which is an ∞-category with no non-trivial isomorphisms. There is a canonical continuous map from the geometric realization $\varphi_P : |N(P)| \to P$: For every non-degenerate simplex of $N(P)$ corresponding to a strictly order-preserving morphism $[n] \to P$, ie. a chain $(p_0 < \cdots < p_n) \subseteq P$, we map the associated simplex $\{(x_0, \ldots, x_n) \in [0,1]^n | \sum x_i = 1\}$ to P via

$$\varphi_P(x_0, \ldots, x_n) := \max\{i \in \{0, \ldots n\} \,|\, t_i \neq 0\}\} \,. \tag{B.3}$$

In particular, $|N(P)|$ is naturally stratified over P and for $P = [n]$, this agrees with the stratification in B.1.12.

Remark. It is a nice exercise to show that this is a well-defined continuous map, understand the stratification in more examples, and to describe the adjoint map $N(P) \to \mathrm{Sing}(P)$.

Observation B.2.3. Postcomposing with, and pulling back along the map φ_P induces an adjunction between slice categories:

$$\begin{tikzcd} \mathrm{Top}_{/P} \arrow[rr, shift right, "- \times_P |N(P)|"'] && \mathrm{Top}_{/|N(P)|} \arrow[ll, shift right, "\varphi_P \circ -"'] \end{tikzcd}$$

Definition B.2.4. Given a simplicial set $(K \to N(P)) \in \mathrm{sSet}_{/P}$ equipped with a map to the nerve of P, we can form the geometric realization $(|K| \to |N(P)|) \in \mathrm{Top}_{/|N(P)|}$. Together with $\phi_P \circ -$ this yields a composition

$$\begin{tikzcd} \mathrm{sSet}_{/N(P)} \arrow[r, shift left, "|-|"] & \mathrm{Top}_{/|N(P)|} \arrow[l, shift left, "\mathrm{Sing}^P"] \arrow[r, shift left, "\varphi_P \circ -"] & \mathrm{Top}_{/P} \arrow[l, shift left, "- \times_P |N(P)|"] \end{tikzcd}$$

where both functors admit right adjoints. The right adjoint Sing^P of the geometric realization (which we identify with the composition of both right adjoints) is constructed as the pullback

$$\begin{array}{ccc} \mathrm{Sing}^P(X) & \longrightarrow & \mathrm{Sing}(X) \\ \downarrow & & \downarrow \\ N(P) & \longrightarrow & \mathrm{Sing}(P)\ . \end{array}$$

Remark. Both of these statements follow from standard theorems on the interaction of adjoints and slice categories, and pasting in the latter case.

Proposition B.2.5. For $(f : X \to P) \in \mathrm{Top}_{/P}$, the constructions $\mathrm{Sing}^{\mathrm{strat}}(X \to P) \cong \mathrm{Sing}^P(X)$ are naturally isomorphic. Similarly, for $(K \to P) \in \mathrm{sSet}_{/P}$, the underlying topological spaces of $|K|_{\mathrm{strat}}$ and $|K \to P|_P$ agree.

Proof. While this can be deduced from abstract nonsense, for the first case this is clear by construction of Sing^P: The pullback in $\mathrm{sSet} = \mathrm{Fun}(\Delta^{op}, \mathrm{Set})$ is computed pointwise, so $\mathrm{Sing}^P(X)$ consists of precisely those simplices σ of $\mathrm{Sing}(X)$ that lie over a simplex of $N(P)$, meaning that they can only go in the direction the edges in $N(P)$ point towards, i.e. upwards in the stratification.

For the geometric realizations, note that both of them as well as the slice projections $\mathrm{sSet}_{/P} \to \mathrm{sSet}$, $\mathrm{Top}_{/P} \to \mathrm{Top}$ preserve colimits, so it is enough to show this on $\Delta^n \to P$. But the underlying space of both realizations by definition is just $|\Delta^n|$ in this case. □

Remark. We have thus learned that the exit-path category $\mathrm{Sing}^{\mathrm{strat}}(X) \simeq \mathrm{Sing}^P(X)$ is equipped with a canonical map to P, and how to calculate the stratified realization.

Theorem B.2.6 ([HA, A.6.4])**.** If $(X \to P)$ is a conically stratified space, the exit path category $\mathrm{Sing}^P(X) \in \mathrm{sSet}$ is a quasicategory.

Definition B.2.7. A functor $F : \mathcal{C} \to \mathcal{D}$ between ∞-categories is called *conservative* if it reflects isomorphisms. This means that if f is a morphism in $\mathcal{C}$ such that $F(f)$ is an isomorphism in $\mathcal{D}$, then f is an isomorphism.

Definition B.2.8. For P a poset, the ∞-category $\mathcal{S}_P$ of *abstract stratified homotopy types* over P is the full subcategory of the slice category $\mathcal{C}at_{\infty/N(P)}$ on conservative functors.

Proposition B.2.9. For $(X \to P) \in \mathrm{Top}_{/P}$, the natural map $\mathrm{Sing}^P(X) \to N(P)$ is conservative. In particular, if $(X \to P)$ is conical, $\mathrm{Sing}^P(X) \in \mathcal{S}_P$.

Proof. Since P is a poset, the only isomorphisms in $N(P)$ are the identities. Therefore, all we need to check is that for each $p \in P$, the fiber $\mathrm{Sing}^P(X) \times_{N(P)} \{p\}$ is an ∞-groupoid. From the definition of $\mathrm{Sing}^P(X)$, we see that morphisms in this fiber are paths in X that stay entirely in X_p, without any conditions from the stratification. This means that they are invertible. In fact, by using the same argument for all simplices in $\mathrm{Sing}^P(X)$, we see that $\mathrm{Sing}^P(X) \times_{N(P)} \{p\} \simeq \mathrm{Sing}(X_p)$ which is a Kan complex (see also [HA, A.7.5]). □

Now, we finally develop the stratified homotopy theory we have promised.

Definition B.2.10. Given stratified spaces $(X \to P)$ and $(Y \to Q)$, stratified maps $f, g : X \to Y$ are called *stratified homotopic* if there is a stratified map

$$H : (X \times \Delta^1 \to Q \times [1]) \to (Y \to Q) \tag{B.4}$$

where Δ^1 is equipped with the above stratification, such that $H|_{X\times\{0\}} = f$ and $H|_{X\times\{1\}} = g$. Further, f is called a *stratified homotopy equivalence* if there is a stratified map $f' : Y \to X$ such that $f \circ f'$ and $f' \circ f$ are stratified homotopic to the respective identity maps.

Definition B.2.11 ([Hai18, 2.1.2])**.** The class of *Joyal-Kan equivalences* is smallest class of weak equivalences in $\mathrm{sSet}_{/N(P)}$ containing both

- Morphisms $(K \to N(P)) \to (L \to N(P))$ where $K \to L$ is a Joyal-equivalence, i.e. a weak equivalence in the Joyal model structure on sSet (a Joyal-equivalence between ∞-categories is an equivalence of ∞-categories),
- For $\Delta^1 \to N(P)$ a constant map, the inclusions $(\{0\} \to N(P)) \hookrightarrow (\Delta^1 \to N(P))$ and $(\{1\} \to N(P)) \hookrightarrow (\Delta^1 \to N(P))$.

Proposition B.2.12 ([Hai18, 2.5.4])**.** For $f : K \to L$ a map of simplicial sets over $N(P)$, if we assume that the fibers of K and L over each $p \in P$ are Kan complexes, then f is a Joyal-Kan equivalence iff it is a Joyal equivalence.

Theorem B.2.13 ([Hai18, 2.5.11])**.** The class W_{JK} of Joyal-Kan equivalences, together with the class of monomorphisms as cofibrations, equips $\mathrm{sSet}_{/P}$ with the structure of a simplicial model category. Its underlying ∞-category, i.e. the localization $N(\mathrm{sSet}_{/P})[W_{JK}^{-1}]$ in the sense of A.2.3, is equivalent to $\mathcal{S}_P$.

Theorem B.2.14 ([Hai18, 3.2.3])**.** Let $\mathrm{Top}^{ex}_{/P}$ be the full subcategory of P-stratified spaces on the spaces $(X \to P)$ where $\mathrm{Sing}^P(X)$ is a quasicategory (in particular, it contains conically stratified spaces by B.2.6); and let W_{ex} be the class of morphisms in it that are sent to Joyal-Kan equivalences by Sing^P. Then, the functor Sing^P induces an equivalence of ∞-categories

$$\mathrm{Sing}^P : N(\mathrm{Top}^{ex}_{/P})[W_{ex}^{-1}] \xrightarrow{\simeq} N(\mathrm{sSet}_{/P})[W_{JK}^{-1}] \simeq \mathcal{S}_P\,. \tag{B.5}$$

In the main text, we use the following corollaries of this result:

Corollary B.2.15. For $(X \to P)$ a conically stratified space, the counit

$$(|\operatorname{Sing}^P(X)|_P \to P) \to (X \to P) \tag{B.6}$$

is in W_{ex}. While the class W_{ex} is not very explicit, it can further by characterized as in [DW21, Section 3.2]. In fact, if X is a triangulable conically stratified space, this counit can even be made into a stratified homotopy equivalence by [DW21, 5.4, 5.9].

Corollary B.2.16. Given a simplicial set $(K \to P) \in \mathrm{sSet}_P$ such that the fibers of K are Kan complexes, a conically stratified space $(X \to P)$ and a stratified homotopy equivalence $(|K|_P \to P) \simeq (X \to P)$, the adjoint map $(K \to P) \to (\operatorname{Sing}^P(X) \to P)$ is a categorical equivalence of underlying simplicial sets.

The latter statement is useful for actually calculating $\operatorname{Sing}^P(X)$. After this technical discussion, let us develop some examples. Recall that we always assume CW complexes are locally finite.

Definition B.2.17. A CW complex X is called *regular* iff the inclusions $\phi : D^n \to X$ of n-cells into X are homeomorphisms onto their image. For arbitrary CW complexes, this is only true in the interior of D^n, and being regular means that this gluing has to be "non-degenerate" along the boundary $\phi_\partial : S^{n-1} \to \mathrm{sk}_{n-1}(X)$ as well.

Proposition B.2.18. If X is a regular CW complex and we denote by $\mathcal{I}_X$ the set of cells in X, then

- $\mathcal{I}_X$ carries a natural partial order,
- There is a canonical stratification $X \to \mathcal{I}_X$ sending each point to the unique cell that contains it in its interior (unless the point is a 0-cell itself, in which case it is sent to this 0-cell),
- This stratification is conical (here, we need X to be locally finite),
- The exit-path category $\operatorname{Sing}^{\mathcal{I}_X}(X) \to \mathcal{I}_X$ is equivalent to the identity map (it does not matter whether we talk about Joyal-Kan or categorical equivalences by B.2.12).

Proof. First, note that a regular CW complex X is in particular normal. This means the set of cells $\mathcal{I}_X$ carries a partial order where $e_1 \leq e_2$ iff, equivalently,

- e_1 is contained in the closure $\overline{e_2}$,
- $e_1 \cap \overline{e_2} \neq \emptyset$,

by [TT18a, 3.1]. This yields a conical stratification on X by [TT18b, 1.7] and a remark in [Lej21, Section 4.2]. To show that the map $\mathrm{Sing}^{\mathcal{I}_X}(X) \to \mathcal{I}_X$ is an equivalence, we proceed by showing it is essentially surjective and fully faithful.

Essentially surjective: In the proof of B.2.9, we saw that the fiber of this map over a cell is just the singular simplicial set of the open cell itself (or, in dimension 0, a point), in particular contractible and non-empty.

Fully faithful: Let e_1 and e_2 be cells in X, and $x \in e_1$, $y \in e_2$. If $e_1 \not\leq e_2$, the mapping space $\mathrm{Map}_{\mathrm{Sing}^S(X)}(x, y)$ is also empty since there can't be a path $\gamma : [0,1] \to X$ from x to y that lies over the arrow $e_1 \to e_2$ in S, as it would have to somehow jump from e_1 to e_2 even though $e_1 \cap \overline{e_2} = \emptyset$, violating continuity.

If $e_1 \leq e_2$, so e_1 lies in the boundary of e_2, we need to show that $\mathrm{Map}_{\mathrm{Sing}^{\mathcal{I}_X}(X)}(x, y)$ is contractible. As in the proof of [HA, A.6.10], we can identify this with $\mathrm{Sing}(P_{x,y})$ with $P_{x,y}$ the space of paths $\gamma : [0,1] \to X$ from x to y such that $\gamma((0,1]) \subseteq e_2$. This only works because we know the stratification is conical so $\mathrm{Sing}^{\mathcal{I}_X}(X)$ is an ∞-category. However $\gamma([0,1]) \subseteq \overline{e_2}$, the image of the gluing map $D^n \to X$ of e_2, which by regularity is a homeomorphism onto its image.

Thus, we can identify $P_{x,y}$ with the space of maps $\gamma : [0,1] \to \mathbb{R}^n$ such that $\gamma(0) = y'$ for some fixed y' with $|y'| = 1$ that corresponds to y, $\gamma(1) = x'$, and $|\gamma(t)| < 1$ for all $0 < t \leq 1$. This can clearly be contracted to the linear path, since the open unit ball is convex. □

Corollary B.2.19. If K is a simplicial complex in the sense of 5.7.1 and we stratify its geometric realization $|K|$ by the poset $\mathcal{I}_K$ of simplices, the exit-path category $\mathrm{Sing}^{\mathcal{I}_K}(|K|) \to \mathcal{I}_K$ is equivalent to the identity $\mathcal{I}_K \to \mathcal{I}_K$.

Proof. By definition of a simplicial complex and its geometric realization, $|K|$ is a regular CW-complex with poset of cells $\mathcal{I}_K$. □

Example B.2.20.

- For a trivially stratified space $X \to \Delta^0$, the exit-path category agrees with the homotopy type $\mathrm{Sing}^{\Delta^0}(X) = \mathrm{Sing}(X)$.
- Using the same argument as in B.2.18, one shows $\mathrm{Sing}^{[1]}(\mathbb{R}_{\geq 0}) \simeq \Delta^1$.
- As a right adjoint, $\mathrm{Sing}^{\mathrm{strat}}$ commutes with products.
- By [AFR15, 3.3.12], if $r : (X \to P) \to (X \to Q)$ is a *refinement*, i.e. a map of stratified spaces determined by the identity on X and an order-preserving surjection $P \to Q$, then the induced functor $\mathrm{Sing}^P(X) \to \mathrm{Sing}^Q(X)$ is a localization (loc. cit. only works in the conically smooth case, but this should hold more generally).

- For $D^n \to [1]$ stratified as a manifold with boundary, choose the triangulation Δ^n. Then,
$$\mathrm{Sing}^{[1]} D^n \simeq (\mathrm{Sing}^{[n]} \Delta^n)[W^{-1}] \simeq \mathcal{P}(\{1,\dots,n\})[W^{-1}] \tag{B.7}$$
where W is the class of face inclusions in Δ^n that do not involve the boundary, and $\mathcal{P}$ denotes the power set ordered by inclusion.

- By [AFT14b, 6.1.4], $\mathrm{Sing}^P(C(X)) \simeq \mathrm{Sing}^P(X)^{\triangleleft}$ in the conically smooth case. In particular, the exit-path category of a basic is
$$\mathrm{Sing}^{P^{\triangleleft}}(\mathbb{R}^i \times C(L)) \simeq \mathrm{Sing}^{\Delta^0} \mathbb{R}^i \times \mathrm{Sing}^{P^{\triangleleft}}(C(L)) \simeq (\mathrm{Sing}^P(L))^{\triangleleft}\,. \tag{B.8}$$

- By [Vol22], the exit-path category of a compact conically smooth stratified space is equivalent to a finite ∞-category.

B.3. Constructible Sheaves

On a topological space X, we can define special classes of sheaves with values in a presentable ∞-category $\mathcal{V}$ that locally do not change:

Definition B.3.1. If $\Gamma^* : \mathcal{V} \to \mathcal{S}h(X,\mathcal{V})$ denotes the left adjoint of the global sections functor, then we call the sheaves $\Gamma^*(V)$ for any $V \in \mathcal{V}$ *constant sheaves.*

Definition B.3.2. Given $F \in \mathcal{S}h(X,\mathcal{V})$, if there is an open cover (U_i) of X such that $F|_{U_i}$ is constant for every i, we call F *locally constant.* The full subcategory on locally constant sheaves will be denoted by $\mathcal{S}h^{lc}(X,\mathcal{V})$.

Definition B.3.3. For $X \to P$ a stratified space, we call a sheaf $F \in \mathcal{S}h(X,\mathcal{V})$ *constructible* if for each $p \in P$, the restriction $F|_{X_p}$ is locally constant. The full subcategory on these will be denoted $\mathcal{S}h^{cbl}(X,\mathcal{V})$ when the stratification is clear.

Definition B.3.4. Similarly, we define *constant hypersheaves* as those that arise as hypercompletions of constant sheaves (or equivalently, via the left adjoint of the global sections functor of $\mathcal{S}h(X)^{hyp}$); *locally constant hypersheaves* as those hypersheaves where the hypercompletions of $F|_{U_i}$ for some open cover are constant hypersheaves; and *constructible hypersheaves* as hypersheaves whose restrictions to strata are locally constant hypersheaves after hypercompleting them.

Warning. Being a locally constant hypersheaf is not equivalent to being locally constant and hypercomplete.

To check whether a given sheaf is locally constant or constructible, we can use the following structure theorem:

Theorem B.3.5 ([PT22] 5.22)**.** Let $X \to P$ be a conically stratified space and $\mathcal{V}$ a presentable stable ∞-category (alternatively, let it be compactly generated). Then, for a sheaf $F \in \mathcal{S}h(X; \mathcal{V})$, the following are equivalent:

- F is a constructible hypersheaf
- For all open subsets $U \subseteq V \subseteq X$ such that the induced map $\mathrm{Sing}^P(U) \to \mathrm{Sing}^P(V)$ is an equivalence, the restriction $F(V) \to F(U)$ is also an equivalence
- For each conical neighborhood $Z \times C(Y)$ in X, any open subsets $U' \subseteq V' \subseteq Z$ such that U, V are weakly contractible, and all $0 < \epsilon < \epsilon'$, application ot F to the inclusions

$$U \times C(Y) \subseteq V \times C(Y)$$
$$Z \times C_{<\epsilon}(Y) \subseteq Z \times C_{<\epsilon'}(Y)$$

yields isomorphisms. Here, $C_{<\epsilon}(Y)$ denotes the open subset of the cone where the real parameter is $< \epsilon$.

Remark. If X is C^0-stratified, every sheaf is hypercomplete by A.5.9, so this becomes a characterization of constructible sheaves.

In the cases of conically smooth stratified spaces and topological manifolds, we can give a more refined characterization:

Proposition B.3.6. For $(X \to P)$ a conically smooth stratified space, a sheaf $F \in \mathcal{S}h(X; \mathcal{V})$ with values in a presentable stable or compactly generated ∞-category is constructible iff for isotopic basics $B \subseteq B' \subseteq M$, by which we mean that the inclusion map $j : B \to B'$ is sent to an isomorphism under $\mathrm{Bsc}(M) \cong \mathrm{Bsc}_{/M} \to \mathcal{B}sc_{/M}$, the image of j under F is an isomorphism in $\mathcal{V}$ as well. In particular, a sheaf on a smooth manifold is locally constant iff it sends every disk inclusion to an isomorphism.

Proof. We showed in 4.3.6 that a sheaf on X is the same thing as a factorization algebra in $\mathcal{V}^{op}$ with symmetric monoidal structure induced by the product in $\mathcal{V}$. Note that this is not a cyclic argument. In other words, a sheaf is a functor $\mathrm{Bsc}_{/M}^{op} \to \mathcal{V}$ compatible with disjoint unions. Now, by the exodromy correspondence B.5.11 and 4.2.1, a sheaf is constructible iff it factors through $\mathrm{Bsc}_{/M} \to \mathcal{B}sc_{/M}$, compare also [AFT14b, 6.1.8]. But using 4.1.10, this is equivalent to saying that it localizes isotopy equivalences of basics. $\square$

Proposition B.3.7. By an analogous proof, a sheaf on a topological manifold M is locally constant iff it sends disk inclusions to isomorphisms. We only need $\mathcal{V}$ to be presentable in this case, since the criterion on $\mathcal{V}$ in [PT22, 5.17] is satisfied because M only has a single stratum.

B.4. Seifert-van-Kampen

To show the Weiss descent property of stratified factorization algebras in section 4.3, we will need a great generalization of the Seifert-van-Kampen theorem. We first remind the reader of the classical statement:

Theorem B.4.1. (Classical Seifert-van-Kampen) Let (X, x) be a path-connected pointed topological space, and $U, V \subseteq X$ path-connected such that $x \in U \cap V$, the intersection is again path-connected, and $U \cup V = X$. Then the following diagram is a pushout square in the category of groups:

$$\begin{tikzcd} \pi_1(U \cap V, x) \arrow[r] \arrow[d] & \pi_1(U, x) \arrow[d] \\ \pi_1(V, x) \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & \pi_1(X, x) \end{tikzcd}$$

Definition B.4.2. Let X be any topological space, $\mathrm{Open}(X)$ its poset of open subsets, and $\mathcal{C}$ an arbitrary 1-category. Then a functor $U : \mathcal{C} \to \mathrm{Open}(X)$ is called a *Seifert-van-Kampen cover*, or shortly SFK cover, if for any $x \in X$, the category $\mathcal{C}_x$, which is the full subcategory on $C \in \mathcal{C}$ with $x \in U(C)$, has weakly contractible nerve as in A.8.4. Put in different words, this means that the geometric realization of its nerve $|\mathrm{N}_\bullet(\mathcal{C}_x)|$ is a contractible CW-complex.

Theorem B.4.3. (Generalized Seifert-van-Kampen, [HA, A.3.1]) If, in the above notation, $U : \mathcal{C} \to \mathrm{Open}(X)$ is a SFK cover, then in the ∞-category of spaces, the following holds:

$$\mathrm{Sing}(X) \cong \operatorname*{colim}_{C \in \mathcal{C}} \mathrm{Sing}(U(C)) \tag{B.9}$$

Proof of B.4.1 using B.4.3. To recover the classical statement from the above generalization, we fist claim that the diagram $U \leftarrow U \cap V \to V$ in $\mathrm{Open}(X)$ is a SFK cover, parametrized by the "walking pushout" category

$$\mathcal{C} = (* \leftarrow * \to *) \quad . \tag{B.10}$$

For any $x \in X \backslash (U \cap V)$, this is clear since $\mathrm{N}_\bullet(\mathcal{C}_x) = \Delta^0$. If $x \in U \cap V$, then $\mathcal{C}_x = \mathcal{C}$, but the geometric realization of $\mathrm{N}_\bullet(\mathcal{C})$ consists of two glued intervals, and is thus contractible. Therefore, we know that $\mathrm{Sing}(X) = \mathrm{Sing}(U) \times_{\mathrm{Sing}(U \cap V)} \mathrm{Sing}(V)$ in $\mathcal{S}$.

We can now apply the fundamental groupoid functor $\pi_1 = h$ on both sides (take the homotopy category); it is the left adjoint ∞-functor of the inclusion of 1-groupoids into ∞-groupoids, so it preserves homotopy colimits and we obtain a groupoid-version of SFK. But all involved spaces are path-connected, so we can replace the fundamental groupoids by the classifying groupoids of the fundamental groups - this is clearly a cofibrant replacement of our diagram, and therefore both a homotopy pushout and an ordinary pushout square. $\square$

Remark. For a more precise and general proof without model category theory, consult [KER, Subsection 012K].

Theorem B.4.4. (Generalized Seifert-van-Kampen for stratified spaces, [HA, A.7.1]) Let $(X \to P)$ be a conically stratified space, and $U : \mathcal{C} \to \mathrm{Open}(X)$ a SFK cover. Then, in the ∞-category $\mathcal{C}at_\infty$,

$$\mathrm{Sing}^P(X) \cong \operatorname*{colim}_{C \in \mathcal{C}} \mathrm{Sing}^P(U(C)) \tag{B.11}$$

B.5. Monodromy and Exodromy

Many central statements across numerous different areas of mathematics follow a common pattern; they are sometimes called Riemann-Hilbert correspondences or Galois correspondences (although the scope of these terms varies a lot between authors). The idea is that they should indicate a correspondence between some sort of sheaves or fibrations on one side, and their transport representation along paths on the other side. Let us start with a very physically relevant example.

B.5.1. Holonomy

Let M be a smooth manifold and G a Lie group.

Definition B.5.1. The *path groupoid* $\mathcal{P}_1(M)$ of M consists of objects being points in M, and morphisms being smooth paths in M that have vanishing derivatives to all orders at their end points (so that their concatenation is again smooth), modulo smooth homotopies covering a vanishing area. It can be refined to a smooth groupoid, i.e. a 1-stack on the site of smooth manifolds.

Definition B.5.2. The delooping BG of G, i.e. the one-object groupoid with the underlying group G as morphism space, can similarly be refined to a smooth groupoid.

Theorem B.5.3 ([BH10, Theorem 1])**.** Connections on the trivial principal G-bundle on M (one can also generalize this to general G-bundles) are in 1-to-1-correspondence with smooth functors

$$\mathrm{hol} : \mathcal{P}_1(M) \to \mathrm{BG} \ . \tag{B.12}$$

Let us not spend too much time on the details of this statement (one has to work over the site of smooth manifolds to capture the smoothness of hol), but focus on the underlying idea: Every G-connection on M is uniquely determined by its holonomy along smooth paths. Note that even for homotopic paths, the holonomy action along them for a particular connection might differ, so that even for contractible loops in M, the holonomy along them might be non-trivial.

B.5.2. Monodromy

We can restrict to a subclass of connections whose holonomy action is homotopy invariant, this are precisely the flat connections:

Theorem B.5.4. For $\mathrm{Flat}_G(M)$ the groupoid with objects principal G-bundles on M equipped with a flat connection, and morphisms of principal bundles (always isomorphisms) that preserve the connection as morphisms, there is an equivalence of categories

$$\mathrm{Flat}_G(M) \simeq \mathrm{Fun}(\pi_{\leq 1}(M), \mathrm{BG}) \tag{B.13}$$

where BG is now an ordinary category. We can bring this into contact with above statement about holonomies by only looking at those $\mathrm{hol} : \mathcal{P}_1(M) \to BG$ that send homotopic paths to the same element of G.

Compare Proposition 1.4.1. We say that a flat connection is uniquely determined by its *monodromy representation*, i.e. by the action of parallel transport along every homotopy class of loops. In fact, we do not even need to be in the smooth setting for such a theorem to hold; it is much more fundamental. Let us recall the following result from covering theory:

Theorem B.5.5. Let X be a locally path-connected, semi-locally simply connected, path connected topological space and denote by $\mathrm{Cov}(X)$ the category of coverings on it and deck transformations, by $\mathrm{Sh}^{lc}(X)$ the category of locally constant 1-sheaves of sets on X (also called *local systems*), and by $\pi_1(X)$-Set the category of sets with an action of the fundamental group. Then, the following correspondence holds:

$$\mathrm{Cov}(X) \simeq \mathrm{Sh}^{lc}(X) \simeq \pi_1(X)\text{-Set} \simeq \mathrm{Fun}(\pi_{\leq 1}(X), \mathrm{Set}) \tag{B.14}$$

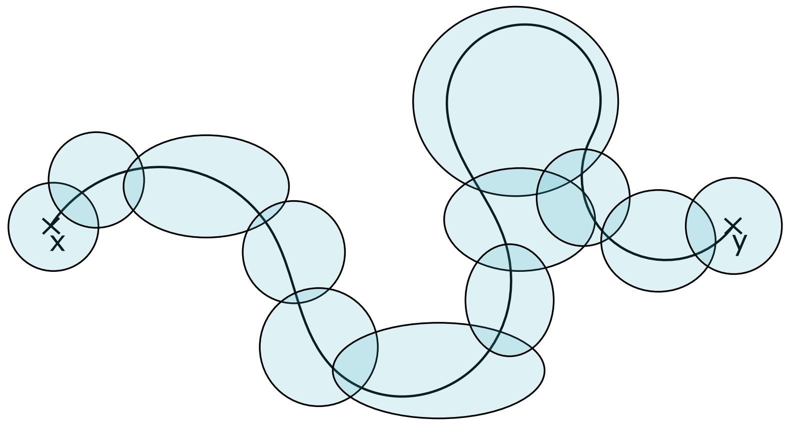
For X not path-connected, the category $\pi(X)$-Set is not equivalent to the rest since it depends on a choice of base point, but apart from that the result still holds. Also, we can replace the category of sets with any presentable 1-category.

Proof Sketch. The first equivalence is induced by constructing the éspace étalé of a locally constant sheaf, which is always a covering, and inversely taking the sheaf of sections of a covering. The last equivalence is also easy to understand, since for X path-connected, the groupoid $\pi_{\leq 1}(X)$ is also connected. It is therefore equivalent to the one-object category associated to the group $\pi_1(X)$, and functors from it into Set are specified by the image of this object, together with an induced $\pi_1(X)$-action.

Finally, we explain how to associate a monodromy representation $m : \pi_{\leq 1}(X) \to \mathrm{Set}$ to any locally constant sheaf F; the converse uses the existence of a universal covering. To each point $x \in X$, we associate the stalk $m(x) := F_x$; so for each path $\gamma : [0,1] \to X$ from x to a point $y \in X$, we need to find a transport map $m(\gamma) : F_x \to F_y$ that is compatible with composition and homotopy invariant. We will construct $m(\gamma)(s_x)$ for a fixed $s_x \in F_x$.

Choose connected open subsets (U_i) in X that cover $\gamma([0,1])$ such that $F|_{U_i}$ is a constant sheaf for each $i \in I$. Since $[0,1]$ is compact, we can reduce to a finite number $U_0, \dots, U_N$ of them such that $x \in U_0$ and $y \in U_N$, see the picture. Note that $F|_{U_i}$ is even a constant presheaf since U_i is connected, so we can canonically identify all stalks in U_i. This allows us to iteratively transport s_x through all of the finitely many U_i until we reach U_N and a germ $m(\gamma)(s_y) \in F_y$. □

Figure B.1.: Parallel transport from x to y by covering the path with small open sets



A similar result also holds in the ∞-setting:

Theorem B.5.6 ([HA, A.4.19])**.** Let X be a topological space that is *locally of singular shape* (this is a difficult property, but topological manifolds and CW complexes are examples). Then,

$$\mathcal{S}h^{lc}(X) = \mathrm{Fun}(\mathrm{Sing}(X), \mathcal{S}) \ . \tag{B.15}$$

More generally, if $\mathcal{V}$ is a presentable ∞-category, then $\mathcal{S}h^{lc}(X;\mathcal{V}) \simeq \mathrm{Fun}(\mathrm{Sing}(X), \mathcal{V})$.

Remark. One can generalize this to spaces X that are locally weakly contractible, if one instead talks about locally constant hypersheaves. We discuss this in B.5.11 below.

Remark. Since every Kan complex is the homotopy colimit of its points, we can rewrite this as

$$\mathcal{S}h^{lc}(X) = \mathrm{Fun}(\operatorname*{colim}_{\mathrm{Sing}(X)} *, \mathcal{S}) \simeq \lim_{\mathrm{Sing}(X)} \mathcal{S} \ . \tag{B.16}$$

This can be interpreted as saying that the category Set acts as a classifying space for covering maps, and the ∞-category of spaces $\mathcal{S}$ acts as a classifying space for local ∞-systems. Analogous monodromy statements can be found all over mathematics:

Theorem B.5.7. For M a smooth manifold and G a Lie group, the isomorphism classes of principal bundles are in 1-to-1-correspondence with homotopy classes of maps $[M, BG]$. In particular, vector bundles are (via their associated principal bundles) in correspondence with $[M, \mathrm{BO}(n)]$. Explicitly, the principal bundle associated to a map $f : M \to BG$ is obtained by pulling back the universal principal bundle $EG \to BG$ along f.

Theorem B.5.8 ([Lur11, Lecture 21])**.** Let R be an associative ring spectrum (an $\mathbb{E}_1$-algebra in $\mathcal{S}p$), X a topological space, and LMod_R the stable ∞-category of R-module spectra. Then,

$$\mathcal{S}h^{lc}(X, \mathrm{LMod}_R) \simeq \mathrm{LMod}_{R\wedge\Omega X} \quad . \tag{B.17}$$

In particular, if $R = HR_0$ is the Eilenberg-MacLane spectrum of an ordinary ring R_0, this allows us to obtain global invariants from sheaves with values in its derived ∞-category $D(R_0) \simeq \mathrm{LMod}_{HR_0}$, for example the L-groups $\mathbb{L}^q(R_0[\pi_1(X)])$ that are important in surgery theory.

Theorem B.5.9 (Fundamental Theorem of Galois Theory, Variation)**.** Let k be a field with separable closure k^{sep}, and $\mathrm{Gal}(k^{sep}|k) = \pi^{\text{ét}}(k)$ its absolute Galois group which agrees with its étale fundamental group. Then, there is an equivalence of categories

$$\mathrm{SAlg}_k^{op} \simeq \pi^{\text{ét}}(k)\text{-Set} \tag{B.18}$$

between separable k-algebras and discrete actions of the absolute Galois group. Finite-dimensional algebras correspond to actions on finite sets.

The Riemann-Hilbert-Correspondence between perverse sheaves and regular holonomic D-modules, and variations of it, are also of this type. More examples related to derived stacks, including a topos-theoretic description of how such situations arise, can be found in 1.4. In particular, using the notation there, let us state the

Theorem B.5.10 (Categorical Geometric Langlands, Best Hope Conjecture [Yooa])**.**
For G a reductive algebraic group with Langlands dual group $\check{G}$, and C a smooth projective curve, the derived category of D-modules on the moduli space $\mathbb{B}\mathrm{un}_G(C) = \underline{Map}(C, \mathbb{B}G)$ is equivalent to the derived category of quasicoherent sheaves on the moduli space of $\check{G}$-local systems over C:

$$D(\mathbb{B}\mathrm{un}_G(C)) \simeq \mathrm{QCoh}(\mathbb{F}\mathrm{lat}_{\check{G}}(C)) \tag{B.19}$$

This statement is provably wrong, but modifications of it are a central object of study in the Langlands Program. In particular, the covariant phase space methods we develop in the first chapter are useful for this purpose, see [EY15].

As a final example, let us point out that the Grothendieck construction A.2.12 can be put in a form resembling such a correspondence (in fact, its proof relies in defining a notion of parallel transport in ∞-categories). Recall from A.2.13 that for $\mathcal{C}$ a small ∞-category,

$$\mathrm{RFib}(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}) \tag{B.20}$$

where just as for classifying spaces, the right fibration corresponding classified by a given monodromy representation $f : \mathcal{C} \to \mathcal{S}$ is constructed via pulling back the universal right fibration $\mathcal{S}_* \to \mathcal{S}$ along f. Similarly,

$$\mathrm{Cart}(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{C}at_\infty) \tag{B.21}$$

is induced by pulling back the universal coCartesian fibration $\mathcal{C}at_{\infty,\mathrm{obj}} \to \mathcal{C}at_\infty$ along a functor $\mathcal{C}^{op} \to \mathcal{C}at_\infty$.

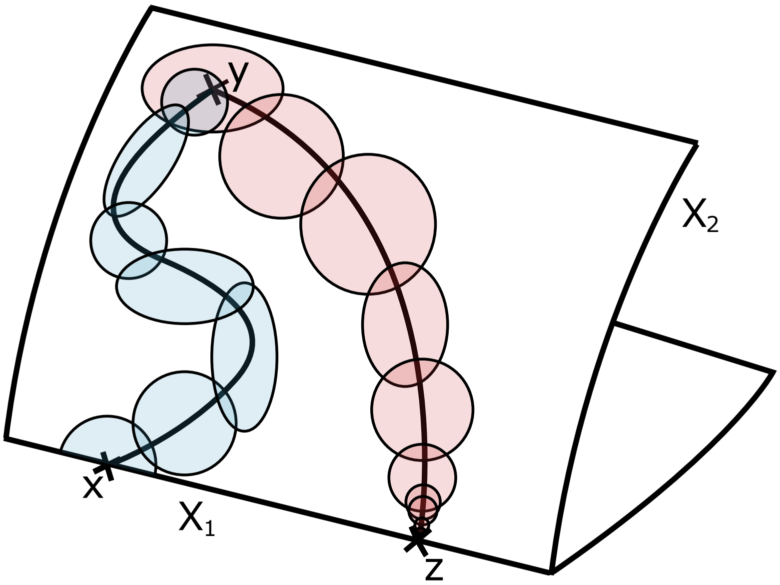
B.5.3. Exodromy

Our next goal is to generalize the monodromy correspondence B.5.6 to constructible sheaves on stratified spaces. Since the abstract homotopy type of a stratified space is described by its exit-path category, which is (on conically stratified spaces) an ∞-category and not a Kan complex like $\mathrm{Sing}(X)$, we expect that there is some directionality involved in the notion of parallel transport that classifies a constructible sheaf. In the classical setting, remember how the monodromy correspondence between locally constant sheaves and representations of the fundamental groupoid was proven by using the local constancy to transport germs along paths inside small open subsets.

Now, suppose we are given an exit path $\gamma : [0,1] \to X$ such that $x := \gamma(0) \in X_1$ and $\gamma((0,1]) \subseteq X_2$, as well as a constructible 1-sheaf $F \in \mathrm{Sh}^{cbl}(X)$ and a germ $s \in \mathcal{F}_x$. By definition of the stalk, there is a small open neighborhood U_0 around y such that s stems from a section of $F(U_0)$, meaning that we can parallel transport s from y to any point in this neighborhood, in particular to some $\gamma(\epsilon)$ with $\epsilon > 0$. From here on, we can work with $F|_{X_2}$ which is locally constant and parallel transport further until we reach $\gamma(1) =: y$, as indicated by the blue open sets in the picture below.

If our path however starts at y and ends in the lower stratum X_1 at z, we might run into a problem as shown by the red sets. Since we can only parallel transport a germ inside of open sets where the respective sheaf is constant, we might never reach X_1 as there does not have to be an open neighborhood around z where $F|_{X_{\leq 2}}$ is constant. We realize that constructible sheaves can only be transported along exit paths, an idea leading us to the exodromy correspondence:

Figure B.2.: Parallel transport is only possible from lower to higher strata



Theorem B.5.11 (Topological Exodromy, [HA, A.9.3] and [PT22])**.**
Let $(X \to P)$ be a paracompact conically stratified space that is locally of singular shape, where P satisfies the ascending chain condition. Also, let $\mathcal{V}$ be either a presentable stable, or a compactly generated ∞-category. Then,

$$\mathcal{S}h^{cbl}(X;\mathcal{V}) \simeq \mathrm{Fun}(\mathrm{Sing}^P(X),\mathcal{V}) \ . \tag{B.22}$$

More generally, if X is a conically stratified space with locally weakly contractible strata and P arbitrary, a similar result holds for constructible hypersheaves:

$$\mathcal{S}h^{hypcbl}(X;\mathcal{V}) \simeq \mathrm{Fun}(\mathrm{Sing}^P(X),\mathcal{V}) \tag{B.23}$$

Here, a topological space is called *locally weakly contractible* if every point possesses an open neighborhood U with trivial homotopy groups (in other words, $\mathrm{Sing}(U)$ is contractible).

Remark. The term *exodromy* stems from applications of this concept to study étale sheaves in algebraic geometry, see [BGH18]. However, the original (topological) statement for ordinary constructible sheaves is due to unpublished work by MacPherson.

Remark. We may use A.9.3 and A.9.4 to rewrite this in analogy with B.16 as

$$\mathcal{S}h^{cbl}(X;\mathcal{V}) \simeq \mathrm{Fun}(\underset{\mathrm{Sing}^P(X)}{\mathrm{laxcolim}}\Delta^0,\mathcal{V}) \simeq \underset{\mathrm{Sing}^P(X)}{\mathrm{laxlim}}\,\mathcal{V} \ . \tag{B.24}$$

In section 2.5, we will add to the terms holonomy, monodromy and exodromy another concept that could be called *multidromy*, which captures the parallel transport properties of locally constant (or constructible, by 5.1.4) factorization algebras.

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Erklärung:

Hiermit erkläre ich, dass ich diese Arbeit selbststädnig verfasst und nur die angegeben Hilfsmittel und Quellen verwendet habe.

Ort, Datum

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