Master Thesis in Mathematics

# L-Groups of Sheaves on Stratified Spaces

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## Abstract

In surgery theory, the obstruction to finding a manifold that is h-cobordant to a given Poincaré complex is an element of the quadratic L-group of locally constant sheaves $L_n^q(\mathbb{Z}[\pi_1 X])$, specifying when global Poincaré duality on this complex can be lifted to a local duality. In the search for a similar statement for stratified spaces, a logical progression would be to look for the obstruction in an L-group of constructible sheaves. Motivated by this thought, the goal of this thesis is to define L-groups of several variations of sheaves, built from Verdier self-dual sheaves in the respective class modulo algebraic bordism, and develop fiber sequences involving the corresponding L-spectra that allow for their computation. This is carried out in the piecewise linear and topological setting, on simplicial complexes and regular CW complexes. The results we obtain exhibit a surprising similarity to statements about Browder-Quinn L-groups that arise in stratified surgery theory.



## Zusammenfassung

In der Chirurgietheorie ist die Obstruktion dazu, eine Mannigfaltigkeit zu finden die h-kobordant zu einem gegebenen Poincaré-Komplex ist, ein Element der quadratischen L-Gruppe von lokal konstanten Garben $L_n^q(\mathbb{Z}[\pi_1 X])$; informell gesagt misst dieses wann wir die globale Poincaré-Dualität auf dem Komplex zu einer lokalen Dualität hochheben können. Wenn wir über Verallgemeinerungen dieser Aussage auf stratifizierte Räume nachdenken, scheint es naheliegend, diese Obstruktion in einer L-Gruppe von konstruierbaren Garben zu suchen. Davon motiviert ist das Ziel dieser Arbeit die Definition von L-Gruppen diverser Klassen von Garben, bestehend aus Äquivalenzklassen von Verdier selbst-dualen Garben modulo algebraischen Bordismen, und das Entwickeln von Fasersequenzen der entsprechenden L-Spektra welche zur Berechnung dieser Gruppen herangezogen werden können. Wir vollziehen dies sowohl im topologischen als auch im PL-Kontext, weiter betrachten wir Simplizialkomplexe und reguläre CW-Komplexe. Unsere Resultate weisen überraschende Parallelen zu ähnlichen Aussagen über die klassischen Browder-Quinn L-Gruppen auf, welche in der stratifizierten Chirurgie-Theorie eine Rolle spielen.

# Contents

# Introduction

A fundamental invariant of a finite-dimensional vector space is its dimension. It is extended to perfect complexes of vector spaces by the Euler characteristic, which takes values in the integers and is invariant under quasi-isomorphisms. For $R$-modules over a commutative ring $R$, this is further refined by the Grothendieck group, as well as higher K-groups, of the perfect derived category $D^{\text{perf}}(R)$: There is a canonical map $K_0(D(R)) \to \mathbb{Z}$ sending the class $[P]$ of a perfect complex $P$ to its Euler characteristic, however generally the $K_0$-group will contain more information than this.

Just as algebraic K-theory generalizes the dimension of a vector space, the goal of *algebraic L-theory* (also called *hermitian K-theory*) is to generalize the signature of a quadratic form on a finite-dimensional vector space. This is done by first defining quadratic forms on chain complexes, and then dividing the space of chain complexes equipped with a quadratic form that exhibits them as self-dual by the relation of algebraic bordism.

Their main application lies in surgery theory, as was clarified in [WR99]. Given a compact oriented topological $n$-manifold $M$, Poincaré duality says that the integration pairing on $C^*(M;\mathbb{Z})$ is non-degenerate, exhibiting $C^*(M,\mathbb{Z}) \simeq \underline{\mathrm{Hom}}(C^*(M,\mathbb{Z}),\mathbb{Z})[-n]$ as self-dual up to a shift so we obtain a class in $\mathbb{L}^s(\mathbb{Z})$. If $n = 4k$ this recovers the signature of $M$, but for $n = 4k+1$ we obtain the new *deRham invariant.*

Conceptually, Poincaré duality follows from the fact that the constant sheaf $\underline{\mathbb{Z}}$ on $M$ is Verdier self-dual up to a shift by $n$, as $\omega_X = \mathbb{D}\underline{\mathbb{Z}} \cong \underline{\mathbb{Z}[-n]}$. It therefore defines an element of the visible symmetric L-group $L^{vs}_n(M,\mathbb{Z})$ of Verdier self-dual locally constant sheaves on $M$, called the *visual symmetric signature* generalizing the ordinary signature if $M$ is not simply connected. Stemming from this observation, the visual quadratic L-group $L^q_n(\mathbb{Z}[\pi_1 M])$ appears in the surgery exact sequence, containing the obstruction to finding a manifold homotopy equivalent/ h-cobordant to a given Poincaré complex $X$, as it controls when the global Poincaré duality on $X$ lifts to a local duality, i.e. can be expressed using Verdier duality as just sketched.

If we however generalize from a topological manifold to a topological pseudomanifold $X$, which is in particular equipped with a stratification, the sheaf $\underline{\mathbb{Z}}$ will generally not be Verdier self-dual anymore. Given that suitable conditions are satisfied, namely we are working with a so-called *Intersection Poincaré space*, there exists a formidable replacement: The intersection homology sheaf $\mathrm{IC}^{\overline{m}}(X;\mathbb{Z})$ of middle perversity $\overline{m}$. This is however not a locally constant, but a constructible sheaf, meaning that it is locally constant on strata of $X$. From this observation, we are lead to the guess that L-groups of constructible sheaves on $X$ might contain the appropriate surgery obstruction in this case, or are at least related to it.

The main goal of this thesis is to define symmetric and quadratic versions of such groups, and understand some of their proporties allowing a partial justification of the above guess, as well as their calculation in some examples. We will not only do this for topologically stratified spaces, but also in the piecewise linear case, for simplicial complexes and (regular) CW complexes. Our main technical tool is the theory of $\infty$-categories and homotopy-coherent algebra, since this allows us to divert many complications from our specific application into this well-developed apparatus. In particular, many results from classical sheaf theory extend to greater generality in this context. Algebraic L-theory in the setting of stable $\infty$-categories with quadratic functors (to be more specific, Poincaré $\infty$-categories) was first introduced in Lurie's Lecture notes [Lur11], and further developed in the series of papers [CDH+20a], [CDH+20b], [CDH+21] by nine different authors.

Apart from these works, our approach was motivated by fiber sequences for Browder-Quinn L-spectra as discussed in [Bro75], [AP17], [Wei94] that seemed similar to the sequences we derive in Corollary 6.5.5. During the creation of this work, we learned about the similar idea of building Witt groups of (constructible) Verdier self-dual sheaves modulo algebraic bordism in [Woo08] and [SW20] in the classical setting, and our results can be regarded as a generalization and further refinement, even though the techniques we use to obtain them are different. In fact, the possibility of an extension of these results using the hermitian K-theory of stable $\infty$-categories was already a remark in [Vol22, Remark 4.9], which we carry out in this work.

In Chapter 1, we lay the technical groundwork in Higher Category Theory and Higher Algebra, in particular we introduce $\infty$-categories, $\infty$-sheaves, stable $\infty$-categories and spectra, (symmetric) monoidal $\infty$-categories, algebra and module objects, and brave new algebra over ring spectra. Our goal is to make this text as self-contained as possible, the reader is only assumed to have a good knowledge of algebraic topology and (ordinary) category theory (including Kan extensions and model categories).

Equipped with these fundamentals, we introduce the algebraic L-theory of Poincaré $\infty$-categories in Chapter 2, as far as it is needed for the subsequent chapters. Also, we discuss the special cases of derived $\infty$-categories of ordinary rings, and perfect modules over ring spectra.

Chapter 3 finishes the theoretical background by defining several ways to decompose stable $\infty$-categories and Poincaré $\infty$-categories. This will later form the categorical counterpart of decomposing a space into strata.

We begin with the actual applications in 4, constructing Poincaré $\infty$-categories of sheaves on simplicial complexes and piecewise linear spaces. This has the technical advantage of being fairly combinatorial, and the results are better behaved than in the topological world. We generalize Lurie's work by allowing for a stratification on our spaces, and show how the L-spectra of constructible sheaves can be iteratively decomposed into strata.

Things become more complicated in the topological case, so we first discuss the case without a stratification in 5. This involves defining Verdier duality and the six-functor formalism for $\infty$-sheaves, and a study of locally constant sheaves and their monodromy representations.

Finally, Chapter 6 extends our results on stratified PL spaces to the topological setting; it requires the introduction of a good deal of stratified homotopy theory and the exodromy correspondence. We finish with a conclusion comparing the different settings.

## What is new?

A large part of this text consists of introductions to the overwhelming amount of background material we need as well as to other expositions, so new developments are mostly relegated to the later sections 3.4, 3.5, 4.5, 4.6, 5.4, 5.5, 6.4, 6.5 and 6.6 which in turn mostly consist of new results, unless otherwise specified. Apart from these chapters, we also want to highlight

- A version of the classical 9-Lemma for (split) (Poincaré-)Verdier sequences 3.3.8,
- A biduality statement for Verdier duality on hypersheaves with perfect stalks and costalks 5.2.12.

Finally, many of the other proofs we give are optimized to our setting, worked out side-remarks from Lurie's notes, or classical proofs that we adapted the $\infty$-setting.

## Notation and Conventions

- We denote the natural numbers including zero by $\mathbb{N}_0$, and excluding zero by $\mathbb{N}^+$ to avoid confusion.
- A topological space is *locally compact* if any open neighborhood of any point contains a compact neighborhood, i.e. we are using the strong version of this notion.
- CW complexes are always locally finite.
- Unless stated otherwise, we use cohomological grading for chain complexes. The grading increases from left to right, and the shift acts as $C[1]_{-1} = C_0$.
- We work with Grothendieck pretopologies instead of Grothendieck topologies.
- In an adjunction, the upper arrow is always the left adjoint.

- We denote Verdier duals of functors by a shriek !, exceptional right adjoints by a minus and exceptional left adjoints by a plus. This distinction from the classical notation using a ! for all of them is necessary since we will have to work with Verdier duals of exceptional adjoints like $f_{+!}$, and even further adjoints that we denote in the following sequence:

$$\cdots \dashv f_{++} \dashv f^{+} \dashv f_{+} \dashv f^{*} \dashv f_{*} \dashv f^{-} \dashv f_{-} \dashv f^{--} \dashv \ldots$$

- Most of the time, we ignore size issues for ordinary and $\infty$-categories; for the situations where they are important we fix a small, a large and a very large Grothendieck universe.

- The term "$\infty$-categories" always refers to $(\infty, 1)$-categories; and "ordinary categories" refers to 1-categories. We generally do not equip categorical constructions with an $\infty$-symbol in front of them, since we almost exclusively work in the $\infty$-setting and want to avoid cluttered notation. The reader is safe to assume we are referring to these higher notions, unless explicitly stated otherwise.

- Our model of choice for $\infty$-categories are quasi-categories, as developed in [Lur09a]. In particular, higher categories are always weak, not strict.

- An $\infty$-category is called (co)complete if it admits small (co)limits, and bicomplete if it admits both. We often suppress the "small" in this statement, but it is always implicitly assumed.

- A functor is left exact if it preserves finite limits, and right exact if it preserves finite colimits. Similarly, precomposing with left cofinal maps leaves limits and precomposing with right cofinal maps leaves colimits invariant.

- We use the word "essential" if a specific property should be completed to a non-evil notion, for example the essential image of a functor $F : \mathcal{C} \to \mathcal{D}$ are those $D \in \mathcal{D}$ that are isomorphic to an object in the image. In 3, we often keep this implicit to not clutter notion, see the remarks there.

- By a non-full subcategory of an $\infty$-category, we always mean a subcategory that is spanned by a subset of objects and morphisms, but still contains *all* of the $n$-morphisms between those for $n \geq 2$. We never leave out those higher morphisms.

- We usually denote (symmetric) monoidal $\infty$-categories as a pair $(\mathcal{V}, \otimes)$ of an $\infty$-category and a product operation, even though more data are actually involved: The associated functor $\mathrm{Fin}_* \to \mathcal{C}at_\infty$ is usually denoted by $v$, and the classifying coCartesian fibration by $\mathcal{V}^{\otimes} \to \mathrm{Fin}_*$.

# Acknowledgments

First of all, I want to thank my supervisor for his time, encouragement, and many helpful discussions. I greatly benefited from the rich mathematical environment at Heidelberg University, allowing for fruitful discussions relevant to this work with Raphael Senghaas, Max Stier, Lukas Waas and numerous others. Several remarks by David Reutter were also helpful during its preparation. Last, but certainly not least, I want to thank my family for their continuous love and support.

# 1 Higher Category Theory

In this chapter, we develop the background on $\infty$-categories that is needed for our further discussion; in particular stable $\infty$-categories, spectra (equipped with homotopy coherent algebraic structure) and higher sheaf theory. We mostly follow [Lur18a] and [Lur17], while describing the structures of interest in a way that keeps technicalities and amount of background material as low as possible.

## 1.1 $\infty$-categories

We assume the reader is familiar with basic category theory (e.g. limits, adjunctions, slice categories), enriched categories and Kan extensions. Let us still, for comparison, repeat the definition of an ordinary category:

**Definition 1.1.1.** A (small) category $\mathcal{C}$ consist of

- A set of objects,
- For any two objects $X, Y \in \mathcal{C}$ a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms between $X$ and $Y$,
- For all $X, Y, Z \in \mathcal{C}$ an associative composition map
$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)\ , \tag{1.1}$$
- For any $X \in \mathcal{C}$ an identity morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ that does not change morphisms under composition.

*Remark.* Being small refers to the fact that objects and morphisms are sets (in a fixed universe), we will also encounter many cases where this is not the case. Still, let us avoid set-theoretic problems unless they are actually important.

**Definition 1.1.2.** A category if called a *groupoid* if every morphism $f : C \to D$ in it is *invertible*, i.e. there exists a $g : D \to C$ such that $f \circ g = \mathrm{id}_D$ and $g \circ f = \mathrm{id}_C$.

**Example 1.1.3.**
- Examples of categories can be found all over mathematics, e.g. the category Set of sets and maps between them, the category Top of topological spaces and continuous maps, the category Ab of abelian groups and homomorphisms or the category Cat of categories and functors.

- There are also important examples of groupoids: For each group $G$, we can construct a groupoid $BG$ with one object $*$ and $\mathrm{Hom}_{BG}(*,*) = G$, where composition is given by the group operation and inverses exist because $G$ has inverses.
- For $X$ a topological space, we may also introduce the *fundamental groupoid* $\pi_{\leq 1}X$ with objects the points of $X$, and morphisms the homotopy classes of paths between the respective points. Composition is given by concatenation of paths, and inverses exist since paths can be followed in the inverse direction.

Some of these examples seem to possess further information that we were not able to capture:

- Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, the category Cat allows for a space of natural transformations $\mathrm{Nat}(F, G)$. In other words, there are morphisms between morphisms, and these can also be composed.
- It is a bit strange that $\pi_{\leq 1}X$ contains a large amount of objects, uncountably many for almost all manifolds; but the morphisms only consist of homotopy classes of paths, instead of actual paths. Why is that? Note that concatenation of paths is, in itself, not associative, but only so up to a reparametrization (i.e. a homotopy). If we want to retain information about individual paths, we therefore need to add information about homotopies into the mix.

Both of these problems can be resolved by 2-categories, also called *bicategories*. They should consist of a set of objects, together with a set of morphisms between any two objects and a set of 2-morphisms between any two morphisms that have a common source and target. Also, they feature composition operations for morphisms and 2-morphisms, as well as associativity constraints and identity (2-)morphisms. Composition of 1-morphisms should only be associative up to an invertible 2-morphisms (the associator), and identity 1-morphisms should only act as identities up to invertible 2-morphisms as well; we see this in the second example since concatenation of paths is not strictly associative. We therefore always speak about *weak* 2-categories, instead of *strict* 2-categories where associativity and identity conditions hold on the nose. Finally, the invertible 2-morphisms in the above definition should be considered as extra data in a 2-category, and they must themselves satisfy higher coherence relations, like the pentagon identity (see [Lur18a, Tag 007Q] for a precise definition).

- The (strict) 2-category ***Cat*** consists of (small) categories as objects, functors as 1-morphisms and natural transformations as 2-morphisms.
- The (weak) 2-category $\pi_{\leq 2}X$ of a topological space $X$ consists of points of $X$ as objects, paths in $X$ as morphisms, and homotopy classes of homotopies as 2-morphisms (we need to think about homotopy classes again to satisfy the strict associativity for 2-morphisms). This is even a *2-groupoid*, since morphisms and 2-morphisms are invertible.

But now, the second example suffers from a similar issue concerning homotopy classes as before. This points us toward a straightforward idea: Why do we not define 3-categories, 4-categories etc., as well as fundamental $n$-groupoids $\pi_{\leq 3}X, \pi_{\leq 4}X, \ldots$ consisting of objects, morphisms, 2-morphisms, 3-morphisms and so on? The reasons why this is not a priori a good idea:

- We need to add composition operations for each type of morphism, that can interact with each other (horizontal composition, whiskering) and satisfy associativity and identity constraints up to higher isomorphisms – that also need to be part of our data! Also, these higher isomorphisms must satisfy their own coherence relations up to even higher isomorphisms, which are subject to even higher coherence relations and so on. Even the definition of a (weak) 3-category is so complicated that it is extremely hard to work with – of course, things are a lot simpler for strict $n$-categories.
- Even if we could define an $n$-category (even $n$-groupoid) $\pi_{\leq n}X$ for each $n \in \mathbb{N}$, this still would not resolve our problem since the $n$-morphisms are still given by homotopy classes of maps.

Surprisingly, it is possible to resolve both issues at once by figuratively going two steps forward and one step back: Things surprisingly become a lot simpler when we do not look at $n$-categories, but at $n$-groupoids, where $m$-morphisms for all $1 \leq m \leq n$ are invertible. Letting $n$ go towards $\infty$, there should be for each topological space $X$ an $\infty$-groupoid $\pi_{\leq\infty}X$ that knows about points, paths, homotopies, homotopies of homotopies etc. in $X$. Since homotopies from the constant path to itself are just embedding of $S^2$ into $X$, and similarly for the other levels, this means that $\pi_{\leq\infty}X$ knows about all homotopy groups and hence, at least if $X$ is a CW complex (by Whitehead), the full homotopy type of $X$. Thus, $\infty$-groupoids, which contain $n$-groupoids as special cases, are intimately related to (CW) topology and homotopy theory – but their definition should still be "algebraic", which can be achieved by working with simplicial complexes as models. A bit of ordinary category theory is necessary to understand it:

**Definition 1.1.4.** We define the *presheaf category* of a given small category $\mathcal{C}$ as $\mathrm{PSh}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$, note that it is never small unless $\mathcal{C} = \emptyset$. There is always a fully faithful, limit-preserving functor $h : \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$, the *Yoneda embedding*, which sends $C \mapsto \mathrm{Hom}_{\mathcal{C}}(-, C)$.

**Theorem 1.1.5** (coYoneda Lemma)**.** For $\mathcal{C}$ a (small) category, any presheaf $F \in \mathrm{PSh}(\mathcal{C})$ may be written as a colimit of representable ones (i.e. those that lie in the image of the Yoneda embedding):

$$\forall C' \in \mathcal{C}: \ F(C') = \operatorname*{colim}_{C \in \mathcal{C}_{/F}} \mathrm{Hom}_{\mathcal{C}}(C', C) = \int^{C \in \mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(C', C) \times F(C) \qquad (1.2)$$

*Remark.* $\mathcal{C}_{/F}$ denotes the generalized slice category (also called comma category) $\mathcal{C} \times_{\mathrm{PSh}(\mathcal{C})} \mathrm{PSh}(\mathcal{C})_{/F}$, which by the Yoneda lemma agrees with the category of elements $\int F$ (i.e. the category of pairs $(C, a)$ with $C \in \mathcal{C}$ and $a \in F(C)$). The latter coend expression is also called the *Ninja Yoneda Lemma* [Lor15], it tells us that the Hom-functor acts as a delta distribution in the coend. We will not use it further, but it is helpful in the proof.

*Proof.* We start by recalling the usual Yoneda Lemma:

$$F(C') = \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(-, C'), F) \tag{1.3}$$

For $G \in \mathrm{PSh}(\mathcal{C})$ any other presheaf, above colimit is characterized by

$$\mathrm{Nat}(\operatorname*{colim}_{C \in \mathcal{C}_{/F}} \mathrm{Hom}_{\mathcal{C}}(-, C), G) \cong \lim_{C \in \mathcal{C}_{/F}} \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(-, C), G) = \lim_{C \in \int F} G(C) \tag{1.4}$$

As the last limit is taken in Set, we may describe it as the set of families $(b_{C,a} \in G(C))_{C \in \mathcal{C}, a \in F(C)}$ such that for any morphism $f : C \to C'$ in $\mathcal{C}$, we have the compatibility $b_{F(C'),F(f)(a)} = G(f)(b_{C,a})$. Rewriting this, we see that $\eta_C : F(C) \to G(C)$ sending $a \mapsto b_{C,a}$ assemble into a natural transformation $F \Rightarrow G$. In other words, the limit agrees with $\mathrm{Nat}(F, G)$, as claimed. □

*Technical Remark.* While this was quite cumbersome, proving the coend expression is a lot easier (and the colimit expression can ultimately be derived from it, using how weighted colimits/ Kan extensions can be written as coends). Let $S \in \mathrm{Set}$, then:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Set}}\left(\int^{C \in \mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(C', C) \times F(C), S\right) &\cong \int_{C \in \mathcal{C}} \mathrm{Hom}_{\mathrm{Set}}\left(\mathrm{Hom}_{\mathcal{C}}(C', C) \times F(C), S\right) \cong \\
&\cong \int_{C \in \mathcal{C}} \mathrm{Hom}_{\mathrm{Set}}\left(\mathrm{Hom}_{\mathcal{C}}(C', C), \mathrm{Hom}_{\mathrm{Set}}(F(C), S)\right) \cong \\
&\cong \mathrm{Nat}\left(\mathrm{Hom}_{\mathcal{C}}(C', -), \mathrm{Hom}_{\mathrm{Set}}(F(-), S)\right) \cong \mathrm{Hom}_{\mathrm{Set}}(F(C'), S)
\end{aligned}$$

**Corollary 1.1.6.** Let $\mathcal{C}, \mathcal{D}$ be small categories, and let $\mathcal{D}$ contain all small colimits. Then, precomposing with the Yoneda embedding $h$ induces an isomorphism

$$\mathrm{Fun}^{\mathrm{colim}}(\mathrm{PSh}(\mathcal{C}), \mathcal{D}) \cong \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \quad , \tag{1.5}$$

where $\mathrm{Fun}^{\mathrm{colim}}$ denotes the colimit-preserving functors and the inverse is given by *Yoneda extension*, i.e. Left Kan Extension $\mathrm{Lan}_h$. In other words, any colimit-preserving functor on the presheaf-category of $\mathcal{C}$ is determined by its action on $\mathcal{C}$, which seems clear since we have shown that $\mathcal{C}$ generates $\mathrm{PSh}(\mathcal{C})$ under colimits.

Intuitively, we should think of $\mathcal{C}$ as a category of model objects, and of $\mathcal{PSh}(\mathcal{C})$ as the category of objects that can possibly be modeled, or tested, by objects in $\mathcal{C}$. The Yoneda extension allows us to define a functor on the simple models, and immediately extend it to this much larger category. After this preparation, let us introduce the most-used theorem in this chapter, which tells us that the extended functor we obtain always possesses a right adjoint:

**Theorem 1.1.7** (Nerve-Realization Paradigm)**.** Let $r : \mathcal{C} \to \mathcal{D}$ be a functor, $\mathcal{C}$ be small, and $\mathcal{D}$ admit all (small) colimits. This functor induces an adjunction

$$\mathrm{PSh}(\mathcal{C}) \underset{|-|}{\overset{N}{\rightleftarrows}} \mathcal{D}$$

where the Yoneda extension $|-| := \mathrm{Lan}_h r$ is called the associated *realization* functor, and $N(D) := \mathrm{Hom}_{\mathcal{D}}(r(-), D)$ the associated *nerve*. In fact, any adjunction containing a presheaf category arises in this way.

*Proof.* The left Kan extension exists because $\mathcal{D}$ has all colimits, we show that $|-| \dashv N$. For $D \in \mathcal{D}, F \in \mathrm{PSh}(\mathcal{C})$, we must construct a natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(\mathrm{Lan}_h r(F), D) \cong \mathrm{Nat}(F, \mathrm{Hom}_{\mathcal{D}}(r(-), D)) \ . \tag{1.6}$$

Both sides send colimits in the argument $F$ to limits, and by the coYoneda lemma above the presheaf $F$ is a colimit of representable presheaves. Without loss of generality, we may therefore assume that $F = \mathrm{Hom}_{\mathcal{C}}(-, C)$ is representable. But then

$$\mathrm{Hom}_{\mathcal{D}}(\mathrm{Lan}_h r(F), D) \cong \mathrm{Hom}_{\mathcal{D}}(r(C), D) \cong \mathrm{Nat}(\mathrm{Hom}(-, C), \mathrm{Hom}_{\mathcal{D}}(r(-), D)) \ , \tag{1.7}$$

where we use 1.1.6 in the first equality (the universal property of presheaf category and Yoneda extension), and the Yoneda lemma in the second. □

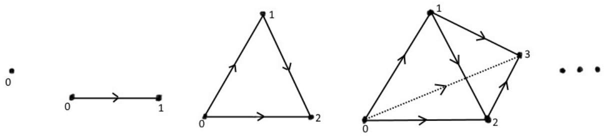
Now, let us apply this to construct models for $\infty$-groupoids and $\infty$-categories:

**Definition 1.1.8.** The *simplex category* $\Delta$ consists of the nonempty finite totally ordered sets $[n] = \{0 < 1 < 2 < \cdots < n\}$, for $n \in \mathbb{N}_0$, as objects; and order-preserving maps as morphisms.

**Definition 1.1.9.** A *simplicial set* is a functor $X : \Delta^{op} \to \mathrm{Set}$. Let us write $\mathrm{sSet} := \mathrm{PSh}(\Delta)$ for their category. We denote $X_n := X([n])$, and the Yoneda embedding $h([n]) = \mathrm{Hom}_{\Delta}(-, [n]) =: \Delta^n$.

By the coYoneda-Lemma, elements of sSet are colimits of representable presheaves $\Delta^n$, and the presheaf category is in some sense freely generated by such colimits. Morphisms between the $\Delta^n$ are, since the Yoneda embedding is fully faithful, the same thing as morphisms in $\Delta$, i.e. order-preserving maps. These can be written as compositions of *face maps* that leave out one number, like $[1] \to [2]$ via $0 \mapsto 0, 1 \mapsto 2$; and *degeneracy maps* that double one number, like $[2] \to [1]$ via $0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 1$. Geometrically, we should imagine $[n]$ and $\Delta^n$ as $n$-simplices, i.e. $n$-dimensional triangles/pyramids with

Figure 1.1: Objects of $\Delta$ regarded as topological simplices

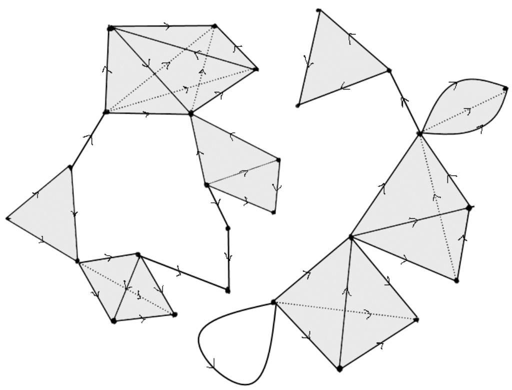



vertices labeled by the numbers 0 to $n$, so that these maps can be identified with face inclusions, and regarding an $n$-Simplex as a degenerate $(n+1)$-simplex (e.g. regarding a line as a triangle with an angle of 0°).

Due to the fact that presheaves are free gluings of these representables, we expect that simplicial sets are abstract gluings of simplices along their faces, in other words a slightly generalized version of simplicial complexes:

**Example 1.1.10.**
- $\Delta^n$ for any $n \geq 0$ are simplicial sets.
- The boundary $\partial\Delta^n$ is the sub-simplicial set of $\Delta^n$ that is obtained when erasing the interior. For example, $\partial\Delta^1 : \Delta^{op} \to \mathrm{Set}$ sends each $[n] \in \Delta$ to the order-preserving maps $[n] \to [1]$ that are not surjective – there are precisely two of those, corresponding to $\partial\Delta^1 \cong \Delta^0 \amalg \Delta^0$.
- The *horn* $\Lambda^n_i$, for $0 \leq i \leq n$, is the sub-simplicial set of $\Delta^n$ that is obtained when erasing both the interior and the face opposite to the vertex $i$.

Figure 1.2: Example of a simplicial set



**Example 1.1.11.** To gain a better understanding for the nerve-realization paradigm, let us define the *barycentric subdivision* $\mathrm{sd}(K)$ of a simplicial set $K$. We start by defining a functor $r_{\mathrm{sd}} : \Delta \to \mathrm{sSet}$ sending $[n]$ the the nerve of the partially ordered set $\mathcal{P}_{>0}([n])$ consisting of nonempty subsets of $[n]$ ordered by inclusion. If we imagine $[n]$ as an $n$-simplex, $r_{\mathrm{sd}}([n])$ can be imagined as describing its subdivision, as it e.g. contains one vertex for every non-degenerate simplex of $\Delta^n$, which we can imagine as sitting in the middle of that simplex. Since sSet has all colimits, we obtain an adjunction

$$\mathrm{sSet} \underset{\mathrm{sd}}{\overset{\mathrm{Ex}}{\leftrightarrows}} \mathrm{sSet}$$

where sd acts as above on every simplex in a simplicial set, and glues the result together in the way the original simplices had been glued together. By definition, $n$-simplices in $\mathrm{Ex}(K)$ are $\mathrm{Ex}(K)_n = \mathrm{Hom}_{\mathrm{sSet}}(\mathrm{sd}(\Delta^n), K)$, which we may as simplices in $K$ that are allowed to be folded around corners or edges. One can show that while $\mathrm{Ex}(K)$ is weakly homotopy equivalent (as in 1.2.16) to $K$, it is more flexible because of this folding, in particular the infinite composition $\mathrm{Ex}^\infty$ is a fibrant replacement functor in the Joyal-model structure as claimed at the end of this section.

**Example 1.1.12.** Let us define a functor $r_{top} : \Delta \to \mathrm{Top}$ sending $[n]$ to the topological $n$-Simplex $|\Delta^n| := \{(x_0, \dots, x_n) \in [0,1]^{n+1} \,|\, x_1 + \cdots + x_n = 1\}$, with the action on morphisms that we geometrically expect. Since Top has all colimits, we may employ the nerve-realization paradigm to obtain an adjunction

$$\mathrm{sSet} \underset{|-|}{\overset{\mathrm{Sing}}{\leftrightarrows}} \mathrm{Top}$$

where $|-|$ is called *geometric realization* and $\mathrm{Sing}(X) = \mathrm{Hom}_{\mathrm{Top}}(|\Delta^\bullet|, X)$ is the *singular simplicial set* of a topological space $X$.

**Definition 1.1.13.** A *Kan complex* is a simplicial set $K$ that satisfies the horn filler property: Any map of simplicial sets $\Lambda^n_i \to K$ can be filled, i.e. extended, to a map $\Delta^n \to K$ such that the following diagram commutes:

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & K \\ \big\downarrow & \nearrow_{\exists} & \\ \Delta^n & & \end{array}$$

**Theorem 1.1.14** (Homotopy hypothesis, [Lur18a, Tag 012Y])**.** For any topological space $X$, the singular simplicial set $\mathrm{Sing}(X)$ is a Kan complex. The adjunction $|-| \dashv \mathrm{Sing}$ induces an equivalence of categories between CW-complexes and Kan complexes. In fact, it even induces a Quillen equivalence between sSet (with the Quillen model structure) and Top that induces above equivalence on homotopy categories.

Kan complexes *are* homotopy types!

This resolves our first problem: Kan complexes should be the same thing as $\infty$-groupoids, if we regard their vertices as objects, edges as morphisms, $n$-simplices as $n$-morphisms. We say that a morphism $h$ in a Kan complex $K$ is a composition of morphisms $f, g$ if there is a 2-simplex $\sigma \in K_2$ such that, identifying $\sigma$ with a map $\Delta^2 \to K$ via the Yoneda lemma, restriction of this map to the boundary component $\{0 < 2\}$ agrees with $h$, while the restrictions to $\{0 < 1\}$ and $\{1 < 2\}$ agree with $f$ and $g$, respectively. We say that $\sigma$ *witnesses* $h$ as a composition $g \circ f$.

Such a composition exists for any morphisms $f : X \to Y$ and $g : Y \to Z$ as can be seen by filling up $\Lambda^2_1$; but since this filling is not required to be unique, the composition of morphisms is not uniquely defined. However, using higher horn fillers, one can show it is unique up to a contractible space of choices. Composition of higher morphisms, as well as associativity, are witnessed by higher horn fillers; and identity $n$-morphisms are induced by the degeneracy maps. In particular, by filling up the other horns $\Lambda^2_0$ and $\Lambda^2_2$ where one edge is degenerate, we see that every morphism in a Kan complex has an inverse.

We have also solved our problem concerning fundamental $\infty$-groupoids if we set $\pi_{\leq\infty}(X) := \mathrm{Sing}(X)$. Because of the homotopy hypothesis, our wish that this should know about the entire homotopy type of a CW-complex comes true. But what about $\infty$-categories?

**Example 1.1.15.** Let $r_{cat} : \Delta \to \mathrm{Cat}$ be the functor that sends the partially ordered set $[n]$ to the corresponding thin category with objects $0, \ldots, n$. Again, we can apply the nerve-realization paradigm to obtain an adjunction

$$\mathrm{sSet} \underset{h}{\overset{\mathrm{N}}{\rightleftarrows}} \mathrm{Cat}$$

where $hX$ is called the *homotopy category* of $X$. The nerve functor $N$ is fully faithful, so categories are a special case of simplicial sets; but $N\mathcal{C}$ is a Kan complex iff $\mathcal{C}$ is a groupoid.

The problem is that if $\mathcal{C}$ is not a groupoid, then the horns $\Lambda^2_0$ and $\Lambda^2_2$ will not always have fillers, since these would require the existence of inverse morphisms (in the case of degenerate simplices). We therefore must relax the horn filler condition:

**Definition 1.1.16.** A simplicial set $X$ is called *quasi-category* if it satisfies the weak horn filler condition: Any *inner horn* $\Lambda^i_n \to X$ with $0 < i < n$ can be extended to $\Delta^n$.

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & K \\ \downarrow & \nearrow_{\exists} & \\ \Delta^n & & \end{array}$$

We will interchangeably also use the terms $\infty$-category or $(\infty, 1)$-category for this construction; the difference in terminology is useful to distinguish the explicit simplicial model we have constructed from the abstract, ontological concept of a higher category that we tried to motivate in the beginning. In particular, morphisms in an $(\infty, 1)$-category can be non-invertible, while one can show that all $n$-vertices in a quasi-category, for $n > 1$, are invertible in some sense – this is what the 1 in the name refers to. Clearly, every Kan complex is a quasi-category; also the nerve of an ordinary category is one (in fact, ordinary categories are precisely those quasi-categories where the choice of an inner horn filler is always unique).

But what about $(\infty, 2)$-categories? To find a common generalization of $(\infty, 1)$-categories and 2-categories, we should as a fist step find a fully faithful functor from 2-categories into simplicial sets, as we did for 1-categories.

**Definition 1.1.17** ([Lur18a, Tag 009T])**.** The *Duskin nerve* of a 2-category is defined via the nerve-realization paradigm; applied to the functor $r_{\textit{2-cat}} : \Delta \to \mathrm{Cat}_2$, which is given by composing $r_{cat}$ with the inclusion of categories into 2-categories. It is fully faithful, and the Duskin nerve of a $(2, 1)$-category is a quasi-category.

However, the Duskin nerve of a 2-category with non-invertible 2-morphisms can never be a quasi-category. One can however proceed as above, and define a class of simplicial sets that contains Duskin nerves to model $(\infty, 2)$-categories. This is a lot more complicated than for quasi-categories, see [Lur18a, Tag 01W6].

There are also notions for $(\infty, k)$-categories, with $k \in \mathbb{N}_0$, but these generally follow a slightly different philosophy in their definition – see [Lur09b] for more. Also, there are currently two different notions of $(\infty, \infty)$-categories, via a projective or an inductive limit in $k$; and both are still poorly developed. We only need $k = 0, 1$ in this text.

| k \ n | $-2$ | $-1$ | 0 | 1 | 2 | ... | $\infty$ |
|---|---|---|---|---|---|---|---|
| 0 | point | boolean | set | groupoid | 2-groupoid | | $\infty$-groupoid |
| 1 | " | " | poset | category | $(2, 1)$-category | | $(\infty, 1)$-category |
| 2 | " | " | " | 2-poset | 2-category | ... | $(\infty, 2)$-category |
| ... | | | | | ... | | |
| $\infty$ | " | " | " | " | | | $(\infty, \infty)$-category ? |

The inclusion functors in vertical and horizontal direction in this chart have adjoint functors that we will make use of regularly. We already know that the nerve functor from categories to $(\infty, 1)$-categories, and the Duskin nerve from 2-categories to $(\infty, 2)$-categories, have left adjoints (the homotopy category and the homotopy 2-category).

**Definition 1.1.18.** Given an $(\infty, 1)$-category $\mathcal{C}$, we can forget all non-invertible 1-morphisms, obtaining its underlying $\infty$-groupoid $\mathcal{C}^{\simeq}$. Similarly, given an $(\infty, 2)$-category $\mathbf{\mathfrak{C}}$, we can forget all non-invertible 2-morphisms, yielding an $(\infty, 1)$-category $\mathrm{Pith}(\mathbf{\mathfrak{C}})$ called its *pith.* These functors are right adjoint to the respective inclusions.

Left adjoints are harder to construct, but also of interest.

- By localizing (as defined in 1.2.3) an $\infty$-category at all 1-morphisms, one obtains a Kan complex, a process called *Quillen fibrant replacement* (equivalently one can apply Kan's $\mathrm{Ex}^{\infty}$-functor we defined above).
- Similarly, a sort of localization of an $(\infty, 2)$-category at all 2-morphisms is called *Joyal fibrant replacement.*

In fact, both of these constructions can be applied to arbitrary simplicial sets; they are fibrant replacements in the Quillen and Joyal model structures on sSet, respectively.

## 1.2 Higher Category Theory

Since $\infty$-categories and ordinary categories both contain objects and (possibly non-invertible) morphisms, the only difference between them is the existence of invertible higher morphisms, i.e. homotopies, homotopies of homotopies and so on, that act as coherence data for composition, associativity and identity constraints for the 1-morphisms in an $\infty$-category. It seems reasonable to assume that, as long as work in a homotopy coherent manner, most concepts from ordinary category theory should translate to $\infty$-categories without much change (similarly, concepts from 2-categories should translate to $(\infty, 2)$-categories). Let $\mathcal{C}, \mathcal{D}$ be $\infty$-categories, then we can define:

**Definition 1.2.1.** The $\infty$-category of functors $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is the internal Hom between them in sSet. In other words, $\mathrm{Fun}(\mathcal{C}, \mathcal{D})_n = \mathrm{Hom}_{\mathrm{sSet}}(\mathcal{C} \times \Delta^n, \mathcal{D})$. Morphisms in this functor $\infty$-category are called natural transformations, and invertible morphisms are called natural isomorphisms.

**Definition 1.2.2.** Functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ define an equivalence of $\infty$-categories if their compositions $F \circ G$ and $G \circ F$ are both naturally isomorphic to the respective identity functors.

**Proposition 1.2.3.** For $\mathcal{C}$ and $\infty$-category and $W$ a set of morphisms in it, there is another $\infty$-category $\mathcal{C}[W^{-1}]$, called the *localization* of $\mathcal{C}$ at $W$, equipped with a functor $\mathcal{C} \to \mathcal{C}[W^{-}1]$ such that for any $\infty$-category $\mathcal{D}$, precomposing with it induces a fully faithful functor

$$\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \tag{1.8}$$

with essential image spanned by those functors $F : \mathcal{C} \to \mathcal{D}$ that send each morphism in $W$ to an isomorphism. By this universal property, $\mathcal{C}[W^{-1}]$ is uniquely determined up to equivalence (unlike for ordinary categories, where it can be made unique up to isomorphism).

**Definition 1.2.4.** For $C, D \in \mathcal{C}$, the *morphism space*

$$\mathrm{Map}(C, D) := \{C\} \times_{\mathcal{C}} \mathrm{Fun}(\Delta^1, \mathcal{C}) \times_{\mathcal{C}} \{D\} \tag{1.9}$$

is always a Kan complex. It is homotopy equivalent to the left and right pinched morphism spaces $\{C\} \times_{\mathcal{C}} \mathcal{C}_{/D}$ and $\mathcal{C}_{C/} \times_{\mathcal{C}} \{D\}$.

*Warning.* For $(\infty, 2)$-categories they are different; one has to work with the left pinched morphism space (which is an $(\infty, 1)$-category).

Proving the statements we make (e.g. that the functor category is again an $\infty$-category) uses a lot of simplicial combinatorics that we will not discuss; see [Lur18a] for more. In particular, we freely use:

- Join and slice constructions for simplicial sets, like $\mathcal{C}_{/C}$ above. Note that there are two simplicial models for those, that are equivalent as $\infty$-categories.
- The opposite simplicial set $\mathcal{C}^{op}$.
- Special kinds of morphisms between simplicial sets, for example trivial fibrations, Kan fibrations, left and right fibrations, Cartesian and coCartesian fibrations, and many more. The last four will be motivated in 1.2.20.

**Example 1.2.5.**

- If $\mathcal{C}$ is the nerve of an ordinary category, then $\mathrm{Map}_{\mathcal{C}}(C, D)$ is a discrete space.
- For $X$ a topological space and $x, y \in X$, the mapping space $\mathrm{Map}_{\mathrm{Sing}(X)}(x, y)$ is the space of paths from $x$ to $y$ in $X$.

We have learned that morphism spaces of $\infty$-categories are Kan complexes. Are $\infty$-categories the same thing as categories enriched over Kan complexes? This can not literally be true, since enriched categories have strict composition maps, while composition in an $\infty$-category is, as we have seen, only defined up to a contractible space of choices. But it is essentially true:

**Definition 1.2.6.** Denote by sSet-Cat the ordinary category of sSet-enriched categories, and by $r_{cube} : \Delta \to \text{sSet-Cat}$ the functor that sends $[n]$ to a simplicially enriched category with objects $0, \ldots, n$ and morphisms between $i, j \in [n]$ given by

$$\underline{\mathrm{Hom}}_{r_{cube}([n])}(i, j) := N(P(\{i, i+1 \ldots, j\}), \subseteq) \in \mathrm{sSet}\ . \tag{1.10}$$

Putting this into words, we take the nerve of the ordinary category associated to the poset of subsets of the set $\{i, i+1 \ldots, j\}$, ordered by inclusion. For intuition: This simplicial set is just a $(i-j+1)$-dimensional cube.

**Theorem 1.2.7.** Applying the nerve-realization paradigm 1.1.7 to $r_{cube}$ yields an adjunction

$$\mathrm{sSet} \underset{\mathrm{Path}}{\overset{\mathrm{N}_{hc}}{\rightleftarrows}} \mathrm{sSet}\text{-}\mathrm{Cat}$$

where $\mathrm{N}_{hc}(\mathcal{C})_n = \mathrm{Fun}(r_{cube}([n]), \mathcal{C})$ is called the *homotopy coherent nerve* of the simplicially enriched category $\mathcal{C}$. This is a Quillen equivalence with respect to certain model structures on both sides, yielding an equivalence of the homotopy categories: Quasi-categories are the same thing as Kan-enriched categories!

*Remark.* Since Kan complexes are the same thing as (good) topological spaces, one could via a change of enrichment also say that quasi-categories are the same thing as topologically enriched categories.

**Proposition 1.2.8** ([Lur18a, Tag 01YL])**.** Similarly, if $\mathcal{C}$ is a simplicially enriched category where all morphism spaces are quasi-categories, then its homotopy coherent nerve is an $(\infty, 2)$-category. Every $(\infty, 2)$-category can be obtained this way up to equivalence of $(\infty, 2)$-categories. However, one can not proceed like this to obtain all $(\infty, 3)$-categories.

**Proposition 1.2.9** ([Lur18a, Tag 01LG])**.** For $\mathcal{C}$ a category enriched over quasi-categories and $X, Y \in \mathcal{C}$, there is an equivalence of the internal Hom with the left pinched mapping spaces in the $(\infty, 2)$-category $N_{hc}(\mathcal{C})$:

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \simeq \mathrm{Hom}^L_{N_{hc}(\mathcal{C})}(X, Y) \simeq \mathrm{Hom}^R_{N_{hc}(\mathcal{C})}(X, Y)^{op} \tag{1.11}$$

In particular, if $\mathcal{C}$ is even enriched over Kan complexes, this is a homotopy equivalence

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \simeq \mathrm{Map}_{N_{hc}(\mathcal{C})}(X, Y) \ . \tag{1.12}$$

**Example 1.2.10.**

- Let Kan be the Kan-enriched category with objects Kan complexes, and morphisms spaces $\underline{\mathrm{Hom}}_{\mathrm{Kan}}(K, L) := \mathrm{Fun}(K, L)$, which is indeed a Kan complex. The homotopy coherent nerve $\mathcal{S} := N_{hc}\,\mathrm{Kan}$ is the $\infty$-*category of spaces.* Its rôle in higher category theory is the same as the rôle of Set in ordinary category theory; one might argue it is the most important $\infty$-category.
- $\mathcal{S}_* := \mathcal{S}_{/\Delta^0}$ is the $\infty$-*category of pointed spaces.* Equivalently, it is the homotopy coherent nerve of the Kan-enriched slice category $\mathrm{Kan}_{/\Delta^0}$.

- Denote by $\mathcal{S}^{\text{fin}}$ the full subcategory of $\mathcal{S}$ on Kan complexes with finitely many non-degenerate simplices, and similarly $\mathcal{S}^{\text{fin}}_*$.
- Let QC be the category enriched over quasi-categories, where objects are quasi-categories and $\underline{\text{Hom}}_{\text{QC}}(\mathcal{C}, \mathcal{D}) := \text{Fun}(\mathcal{C}, \mathcal{D})$. The homotopy coherent nerve $\boldsymbol{\mathcal{C}at}_\infty := N_{hc}(\text{QC})$ is the *$(\infty, 2)$-category of all $\infty$-categories.*
- If we denote by $\text{QC}^{\simeq}$ the Kan-enriched category with objects quasi-categories, and morphisms given by the Kan complexes $\underline{\text{Hom}}_{\text{QC}^{\simeq}}(\mathcal{C}, \mathcal{D}) := \text{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}$, then the homotopy coherent nerve is $\mathcal{C}at_\infty := N_{hc}\,\text{QC}^{\simeq} \simeq \text{Pith}(\boldsymbol{\mathcal{C}at}_\infty)$, the *$\infty$-category of all $\infty$-categories.*
- The homotopy coherent nerve of the quasi-category-enriched slice category $\boldsymbol{\mathcal{C}at}_{\infty,\text{obj}} := N_{hc}\,\text{QC}_{/\Delta^0}$ is the *$(\infty, 2)$-category of lax-pointed $\infty$-categories.* We will denote its pith $\infty$-category by $\mathcal{C}at_{\infty,\text{obj}}$.

After these very foundational definitions, let us introduce some universal constructions for $\infty$-categories:

**Definition 1.2.11.** Let $K \in \text{sSet}$, and $p : K \to \mathcal{C}$ be a morphism of simplicial sets, that we interpret as a diagram in the $\infty$-category $\mathcal{C}$. Denote by $K^{\triangleleft}$ the left cone on $K$, formed by adding an initial object to it (i.e. taking the join $\Delta^0 \star K$). The *limit cone* of this diagram, if it exists, is a morphism $\overline{p} : K^{\triangleleft} \to \mathcal{C}$ with $p(-\infty) := \lim(p)$, that induces for all $C \in \mathcal{C}$ a homotopy equivalence

$$\text{Map}(C, \lim(p)) \simeq \text{Nat}(\underline{C}, p) \tag{1.13}$$

where $\underline{C} : K \to \mathcal{C}$ it the constant diagram on $C$. Note how this agrees with the ordinary limit if $\mathcal{C}$ is a 1-category. Oppositely, we can define a $\text{colim}(p)$ by extending $p$ to $K^{\triangleright}$ such that

$$\text{Map}(\text{colim}(p), C) \simeq \text{Nat}(p, \underline{C}) \;. \tag{1.14}$$

Special cases of this construction yield (as in ordinary category theory) products, coproducts; pullbacks, pushouts; final, initial and zero objects; kernels, cokernels. While coproducts and products can be treated with similar intuition as in ordinary categories, pullbacks and pushouts behave like homotopy pullbacks and pushouts. For example, (co)limits in $\mathcal{S}$ are precisely homotopy (co)limits of topological spaces by [Lur09a, 4.2.4.1]; and kernels in the $\infty$-category chain complexes are mapping cones, see 1.5.4. To make this distinction clear, kernels are also called *fibers* in this setting, and cokernels are called *cofibers.*

**Lemma 1.2.12.** Just as every set is a colimit (coproduct, since the indexing category is discrete) over its elements regarded as one-element sets; every Kan complex $K$ is the colimit over the functor $\underline{\Delta^0} : K \to \mathcal{S}$ constant on $\Delta^0$.

*Proof.* We need to show that the induced map $\mathrm{Map}_{\mathcal{S}}(K, C) \to \mathrm{Nat}(\underline{\Delta^0}, \underline{C})$ is a homotopy equivalence for each $C \in \mathcal{S}$. We have defined $\mathcal{S}$ as the homotopy coherent nerve of the Kan-enriched category Kan of Kan complexes, so the former mapping space is given by the Kan complex $\mathrm{Fun}(K, C)$. On the right,

$$\mathrm{Nat}(\underline{\Delta^0}, \underline{C}) \simeq \mathrm{Fun}_{\underline{\Delta^0}, \underline{C}}(\Delta^1, \mathrm{Fun}(K, \mathcal{S})) \simeq \mathrm{Fun}(K, \mathrm{Fun}_{\Delta^0, C}(\Delta^1, \mathcal{S})) \simeq \mathrm{Fun}(K, \mathrm{Map}_{\mathcal{S}}(\Delta^0, C))$$

where in the beginning and end we use one of the equivalent definitions for the mapping space, and in the middle identify both sidese as the same full subcategory of $\mathrm{Fun}(\Delta^1 \times K, \mathcal{S})$. We are finished once we notice $\mathrm{Map}_{\mathcal{S}}(\Delta^0, C) = \mathrm{Fun}(\Delta^0, C) \simeq C$ since on simplices $\mathrm{Fun}(\Delta^0, C)_n = \mathrm{Hom}_{\mathrm{sSet}}(\Delta^0 \times \Delta^n, C) = C_n$ by the Yoneda lemma. □

We need to be a bit more careful when defining cofinal functors and filtered (co)limits:

**Definition 1.2.13.** A morphism of simplicial sets $f : L \to K$ is called *right cofinal* if for any $\infty$-category $\mathcal{C}$, every diagram $p : K \to \mathcal{C}$ and any $C \in \mathcal{C}$, precomposing with $f$ induces a homotopy equivalence of Kan complexes

$$\mathrm{Nat}(p, \underline{C}) \simeq \mathrm{Nat}(p \circ f, \underline{C}) \ , \tag{1.15}$$

where $\underline{C}$ denotes the constant functors $K \to \mathcal{C}$ or $L \to \mathcal{C}$ with value $C$, respectively.

**Proposition 1.2.14.** In particular, if in the above situation $p$ admits a colimit, then by definition 1.2.11 this is equivalent to $\mathrm{Map}_{\mathcal{C}}(\mathrm{colim}\, p, C) = \mathrm{Map}_{\mathcal{C}}(\mathrm{colim}(p \circ f), C)$. In other words, right cofinal morphisms are precisely those that preserve (universal properties of) colimits! Similarly, left cofinal morphisms are those that preserve limits.

*Remark.* There are many equivalent characterizations of cofinality (see [Lur18a, Tag 02NR]) that are often easier to check than ours, most prominently:

**Theorem 1.2.15** (Quillen's Theorem A, [Lur18a, Tag 02NY])**.**
A morphism of simplicial sets $F : \mathcal{C} \to \mathcal{D}$ with $\mathcal{D}$ an $\infty$-category is

- left cofinal iff, for all $D \in \mathcal{D}$, the fiber $\mathcal{C}_{/D} := \mathcal{C} \times \mathcal{D}_{/D}$ is weakly contractible,
- right cofinal iff, for all $D \in \mathcal{D}$, the fiber $\mathcal{C}_{D/} := \mathcal{C} \times \mathcal{D}_{D/}$ is weakly contractible.

**Definition 1.2.16.** Here, a simplicial set $K$ is *weakly contractible* iff the geometric realization $|K|$ is contractible, or equivalently (by the adjunction $|-| \dashv \mathrm{Sing}$), the space of maps $\underline{\mathrm{Hom}}(K, X)$ into any Kan complex $X$ is contractible. Similarly, we define *weak homotopy equivalences* as those maps of simplicial set that become homotopy equivalences after applying $|-|$, or $\underline{\mathrm{Hom}}(-, X)$ for any Kan complex. They are the weak equivalences in the Quillen model structure on sSet.

**Definition 1.2.17** ([Lur18a, Tag 02PB])**.** An $\infty$-category $\mathcal{C}$ is called *filtered* if for each simplicial set $K$, any map $K \to \mathcal{C}$ can be extended to a map $K^{\triangleright} \to \mathcal{C}$.

**Proposition 1.2.18.** An $\infty$-category $\mathcal{C}$ is filtered iff for any simplicial set $K$, the diagonal map $\mathcal{C} \to \mathrm{Fun}(K, \mathcal{C})$ that sends $C$ to the constant functor $\underline{C} : K \to \mathcal{C}$ is right cofinal.

This generalizes filtered diagrams in ordinary categories; and colimits parametrized by filtered simplicial sets have similarly nice properties as filtered colimits in ordinary categories.

Using the mapping space construction in a similar way, one can define adjunctions, Kan extensions, and so on. Almost all the usual formulae for limits and colimits still hold. Generally, almost all theorems from category theory still hold, like the Yoneda lemma, colimits commuting with colimits, uniqueness of colimits and adjoints and so on.

There are notions of accessible, presentable (sometimes also called locally presentable), and compactly generated $\infty$-categories mimicking the ordinary notions. Intuitively, an $\infty$-category is accessible iff it is somehow controlled by a small collection of (compact) objects, even though it is not small itself (e.g. how the ordinary category of $\mathbb{R}$-vector spaces is the Ind-completion of the category of finite-dimensional $\mathbb{R}$-vector spaces); it is presentable iff it is accessible and has all colimits (automatically also all limits), and compactly generated iff it is accessible and some further size conditions are imposed on how it is controlled by this small class of objects. For more details, compare 3.2.14 and the Remark thereafter.

In fact, presentable $\infty$-categories turn out to be "the same thing" as combinatorial model categories! Higher category theory therefore trivializes many cumbersome model category calculations. Another strong appeal are useful representability criteria:

**Theorem 1.2.19** (Adjoint Functor Theorem, [Lur09a, 5.4.2.5])**.** A functor $F : \mathcal{C} \to \mathcal{D}$ between presentable $\infty$-categories has

- a right adjoint iff it preserves (small) colimits,
- a left adjoint iff it preserves (small) limits *and* $\kappa$-small filtered colimits for some regular cardinal $\kappa$.

In fact, the first claim only requires $\mathcal{D}$ to be essentially locally small.

Finally, let us give a short comment on why higher category is so technically difficult (and why loc. cit. is almost 1000 pages long). Giving a functor between two $\infty$-categories, and checking that it is indeed a functor, can be extremely difficult to do explicitly, for example it is very hard to see why the mapping space construction we gave above is functorial in its arguments. However, to even define a Yoneda embedding, see that limits are functorial and so on, we need to understand this. There is a very elegant, roundabout way to define the mapping space functor:

**Theorem 1.2.20** (Grothendieck construction, [Lur09a, Section 3.2])**.** For a fixed $\infty$-category $\mathcal{C}$, functors $F : \mathcal{C} \to \mathcal{C}at_\infty$ are essentially the same thing as $\infty$-categories $\mathcal{M}$ equipped with a functor $p : \mathcal{M} \to \mathcal{C}$ that is a so-called *coCartesian fibration* (we define those in a moment). More explicitly, the fiber of $p$ over an object $C \in \mathcal{C}$ is equivalent to the $\infty$-category $F(C)$, and the action of $F$ on morphisms in $\mathcal{C}$ is encoded via a version of parallel transport that lifts morphisms of $\mathcal{C}$ to a certain class of morphisms in $\mathcal{M}$, called *p-coCartesian morphisms.* Conversely, one can obtain $p$ from $F$ as the pullback

$$p : \mathcal{M} := \mathcal{C}at_{\infty,\mathrm{obj}} \times_{\mathcal{C}at_\infty} \mathcal{C} \xrightarrow{\mathrm{pr}_2} \mathcal{C} \, . \tag{1.16}$$

Similarly, functors $F : \mathcal{C}^{op} \to \mathcal{C}at_\infty$ are essentially the same thing as *Cartesian fibrations* over $\mathcal{C}$, i.e. functors $p : \mathcal{M} \to \mathcal{C}$ such that $p^{op} : \mathcal{M}^{op} \to \mathcal{C}^{op}$ is a coCartesian fibration. To give a precise version of these statements, [GHN15, A.32] shows that there is an equivalence of $\infty$-categories

$$\mathrm{Fun}(\mathcal{C}, \mathcal{C}at_\infty) \simeq \mathcal{C}at_{\infty/\mathcal{C}}^{\mathrm{coCart}} \tag{1.17}$$

where $\mathcal{C}at_{\infty/\mathcal{C}}^{\mathrm{coCart}}$ is the non-full subcategory of the slice category over $C$ on the coCartesian fibrations, and functors preserving coCartesian morphisms. This equivalence is functorial in $C$.

*Remark.* The total space $\mathcal{M}$ of the coCartesian fibration associated to a functor $\mathcal{C} \to \mathcal{C}at_\infty$ is classically also called its *category of elements.* We could interpret this result as saying that the category $\mathcal{C}at_\infty$ acts as a classifying space for coCartesian fibrations. Let us sketch how those are defined; we also recommend [Lur18a, Tag 01J2] for more on the Grothendieck construction.

**Definition 1.2.21.** A map of simplicial sets $f : K \to L$ is called an

- *inner fibration* if for each $0 < i < n$
- *right fibration* if for each $0 < i \leq n$
- *left fibration* if for each $0 \leq i < n$
- *Kan fibration* if for each $0 \leq i \leq n$

with $n \in \mathbb{N}_0$, any commuting square of the form

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & K \\ \big\downarrow & \nearrow & \big\downarrow \\ \Delta^n & \longrightarrow & L \end{array}$$

has a horn filler/ lift as indicated. It is called a *trivial Kan fibration* if even the inclusion $\partial\Delta^n \hookrightarrow \Delta^n$ can be filled in this manner.

**Example 1.2.22.** For $K$ any simplicial set, the terminal map $K \to \Delta^0$ is an inner fibration iff $K$ is an $\infty$-category; it is a left/ right/ Kan fibration iff $K$ is a Kan complex; and it is a trivial Kan fibration iff $K$ is a contractible Kan complex.

**Example 1.2.23.** If $f : K \to L$ is an inner fibration and $L$ is an $\infty$-category, then $K$ is so as well, since the terminal map factors as $K \to L \to \Delta^0$ and the composition of inner fibrations is still an inner fibration (applying the lifting condition twice).

**Definition 1.2.24.** A map of simplicial sets $U : \mathcal{M} \to \mathcal{C}$ is a *coCartesian fibration* if it is an inner fibration, and for any vertex $X' \in \mathcal{M}$ and any edge $e : U(X') \to Y$ in $\mathcal{C}$, there is a vertex $Y' \in \mathcal{M}$ and a $U$-coCartesian edge $e' : X' \to Y'$ such that $U(e') = e$. In other words, any edge in $\mathcal{C}$ can be lifted to a $U$-coCartesian edge in $\mathcal{M}$ in covariant direction.

If we assume that $\mathcal{C}$ is an $\infty$-category (and $\mathcal{M}$ automatically as well), an edge $e' : X' \to Y'$ in $\mathcal{M}$ is called $U$-coCartesian iff for each $W' \in \mathcal{M}$, the commuting square

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{M}}(W', X') & \xrightarrow{e' \circ -} & \mathrm{Map}_{\mathcal{M}}(W', Y') \\ \downarrow U & & \downarrow U \\ \mathrm{Map}_{\mathcal{C}}(U(W'), U(X')) & \xrightarrow{U(e') \circ -} & \mathrm{Map}_{\mathcal{C}}(U(W'), U(Y')) \end{array}$$

is a pullback square in $\mathcal{S}$ (not just of simplicial sets).

*Remark.* Our definition of $U$-coCartesian edges is wrong if $\mathcal{C}$ is no $\infty$-category or $U$ is not an inner fibration (in particular, postcomposition does not make sense in any simplicial set). It is equivalent to the correct definition in our case by [Lur18a, Tag 01TL].

**Corollary 1.2.25** (Grothendieck Construction over Spaces)**.** For a fixed $\infty$-category $\mathcal{C}$, functors $\mathcal{C} \to \mathcal{S}$ are essentially the same thing as left fibrations over $\mathcal{C}$, with a similar explicit description of this correspondence as above. Oppositely, functors $\mathcal{C}^{op} \to \mathcal{S}$ are the same thing as right fibrations.

*Remark.* This would be immediate if we could show that a left fibration is precisely a coCartesian fibrations where all fibers are Kan complexes, and similarly for right fibrations. See [Lur18a, Tag 01UM] for this.

*Remark.* One can deduce that for an $\infty$-groupoid $K$, the $\infty$-categories $\mathrm{Fun}(K, \mathcal{S}) \simeq \mathcal{S}_{/K}$ are equivalent. This relies on a model categorical argument; given an arbitrary map of Kan complexes $M \to K$, we can always replace it by a left fibration that is weakly homotopy equivalent to $M$. This induces an equivalence between $\mathcal{S}_{/K}$ and its full subcategory spanned by the left fibrations, so we can apply 1.2.25. Note that this argument would break down for $K$ an arbitrary $\infty$-category, where we would have to work with this subcategory.

**Example 1.2.26.** One can show that for $\mathcal{C}$ an $\infty$-category and $C \in \mathcal{C}$, the projection $\mathcal{C}_{C/} \to \mathcal{C}$ out of the slice category is a left fibration [Lur18a, Tag 018F]. The associated functor $\mathcal{C} \to \mathcal{S}$ sends $D$ to the fiber $\mathcal{C}_{C/} \times_{\mathcal{C}} \{C\} \simeq \mathrm{Map}_{\mathcal{C}}(C, D)$, so it can be used together with the analogous observation for the right fibration $\mathcal{C}_{/C} \to \mathcal{C}$ to define the mapping space functor.

## 1.3 Sheaves and $\infty$-Topoi

**Definition 1.3.1.** Let $\mathcal{C}$ be an $\infty$-category, then denote by $\mathcal{PSh}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ its *presheaf category*, and by $h : \mathcal{C} \to \mathcal{PSh}(\mathcal{C})$ the fully faithful Yoneda embedding.

As in ordinary category theory, we often want to restrict our attention to a full subcategory of $\mathcal{PSh}(\mathcal{C})$ that contains *sheaves*, which are presheaves that satisfy descent with respect to a particular notion of covering.

**Definition 1.3.2.** A *Grothendieck pretopology* $\tau$ on $\mathcal{C}$ consists of, for every $U \in \mathcal{C}$, a set of *coverings* $\mathrm{Cov}_\tau(U)$ whose elements are families $(U_i \to U)_{i \in I}$ with $U_i \in \mathcal{C}$, such that the following hold:

- Given an isomorphism $U' \to U$, the one-element family $(U' \to U)$ is a covering.
- For any morphism $V \to U$, the pullbacks $(U_i \times_U V \to V)_i$ exist and form a covering again.
- If for any $i$, the family $(U_{ij} \to U_i)_j$ is a covering, then the composition $(U_{ij} \to U)_{ij}$ is a covering.

*Technical Remark.* While every *Grothendieck topology*, as in [Lur09a, 6.2.2.1], is a Grothendieck pretopology, the latter or usually much smaller. However, every pretopology specifies a unique topology by defining the covering sieves as those that contain a whole covering family, see [Pst18, A.5]. We will therefore work with this simpler notion.

**Definition 1.3.3.** An $\infty$-*site* $\mathcal{C}_\tau$ is an $\infty$-category $\mathcal{C}$ equipped with a Grothendieck pretopology $\tau$. Since covering families are invariant under isomorphisms (combining the first and third axiom), it is enough to specify the pretopology on the homotopy category.

**Definition 1.3.4.** Given a covering $(U_i \to U)$, we define its *Čech nerve* $C(U_i \to U) \in \mathrm{Fun}(\Delta^{op}, \mathcal{PS}h(\mathcal{C}))$ as the simplicial diagram

$$\cdots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \coprod_{i,j,k} h(U_i) \underset{h(U)}{\times} h(U_j) \underset{h(U)}{\times} h(U_k) \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \coprod_{i,j} h(U_i) \underset{h(U)}{\times} h(U_j) \longrightarrow \coprod_i h(U_i)$$

which by functoriality of $h$ possesses a canonical morphism to $h(U)$.

**Definition 1.3.5.** A *sheaf* on an $\infty$-site $\mathcal{C}$ is a presheaf $F : \mathcal{C}^{op} \to \mathcal{S}$ that is local with respect to these morphisms; that is for every covering $(U_i \to U)$,

$$\lim_{\Delta^{op}} \mathrm{Map}_{\mathcal{PS}h(\mathcal{C})}(C(U_i \to U), F) \overset{!}{=} \mathrm{Map}_{\mathcal{PS}h(\mathcal{C})}(h_U, F) = F(U)\ . \tag{1.18}$$

In other words, we require

$$F(U) = \lim_{\Delta^{op}} \left( \prod_i F(U_i) \longrightarrow \prod_{i,j} F(U_i \times_U U_j) \;\substack{\longrightarrow\\\longrightarrow}\; \cdots \right)\ . \tag{1.19}$$

*Technical Remark.* We denote the full subcategory on them by $\mathcal{S}h(\mathcal{C}_\tau)$, leaving out the topology if it is clear. This is equivalent to the definition in [Lur09a] by [Pst18, A.8, A.9].

**Theorem 1.3.6** ([Lur09a, 6.2.2.7])**.** For any $\infty$-site $\mathcal{C}$, there is a *sheafification* functor $(-)^{sh}$, which can be constructed as a transfinite composition of a *plus construction* (mimicking the classical double-plus-construction) is left adjoint to the canonical inclusion

$$\mathcal{S}h(\mathcal{C}) \; \underset{i}{\overset{(-)^{sh}}{\rightleftarrows}} \; \mathcal{PS}h(\mathcal{C})\ .$$

This leads to a general axiom for $\infty$-categories that "look like" categories of sheaves:

**Definition 1.3.7.** An *$\infty$-topos* $\mathcal{X}$ is an $\infty$-category that can be written as a left exact accessible localization of a presheaf category. In other words, there must exist a (small) $\infty$-category $\mathcal{C}$ and an adjunction

$$\mathcal{X} \; \underset{i}{\overset{L}{\rightleftarrows}} \; \mathcal{PS}h(\mathcal{C})$$

such that $i$ is fully faithful and preserves $\kappa$-filtered colimits for some regular cardinal $\kappa$, and $L$ preserves finite limits.

*Technical Remark.* The accessibility condition (preserving $\kappa$-filtered colimits) is equivalent to ensuring $\mathcal{X}$ is again presentable. It is currently not known whether it is automatic (as it is in the case of $n$-topoi).

*Remark.* This definition is extrinsic, since it tells us how to construct $\infty$-topoi, but not how to check if a specific $\infty$-category is one. There are also several intrinsic definitions, for example the *Giraud-Rezk-Lurie axioms.*

*Warning.* Not every left exact accessible reflective localization of a presheaf category arises as sheaves with respect to a Grothendieck category! It is not even known whether any $\infty$-topos can be written as sheaves on an $\infty$-site at all.

**Example 1.3.8.** Since identity functors are always left exact accessible localizations, presheaf categories are always $\infty$-topoi. In particular, $\mathcal{S} = \mathcal{P}Sh(*)$ is an $\infty$-topos. Also, for any $\infty$-site $\mathcal{C}_\tau$, the sheaves $\mathcal{S}h(\mathcal{C}_\tau)$ form an $\infty$-topos using the adjunction 1.3.6.

**Example 1.3.9.** For $\mathcal{X}$ an $\infty$-topos and $C$ an object in it, the *slice topos* $\mathcal{X}_{/C}$ is again an $\infty$-topos.

**Definition 1.3.10.** A *geometric morphism* between $\infty$-topoi is an adjunction

$$\mathcal{X} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{Y}$$

where $f^*$ preserves finite limits. Let us denote the subcategory of $\mathcal{C}at_\infty$ on $\infty$-topoi and geometric morphisms by $\mathcal{L}\mathrm{Top}$.

**Proposition 1.3.11.** $\mathcal{S}$ is the terminal object of $\mathcal{L}\mathrm{Top}$. This means that every $\infty$-topos $\mathcal{X}$ is equipped with an essentially unique adjunction

$$\mathcal{X} \underset{\Gamma_*}{\overset{\Gamma^*}{\rightleftarrows}} \mathcal{Y}$$

In particular, for $*$ the terminal object, $\Gamma_* = \mathrm{Map}_{\mathcal{X}}(*, -)$ and if $\mathcal{X} = \mathcal{S}h(\mathcal{C})$ over a $\infty$-site, $\Gamma^*(K) = (C \mapsto K)^{sh}$. Also, note that since $\Gamma^*$ preserves colimits and every Kan-complex is the colimit over its points, $\Gamma^*$ can be understood via its value on $\Delta^0$.

**Definition 1.3.12** ([Pst18, A.10 and A.12])**.** Given $\infty$-sites $\mathcal{C}$ and $\mathcal{D}$, a *morphism of sites* is a functor $F : \mathcal{C} \to \mathcal{D}$ that sends coverings to coverings.

Further, $F$ has the *covering lifting property* if for any $U \in \mathcal{C}$ and $(V_i \to F(U))$ covering family in $\mathcal{D}$, there is a covering $(U_j \to U)$ in $\mathcal{C}$ such that for every $j$ one can find an $i$ such that one can factor $F(U_j) \to V_i \to F(U)$.

**Proposition 1.3.13** ([Pst18, A.11 and A.13])**.** For any morphism of sites $F : \mathcal{C} \to \mathcal{D}$, precomposition $F_* := - \circ F$ preserves sheaves and, together with sheafification of the Left Kan Extension along it $F^* = (-)^{sh} \circ \mathrm{Lan}_F$, induces an adjunction

$$\mathcal{S}h(\mathcal{D}) \underset{F_*}{\overset{F^*}{\rightleftarrows}} \mathcal{S}h(\mathcal{D}) \ .$$

If $F$ has the covering lifting property, then $F_*$ commutes with sheafification, in particular it preserves colimits and admits another left adjoint $F^- : \mathcal{S}h(\mathcal{C}) \to \mathcal{S}h(\mathcal{D})$.

Let $\mathcal{D}$ be an arbitrary $\infty$-category.

**Definition 1.3.14.** A functor $F : \mathcal{C}^{op} \to \mathcal{D}$ is a *$\mathcal{D}$-valued sheaf on* $\mathcal{C}$ if, for any $D \in \mathcal{D}$, the composition $\mathrm{Map}_{\mathcal{D}}(D, F(-)) : \mathcal{C}^{op} \to \mathcal{S}$ is a sheaf on $\mathcal{C}$. We denote the subcategory on them by $\mathcal{S}h(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{D})$.

*Remark.* Again, explicitly we impose that for any cover $(U_i \to U)$, we have Čech descent:

$$F(U) = \lim_{\Delta^{op}} \left( \prod_i F(U_i) \longrightarrow \prod_{i,j} F(U_i \times_U U_j) \rightrightarrows \cdots \right) . \tag{1.20}$$

In particular, this limit should exist in $\mathcal{D}$.

**Proposition 1.3.15** ([Lur18b, 1.3.1.7])**.** If $\mathcal{D}$ has all limits, there is an equivalence

$$\mathcal{S}h(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}^{\lim}(\mathcal{S}h(\mathcal{C})^{op}, \mathcal{D}) \ , \tag{1.21}$$

where $\mathrm{Fun}^{\lim}$ denotes the subcategory of Fun on the limit-preserving functors.

This description can be further refined when we restrict to the class of *presentable* $\infty$-categories, which generalizes the class of (locally) presentable ordinary categories. To put it loosely, an $\infty$-category is presentable if is *accessible*, that is, generated under colimits by a small subcategory of compact objects; and it has all colimits (and automatically all limits).

**Theorem 1.3.16** ([Lur17, 4.8.1.17])**.** For $\mathcal{C}$ and $\mathcal{D}$ any presentable $\infty$-categories, one can define their tensor product $\mathcal{C} \otimes \mathcal{D} := \mathrm{Fun}^{\lim}(\mathcal{C}^{op}, \mathcal{D})$ that is again a presentable $\infty$-category, and a natural functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ such that for any presentable $\mathcal{E}$,

$$\mathrm{Fun}^{\mathrm{colim}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}^{\mathrm{colim,colim}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \ . \tag{1.22}$$

Here, $\mathrm{Fun}^{\mathrm{colim,colim}}$ denotes functors that preserve colimits in both variables.

*Remark.* Even though it does not seem that way, $\mathcal{C}\otimes\mathcal{D}$ is symmetric in $\mathcal{C}$ and $\mathcal{D}$ because of this universal property. Alternatively, a limit-preserving functor from $\mathcal{C}^{op} \to \mathcal{D}$ is a colimit-preserving functor $\mathcal{C} \to \mathcal{D}^{op}$, which by the Adjoint-Functor-Theorem 1.2.19 possesses an (essentially unique) right adjoint $\mathcal{D}^{op} \to \mathcal{C}$ automatically preserving limits.

**Corollary 1.3.17.** For $\mathcal{D}$ a presentable $\infty$-category, $\mathcal{S}h(\mathcal{C},\mathcal{D}) \simeq \mathcal{S}h(\mathcal{C}) \otimes \mathcal{D}$.

**Corollary 1.3.18.** If $\mathcal{D}$ is a presentable (and/ or stable) $\infty$-category, then $\mathcal{S}h(\mathcal{C},\mathcal{D})$ is presentable (and/ or stable) as well.

*Proof.* For $\mathcal{D}$ presentable, $\mathcal{S}h(\mathcal{C},\mathcal{D}) = \mathcal{S}h(\mathcal{C}) \otimes \mathcal{D}$ is presentable by 1.3.16.

If $\mathcal{D}$ is stable, then $\mathrm{Fun}(\mathcal{C}^{op},\mathcal{D})$ is stable because limits and colimits in a functor category are computed pointwise, so we need to show that the sheaves form a stable subcategory in the sense of 1.5.7. This follows since the sheafification functor is left exact, so that the category of sheaves is in particular closed under fibers and contains the zero object. □

## 1.4 Sheaves on Topological Spaces

Let us apply this machinery to the probably most interesting case:

**Definition 1.4.1.** For $X$ a topological space and $\mathcal{D}$ a complete $\infty$-category, equip the thin category of open subsets $\mathrm{Open}(X)$ with the Grothendieck pretopology $\tau$ where covering families are open coverings. We denote

$$\mathcal{S}h(X) := \mathcal{S}h(\mathrm{Open}(X)_\tau)\,, \quad \mathcal{S}h(X;\mathcal{D}) := \mathcal{S}h(\mathrm{Open}(X)_\tau;\mathcal{D})\,. \tag{1.23}$$

*Remark.* In particular, a functor $F : \mathrm{Open}^{op}(X) \to \mathcal{D}$ is an $\infty$-sheaf if for any open cover $(U_i \subseteq U)$,

$$F(U) = \lim_{\Delta^{op}} \left( \prod_i F(U_i) \longrightarrow \prod_{i,j} F(U_i \cap U_j) \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \cdots \right)\,. \tag{1.24}$$

There are several different ways to intuitively make sense of this descent condition. First of all, note the similarity with the Čech complex which also involves comparing sections at higher intersections of the $U_i$. One can show that for every ordinary sheaf $F_0 \in \mathrm{Sh}(X;\mathbb{Z})$, the derived sections $R\Gamma(-,F_0) \in \mathcal{S}h(X;D(\mathbb{Z}))$ form an $\infty$-sheaf; by 1.5.18 we will see that the descent condition in this case is equivalent to the statement that sheaf cohomology of $F_0$ agrees with the Čech hypercohomology of $R\Gamma(-,F_0)$ on any cover (which follows from the Čech-to-sheaf-cohomology spectral sequence). In fact, $\mathcal{S}h(X;D(\mathbb{Z}))$ and the derived category of ordinary sheaves $D(\mathrm{Sh}(X,\mathrm{Ab}))$ almost agree as discussed in 5.1.9, the only difference appears for very "infinite-dimensional" spaces and is remedied by the notion of hypercompletion we define shortly.

As a second example, suppose we are given a collection of topological spaces $(X_i)_{i\in I}$ and open subsets $U_j^{(i)} \subseteq X_i$ for $i, j \in I$, together with homeomorphisms $\phi_{ij} : U_j^{(i)} \cong U_i^{(j)}$ such that $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ on the respective intersections. Then, we can glue the spaces $X_i$ together along the gluing maps $\phi_{ij}$. Note how this involves triple intersections, unlike the ordinary sheaf condition which only compares sections on intersections of two open sets in a covering. A similar descent via triple intersections holds for the functors of points of stacks in algebraic geometry. Since descent for $\infty$-sheaves involves intersections of arbitrary order, they are sometimes called *higher stacks*.

**Proposition 1.4.2.** A continuous map of topological spaces $f : X \to Y$ induces a geometric morphism

$$(f_* \dashv f^*) : \mathcal{S}h(X) \to \mathcal{S}h(Y) . \tag{1.25}$$

In particular, for $F \in \mathcal{S}h(X)$ and $x \in X$ with inclusion $x : \{x\} \to X$ we can define the *stalk* $x^*F$ of $F$ at $x$.

*Proof.* This follows immediately from 1.3.13, since the inverse image $f^{-1} : \mathrm{Open}(Y) \to \mathrm{Open}(X)$ is a morphism of sites by definition of continuity. □

An $\infty$-topos $\mathcal{X}$ can be understood as an exotic world to do topology in; in particular the terminal $\infty$-topos $\mathcal{S}$ describes usual topology (of CW complexes), and the topoi $\mathcal{S}h(X)$ describe (if $X$ is paracompact and Hausdorff) topology relative to $X$. In particular, one can define homotopy groups of objects in every $\infty$-topos. However, not all results from usual topology still hold – in particular, the theorem of Whitehead can break down:

**Definition 1.4.3.** A morphism $f : X \to Y$ in an $\infty$-topos $\mathcal{X}$ is called $\infty$-*connected* if it induces an isomorphism on all homotopy groups internal to $\mathcal{X}$ (we do not define what this means).

**Proposition 1.4.4** ([Lur17, A.3.9])**.** A morphism $f : F \to G$ in $\mathcal{S}h(X)$ is $\infty$-connected iff for any $x \in X$, it induces an isomorphism on stalks $x^*f : x^*F \xrightarrow{\cong} x^*G$.

*Proof Sketch.* The $\Rightarrow$ direction is immediate since the pullback $x^*$ preserves homotopy groups and hence $\infty$-connected morphisms, but in $\mathcal{S}$ we can apply the Whitehead theorem. For the $\Leftarrow$ direction, we need to show that $f$ induces isomorphisms on all homotopy groups $\pi_i F \to \pi_i G$. Those are however ordinary sheaves, so an isomorphism on stalks is already an isomorphism. □

**Definition 1.4.5.** Let $\mathcal{X}$ be an $\infty$-topos, then an object $X \in \mathcal{X}$ is *hypercomplete* if it is local with respect to $\infty$-connected morphisms, meaning that for any $\infty$-connected $f : C \to D$, precomposing with $f$ induces a homotopy equivalence

$$- \circ f : \mathrm{Map}_{\mathcal{X}}(D, X) \xrightarrow{\simeq} \mathrm{Map}_{\mathcal{X}}(C, X) \ . \tag{1.26}$$

**Theorem 1.4.6.** The full subcategory on the hypercomplete objects $\mathcal{X}^{hyp}$ is again an $\infty$-topos, the *hypercompletion* of $\mathcal{X}$. The inclusion $\mathcal{X}^{hyp} \subseteq \mathcal{X}$ is in fact a geometric morphism, so that any $X \in \mathcal{X}$ has an associated *hypercompletion* $X^{hyp}$.

The functor $(-)^{hyp}$ is a reflection on the subcategory of $\mathcal{L}\mathrm{Top}$ on the hypercomplete $\infty$-topoi, in the sense that for $\mathcal{Y}$ hypercomplete, the spaces of geometric morphisms $\mathrm{Map}_{\mathcal{L}\mathrm{Top}}(\mathcal{Y}, \mathcal{X}) \cong \mathrm{Map}_{\mathcal{L}\mathrm{Top}}(\mathcal{Y}, \mathcal{X}^{hyp})$ agree.

**Definition 1.4.7.** For $\mathcal{C}$ an $\infty$-site, we call a $\mathcal{D}$-valued sheaf $F \in \mathcal{S}h(\mathcal{C}, \mathcal{D})$ *hypercomplete* if for any $D \in \mathcal{D}$, the composition $\mathrm{Map}_{\mathcal{D}}(D, F) \in \mathcal{S}h(\mathcal{C})$ is hypercomplete. Denote their full subcategory by $\mathcal{S}h^{hyp}(\mathcal{C}, \mathcal{D})$.

**Definition 1.4.8.** We call hypercomplete $\mathcal{D}$-valued sheaves on $X$ *hypersheaves*, and denote their category by $\mathcal{S}h^{hyp}(X, \mathcal{D})$

**Proposition 1.4.9.** If $X$ is paracompact Hausdorff and of finite covering dimension, every sheaf on $X$ is hypercomplete.

*Proof.* This is very technical and only added for lack of reference. [Lur09a, 7.1.1.1] assures that we can find a basis $U_i$ for the topology of $X$, such that every $U_i$ is itself open, paracompact Hausdorff and of finite covering dimension; therefore [Lur09a, 7.2.3.6] tells us that $\mathcal{S}h(U_i)$ has finite homotopy dimension. Since the Yoneda embeddings $h_{U_i} \in \mathcal{S}h(X)$ generate $\mathcal{S}h(X)$ under colimits and the slice topoi $\mathcal{S}h(X)_{/U_i} \simeq \mathcal{S}h(U_i)$, we even know that $\mathcal{S}h(X)$ is locally of finite homotopy dimension. Because of [Lur09a, 7.2.1.12], every $\infty$-topos that is locally of finite homotopy dimension is hypercomplete. ☐

## 1.5 Stable $\infty$-categories and Spectra

**Definition 1.5.1.** A *zero object* 0 in an $\infty$-category $\mathcal{C}$ is an object that is both initial and final; in other words for any $C \in \mathcal{C}$,

$$\mathrm{Map}_{\mathcal{C}}(C, 0) \simeq \mathrm{Map}_{\mathcal{C}}(0, C) \simeq \Delta^0 \tag{1.27}$$

are contractible. Since this is a universal property, a zero object is (if it exists) unique up to a contractible space of choices. Also, for $C, D \in \mathcal{C}$, the composition of the essentially unique maps $C \to 0 \to D$ specifies a zero-morphism in every mapping space of $\mathcal{C}$, so they become pointed spaces.

**Definition 1.5.2.** If $\mathcal{C}$ has a zero object 0, and $f : C \to D$ is a morphism in $\mathcal{C}$, then its *fiber* $\mathrm{fib}(f)$ is the equalizer of $f$ and the zero morphism 0 from $C$ to $D$ (just like the kernel in ordinary category theory). Similarly, its *cofiber* $\mathrm{cofib}(f)$ is the coequalizer of $f$ and 0. Sequences of the form

$$\mathrm{fib}(f) \to \mathcal{C} \xrightarrow{f} \mathcal{D}$$
$$\mathcal{C} \xrightarrow{f} \mathcal{D} \to \mathrm{cofib}(f)$$

(up to isomorphism) are called *fiber sequences* and *cofiber sequences*, respectively.

**Definition 1.5.3.** An $\infty$-category $\mathcal{C}$ is called *stable* if:

- It has a zero object 0,
- Every morphism in $\mathcal{C}$ has a fiber and a cofiber,
- Any cofiber sequence is also a fiber sequence.

One can show, using these axioms, that:

- A sequence is a fiber sequence iff it is a cofiber sequences
- All finite limits and colimits exist in $\mathcal{C}$
- A square is a pushout square iff it is a pullback square
- The *loop space* functor $\Omega : \mathcal{C} \to \mathcal{C}$ sending $C \mapsto 0 \times_C 0$ is an equivalence of categories, with inverse the *suspension* functor $\Sigma : C \mapsto 0 \amalg_C 0$.
- The homotopy category $h\mathcal{C}$ has a natural Ab-enrichment.

**Theorem 1.5.4** ([Lur17, 1.1.2.14])**.** If $\mathcal{C}$ is a stable $\infty$-category, then the homotopy category $h\mathcal{C}$ is a triangulated category. In particular, (co-)fiber sequences in $\mathcal{C}$ become triangles in $h\mathcal{C}$, cofibers become (functorial) mapping cones and $\Sigma$ becomes the shift functor [1].

The upshot: Stable $\infty$-categories are equipped to take over the rôle of triangulated categories (and their dg enhancements), just like presentable $\infty$-categories took over the role of (combinatorial) model categories. This is extremely nice, since their definition is just a homotopy coherent formulation of the axioms of an abelian category, in particular very simple compared to Verdier's definition. Similarly, presentable stable $\infty$-categories are one analogon of Grothendieck abelian categories.

**Lemma 1.5.5.** For $f : C \to D$ a morphism in a stable $\infty$-category $\mathcal{C}$, the fiber and cofiber $\mathrm{fib}(f)[1] \cong \mathrm{cofib}(f)$ agree up to a shift.

*Proof.* This follows from the commutative diagram

$$
\begin{array}{ccccc}
\mathrm{fib}(f) & \longrightarrow & C & \longrightarrow & 0 \\
\big\downarrow & & \big\downarrow{\scriptstyle f} & & \big\downarrow \\
0 & \longrightarrow & D & \longrightarrow & \mathrm{cofib}(f)
\end{array}
$$

where all small squares are pushouts or equivalently pullbacks, so the big square is a pushout as well. □

**Proposition 1.5.6.** For $F : \mathcal{C} \to \mathcal{D}$ a functor between stable $\infty$-categories, the following are equivalent:

- $F$ is *left exact*, i.e. preserves limits parametrized by finite simplicial sets
- $F$ is *right exact*, i.e. preserves colimits parametrized by finite simplicial sets
- $F$ preserves the zero object and (co)fiber sequences

It is then called *exact*, and we obtain an $\infty$-category $\mathcal{C}at_\infty^{ex}$ of stable $\infty$-categories and exact functors as a (non-full) subcategory of $\mathcal{C}at_\infty$.

*Proof.* It suffices to show that the last point implies the first, since the converse is clear and the second point follows using a dual argument. By [Lur09a, 4.4.3.2], all finite limits are generated by the terminal object, equalizers and products. But $\mathrm{equ}(f, g) = \mathrm{fib}(f - g)$ and $X \oplus Y = \mathrm{fib}(X \xrightarrow{0} Y[1])$ as well as $Y[1] = \mathrm{cofib}(Y \to 0)$ can all be expressed using 0 and (co)fibers, which $F$ preserves. □

**Proposition 1.5.7** ([Lur17, 1.1.3.2])**.** A full subcategory $\mathcal{D}$ of a stable $\infty$-category $\mathcal{C}$ that contains the zero object, and is closed under fibers and cofibers, is itself stable. We call this a *stable subcategory.*

*Proof.* $\mathcal{D}$ has a zero object, fibers and cofibers since they can be calculated in $\mathcal{C}$ because their universal properties restrict. For the same reason, fiber or cofiber sequences in $\mathcal{D}$ are precisely fiber or cofiber sequences in $\mathcal{C}$ where every object lies in $\mathcal{D}$, so the notions coincide. □

**Proposition 1.5.8** ([Lur17, 1.1.3.1])**.** If $\mathcal{C}$ is a stable $\infty$-category and $K$ a simplicial set, then the $\infty$-category $\mathrm{Fun}(K, \mathcal{C})$ is also stable.

*Proof.* The limits and colimits involved in the definition of a stable $\infty$-category can be calculated pointwise in $\mathrm{Fun}(K, \mathcal{C})$, as shown in [Lur09a, 5.1.2.2]. □

**Proposition 1.5.9** ([Gro16])**.** An $\infty$-category $\mathcal{C}$ is stable iff it admits finite limits and colimits, and finite limits and colimits commute.

Let us construct a few examples.

**Definition 1.5.10.** A *simplicial abelian group* is an ordinary functor in $\mathrm{sAb} := \mathrm{Fun}(\Delta^{op}, \mathrm{Ab})$. Forgetting about the group operation, it has an underlying simplicial set, which can be shown to automatically be a Kan complex. Conversely, every $X \in \mathrm{sSet}$ defines a simplicial abelian group $\mathbb{Z}X : \Delta^{op} \to \mathrm{Ab}$ by composing with the free $\mathbb{Z}$-module functor.

**Definition 1.5.11.** For $X$ a simplicial abelian group, let its *Moore complex* $C_*(X)$ be the chain complex (in homological convention) with $C_n(X) = X_n$ concentrated in non-negative degrees and differential induced by the face maps $\delta_i : X_n \to X_{n-1}$ in $X$:

$$\forall c \in X_n : \; dc := \sum_{i=0}^{n} (-1)^i \delta_i X_n \in C_*(X)_{n-1} \tag{1.28}$$

The *normalized Moore complex* $N_*(X)$ is the subcomplex of $C_*(X)$ spanned by the non-degenerate simplices, where all contributions form degenerate simplices in the differential are set to 0.

Both $C_*$ and $N_*$ are additive functors and preserve colimits. Therefore, $N_*$ is the left Kan Extension along the (Ab-enriched) Yoneda embedding of its restriction to $\Delta$ (by 1.1.6), and therefore arises by applying the nerve-realization paradigm 1.1.7 to this restriction. Hence, it has a right adjoint $K : \mathrm{Ch}(\mathbb{Z})_{\geq 0} \to \mathrm{sAb}$ sending $C \mapsto \mathrm{Hom}(N_*(-), C)$. We call $K(C)$ the *Eilenberg-MacLane space* of $C$, in particular for $A$ an abelian group, $K(A, n) := |K(A[n])|$ is the Eilenberg-MacLane space from topology.

**Theorem 1.5.12** (Dold-Kan correspondence)**.** The functors $N_*$ and $K$ form an equivalence of categories between non-negatively graded chain complexes and simplicial abelian groups, $\mathrm{Ch}(\mathbb{Z})_{\geq 0} \simeq \mathrm{sAb}$. This can be generalized to any abelian category, instead of Ab.

**Example 1.5.13.** Applying $C_*$ to $\mathbb{Z}\,\mathrm{Sing}(X)$ yields the singular chain complex of a topological space $X$.

**Construction 1.5.14.** Let $\mathcal{C}$ be a differential graded (dg) category (a category enriched over $\mathrm{Ch}(\mathbb{Z})$). Truncating the morphism complexes at 0 and applying $K$ yields a category enriched over simplicial abelian groups, and forgetting the group structure yields a Kan-enriched category because of 1.5.10. Finally, applying the homotopy coherent nerve yields an $\infty$-category $N_{dg}\mathcal{C}$, called the dg-nerve of $\mathcal{C}$.

*Remark.* There is an equivalent construction of the dg-nerve that is easier to compute; the shortest way to define it is to apply the nerve-realization paradigm to a functor that realizes objects of $\Delta$ as $A_\infty$-categories, see [Fao13]. This paper also shows that if $\mathcal{C}$ is a pretriangulated dg-category, then $N_{dg}(\mathcal{C})$ is stable.

**Example 1.5.15.** For $R$ any commutative ring, let $\mathrm{Ch}(R)$ be the dg-category of chain complexes of $R$-modules. Denote by $\mathcal{C}\mathrm{h}(R) := N_{dg}\,\mathrm{Ch}(R)$ its dg-nerve, the stable $\infty$-category of chain complexes. Explicitly, its

- Objects are chain complexes of $R$-modules
- Morphisms are chain maps
- 2-morphisms are chain homotopies
- 3-morphisms are chain homotopies between chain homotopies, and so on.

**Example 1.5.16.** Localization (as in 1.2.3) of $\mathcal{C}\mathrm{h}(R)$ at the quasi-isomorphisms yields the *derived $\infty$-category* $D(R)$ of $R$. Similarly for the bounded variants $D^b(R)$, $D^+(R)$ and $D^-(R)$, all of which are stable.

More generally, one can define the derived $\infty$-category $D(\mathcal{A})$ of any abelian category $\mathcal{A}$ by inverting the quasi-isomorphisms in the $\infty$-category of chain complexes in $\mathcal{A}$. This is particularly well-behaved for Grothendieck abelian categories, where $D(\mathcal{A})$ actually is a presentable $\infty$-category by [Lur17, 1.3.5.21].

**Example 1.5.17.** For $R$ a ring, we denote by $D^{\mathrm{fp}}(R)$ the smallest full subcategory of $D(R)$ generated by $R[0]$ by shifts and fibers, i.e. the smallest stable subcategory containing $R[0]$ as in 1.5.7. Similarly, we define by $D^{\mathrm{perf}}(R)$ the smallest full subcategory spanned by $R[0]$ under shifts, fibers and direct summands (i.e. retracts by 2.1.11).

*Remark.* By [Lur17, 1.3.5.21], the derived $\infty$-category $D(\mathcal{A})$ of a Grothendieck abelian category possesses a canonical *t-structure*, i.e. two full subcategories $D_{\geq 0}(\mathcal{A})$ and $D_{\leq 0}(\mathcal{A})$ such that

- For $Y \in D_{\geq 0}(\mathcal{A})$ and $Z \in D_{\leq 0}(\mathcal{A})$, the mapping space $\mathrm{Map}_{D(\mathcal{A})}(Y, Z[-1])$ is contractible,
- $Y[1] \in D_{\geq 0}(\mathcal{A})$ and $Z[-1] \in D_{\leq 0}(\mathcal{A})$, and
- For $X \in D(A)$, there exist $X' \in D_{\geq 0}(\mathcal{A})$ and $X'' \in D_{\leq 0}(\mathcal{A})[-1]$ and a fiber sequence $X' \to X \to X''$.

Explicitly, $D_{\geq 0}(\mathcal{A})$ consists of chain complex concentrated in non-negative, and $D_{\leq 0}(\mathcal{A})$ of chain complexes in non-positive degree (equivalently, with homology concentrated in the respective degrees). The *heart* of this t-structure is

$$D(A)^{\heartsuit} := D_{\geq 0}(\mathcal{A}) \cap D_{\leq 0}(\mathcal{A}) \simeq \mathcal{A} \,. \tag{1.29}$$

Generally, the heart of any t-structure on a stable $\infty$-category is an abelian 1-category.

**Example 1.5.18.** For $\mathcal{A}$ a Grothendieck abelian category, using 1.5.4 we see that cofibers/ fibers in $D(\mathcal{A})$ agree with the mapping cone/ cocone. Also, (co)products are given by the direct sum. Every finite limit or colimit can however be expressed using these constructions (as there is a zero object), so in theory we know how to calculate them. Spelling out the combinatorics, we see that they are calculated using the *bar construction* (see [CG16, C.5.11] or [Rie14]). This is a well-known fact in model category theory, where this construction calculates homotopy (co)limits.

Let us spell this out for a simplicial diagram $F : \Delta^{op} \to D(\mathcal{A})$, to better understand the descent condition for $\infty$-sheaves with values in $D(\mathcal{A})$. We can associate to $F$ a Čech double complex

$$\begin{array}{ccccccccc}
 & & \dots & & \dots & & \dots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & F([0])_1 & \longleftarrow & F([1])_1 & \longleftarrow & F([2])_1 & \longleftarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & F([0])_0 & \longleftarrow & F([1])_0 & \longleftarrow & F([2])_0 & \longleftarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & F([0])_{-1} & \longleftarrow & F([1])_{-1} & \longleftarrow & F([2])_{-1} & \longleftarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \dots & & \dots & & \dots & &
\end{array}$$

where the vertical maps are the differentials in $F([n])$, and the horizontal maps are induced by alternating sums over the boundary maps. The $\infty$-limit, or homotopy limit, over $F$ is the total complex of this double complex (using the direct product should it not be bounded). Similarly for homotopy colimits over $\Delta$, using the direct sum. In short, the descend condition for an $\infty$-sheaf $F : \mathrm{Open}(X)^{op} \to D(\mathcal{A})$ on a space $X$ assures that for $(U_i)$ a cover of $U$,

$$F(U) \cong \check{C}((U_i), F) \tag{1.30}$$

is quasi-isomorphic to the Čech hypercohomology of $F$ on the cover $U$. In particular, as we have discussed after 1.24, the Čech-to-sheaf-cohomology spectral sequence is associated to this double complex.

**Example 1.5.19.** Let $\mathcal{S}_*^{\mathrm{fin}}$ be the $\infty$-category of finite pointed spaces from 1.2.10. Denote by $\mathrm{Exc}^*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})$ the full subcategory of $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})$ on functors that are

- *reduced*, i.e. preserve the final object, and
- *excisive*, i.e. send pushout squares to pullback squares.

This is the stable $\infty$-category $\mathcal{S}p$ of *spectra*. Its homotopy category agrees with the triangulated category of (symmetric) spectra. More generally, for $\mathcal{C}$ an $\infty$-category with finite limits, we denote by $\mathcal{S}p(\mathcal{C})$ the stable $\infty$-category of *spectrum objects* in $\mathcal{C}$, also called *spectrification*. It is defined as

$$\mathcal{S}p(\mathcal{C}) := \mathrm{Exc}^*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}) \ . \tag{1.31}$$

**Definition 1.5.20.** Denote by $\Omega^\infty : \mathcal{S}p \to \mathcal{S}$ the functor that evaluates a pointed excisive functor $\mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}$ at the pointed space $S^0$.

**Proposition 1.5.21** ([Lur17, 1.4.2.24])**.** The functors $\Omega^\infty \circ \Sigma^n : \mathcal{S}p \to \mathcal{S}_*$ evaluating a reduced, excisive functor $\mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}_*$ at the sphere $S^n$ exhibit the stable $\infty$-category of spectra as the limit of the following diagram in $\mathcal{C}at_\infty$:

$$\mathcal{S}p \simeq \lim_{\mathbb{N}} \left( \cdots \to \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \right) \ . \tag{1.32}$$

In other words, we obtain $\mathcal{S}p$ from the $\infty$-category $\mathcal{S}_*$ of pointed spaces by inverting the suspension functor.

*Remark.* Given a sequence $(E_i)_{i\in\mathbb{N}}$ of pointed spaces together with homotopy equivalences $\Omega E_i \cong E_{i-1}$ for all $i > 0$, we denote the corresponding spectrum under the above correspondence by

$$\mathbb{E} = [E_0, E_1, E_2, E_3, \ldots] \ . \tag{1.33}$$

This is the usual way spectra are introduced (in fact since the above maps must be equivalences, we are working with $\Omega$-*spectra*; we are only interested in these fibrant-cofibrant objects of the model category of spectra as they are the objects of the underlying $\infty$-category). In particular this explains the name $\Omega^\infty$, since $\Omega^\infty\mathbb{E} = E_0 = \Omega E_1 = \Omega^i E_i$ for all $i$ is an infinite loop space.

**Example 1.5.22.** For $A$ an abelian group, the Eilenberg-MacLane space $K(A, n)$ is the unique space satisfying $\pi_k K(A, n) = A$ iff $k = n$, and 0 otherwise. By this uniqueness, we must have $\Omega K(A, n) = K(A, n-1)$ since $\Omega$ shifts the homotopy groups by one, so

$$HA := [K(A, 0), K(A, 1), K(A, 2), \ldots] \tag{1.34}$$

is a spectrum, the *Eilenberg-MacLane spectrum* associated to $A$. We can replace $A$ by a chain complex using 1.5.12, see 1.7.1.

**Corollary 1.5.23** ([Lur17, 1.4.4.4])**.** The stable $\infty$-category $\mathcal{S}p$ is presentable, and the functor $\Omega^\infty : \mathcal{S}p \to \mathcal{S}$ admits a left adjoint $\Sigma^\infty_+$ defined as

$$\Sigma^\infty_+(X) = [X_+, \Sigma X_+, \Sigma^2 X_+, \ldots] \tag{1.35}$$

with $X_+$ the space $X$ with an added disjoint base point. Similarly, $\Omega^\infty : \mathcal{S}p \to \mathcal{S}_*$ has a left adjoint $\Sigma^\infty$ where no extra basepoint needs to be added. The *sphere spectrum* is defined as

$$\mathbb{S} := \Sigma^\infty_+ \Delta^0 = \Sigma^\infty S^0 \,, \tag{1.36}$$

equivalently it corresponds to the pointed excisive functor $\mathcal{S}^{\text{fin}}_* \hookrightarrow \mathcal{S}_*$ given by the canonical inclusion.

*Remark.* To be precise, $\Sigma^\infty_+$ as defined above is no spectrum in our sense, since the unit $X_+ \to \Omega\Sigma X_+$ is generally no homotopy equivalence. We have to replace it by an $\Omega$-spectrum, in the model category language, for example

$$Q\Sigma^\infty_+ X := [QX_+, Q\Sigma X_+, Q\Sigma^2 X_+, \dots] \tag{1.37}$$

with $QX_+ := \operatorname*{colim}_{k\in\mathbb{N}} \Omega^k \Sigma^k X_+$ the *free infinite loop space* on $X_+$.

**Definition 1.5.24.** As a variant, we introduce the $\infty$-category of *finite spectra* as the colimit

$$\mathcal{S}p^{\text{fin}} := \operatorname*{colim}_{\mathbb{N}} \left( \mathcal{S}^{\text{fin}}_* \xrightarrow{\Sigma} \mathcal{S}^{\text{fin}}_* \xrightarrow{\Sigma} \dots \right) \tag{1.38}$$

which can be embedded into $\mathcal{S}p$ using a Yoneda-argument [Lur17, Introduction to 1.4]. In fact, this is the smallest stable subcategory of $\mathcal{S}p$ containing $\mathbb{S}$.

To be more explicit about this embedding, a finite spectrum may be thought of as a pair $(K, n)$ where $n \in \mathbb{N}$ and $K$ is a finite space. We identify it with the spectrum $\Sigma^n \Sigma^\infty_+ K$, and since $\Sigma \dashv \Omega$ this can be checked to induce a fully faithful functor – see also the classical discussion in [Sch12, Section 7.1].

**Definition 1.5.25.** A *point* of a spectrum $\mathbb{E}$ is a point of the underlying space $\Omega^\infty \mathbb{E}$. Equivalently, it is an $\mathbb{S}$-point of $\mathbb{E}$, since

$$\operatorname{Map}_{\mathcal{S}p}(\mathbb{S}, \mathbb{E}) = \operatorname{Map}_{\mathcal{S}p}(\Sigma^\infty_+ \Delta^0, \mathbb{E}) \simeq \operatorname{Map}_{\mathcal{S}}(\Delta^0, \Omega^\infty \mathbb{E}) \simeq \Omega^\infty \mathbb{E} \,. \tag{1.39}$$

**Proposition 1.5.26** (Universal Property of $\mathcal{S}p$, [Lur17, 1.4.4.5])**.** For any presentable stable $\infty$-category $\mathcal{D}$, the functor $\Sigma^\infty_+$ induces equivalences

$$\begin{aligned} \operatorname{Fun}^{\lim}(\mathcal{D}, \mathcal{S}p) \simeq \operatorname{Fun}^{\lim}(\mathcal{D}, \mathcal{S}) \simeq \mathcal{D} \\ \operatorname{Fun}^{\operatorname{colim}}(\mathcal{S}p, \mathcal{D}) \simeq \operatorname{Fun}^{\operatorname{colim}}(\mathcal{S}, \mathcal{D}) \simeq \mathcal{D} \end{aligned} \tag{1.40}$$

where $\operatorname{Fun}^{\operatorname{colim}}$ and $\operatorname{Fun}^{\lim}$ denote limit- and colimit preserving functors. Note that both rows are equivalent by the adjoint functor theorem, and the last equivalence follows from the universal property of the presheaf category $\mathcal{PS}h(\Delta^0) \simeq \mathcal{S}$ in 1.1.6.

*Remark.* One can define $\mathcal{S}p$ as the unique $\infty$-category satisfying this universal property.

**Definition 1.5.27.** There is a notion [Lur09a, 6.5.1.1] of *homotopy groups* of objects in general presentable $\infty$-categories, in particular in spectra. Using the description as a sequential limit of $\mathcal{S}_*$, one can show that for $\mathbb{E} = [E_0, E_1, \dots] \in \mathcal{S}p$ they agree with the stable homotopy groups

$$\pi_n \mathbb{E} := \operatorname*{colim}_{m \in \mathbb{N}} \pi^{n+m}(E_m) \ . \tag{1.41}$$

**Proposition 1.5.28** ([Lur17, 1.4.3.6])**.** There is a canonical t-structure on $\mathcal{S}p$, with $\mathcal{S}p_{\leq 0}$ the full subcategory on those spectra $\mathbb{E}$ with contractible underlying space $\Omega^\infty \mathbb{E}$ and $\mathcal{S}p_{\geq 0}$ determined as its orthogonal full subcategory. By Whitehead, we could equivalently define $\mathcal{S}p_{\geq 0}$ to consist of *connective spectra*, i.e. those whose homotopy groups $\pi_{-n}\mathbb{E} = 0$ for $n > 0$; and $\mathcal{S}p_{\leq 0}$ as the *coconnective spectra* with $\pi_n \mathbb{E} = 0$. The heart $\mathcal{S}p^\heartsuit \simeq \mathrm{Ab}$ is hence the category of abelian groups, as it spanned by spectra with a single non-vanishing homotopy groups, i.e. Eilenberg-MacLane spectra.

*Remark.* All of the above results are still true if we replace $\mathcal{S}$ by any presentable $\infty$-category $\mathcal{C}$, and $\mathcal{S}p$ with $\mathcal{S}p(\mathcal{C})$.

The $\infty$-category $\mathcal{S}p$ plays a similar role in the theory of stable $\infty$-categories as $\mathcal{S}$ plays for general $\infty$-categories:

**Proposition 1.5.29.** For $\mathcal{C}$ a stable $\infty$-category and objects $C, D \in \mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(C, D)$ can be refined to a *mapping spectrum* $\mathrm{map}_{\mathcal{C}}(C, D)$ such that

$$\Omega^\infty \mathrm{map}_{\mathcal{C}}(C, D) = \mathrm{Map}_{\mathcal{C}}(C, D) \ . \tag{1.42}$$

In other words, every stable $\infty$-category is enriched over spectra.

*Proof Sketch.* We can construct the spectrum $\mathrm{map}_{\mathcal{C}}(C, D)$ as the infinite loop space

$$[\mathrm{Map}_{\mathcal{C}}(C, D), \mathrm{Map}_{\mathcal{C}}(C, \Sigma D), \mathrm{Map}_{\mathcal{C}}(C, \Sigma^2 D), \dots] \tag{1.43}$$

which is clearly functorial in $C$ and $D$, and has the underlying space $\mathrm{Map}_{\mathcal{C}}(C, D)$. It is also compatible with composition, but since we have not defined enrichments of $\infty$-categories we will not go into further details. □

## 1.6 Ring and Module Spectra

**Definition 1.6.1.** Let $\mathrm{Fin}_*$ be the ordinary category (or via its nerve, $\infty$-category) of finite pointed sets $\langle n \rangle = \{*, 1, 2, \dots, n\}$ with pointed maps. In particular, denote by $\rho_i : \langle n \rangle \to \langle 1 \rangle$ the map that sends $i \mapsto 1$ and all other elements to $*$, for $i = 1, \dots, n$.

**Definition 1.6.2.** A *symmetric monoidal* $\infty$*-category* with underlying $\infty$-category $\mathcal{V}$ is a functor $v : \mathrm{Fin}_* \to \mathcal{C}at_\infty$ such that for each $n$, the functors $v(\rho_i)$ are the projections exhibiting $v(\langle n \rangle)$ as the product $\mathcal{V}^{\times n}$.

**Construction 1.6.3.** For $\mathcal{V}$ a symmetric monoidal $\infty$-category, the unique morphism $u : \langle 0 \rangle \to \langle 1 \rangle$ in $\mathrm{Fin}_*$, and the morphism $t : \langle 2 \rangle \to \langle 1 \rangle$ sending $* \mapsto *$ and everything else to 1, induce morphisms

$$1_{\mathcal{V}} : \Delta^0 \to \mathcal{V}\,, \quad \otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V} \tag{1.44}$$

called the *unit object* and *tensor product* of $\mathcal{V}$. Often, we therefore denote a symmetric monoidal $\infty$-category by $(\mathcal{V}, \otimes)$ for clarity. Other morphisms in $\mathrm{Fin}_*$ induce higher coherence relations on $1_{\mathcal{V}}$ and $\otimes$, in particular the morphism $\langle 2 \rangle \to \langle 2 \rangle$ interchanging 1 and 2 induces a symmetric braiding $V \otimes W \cong W \otimes V$.

*Remark.* By 1.2.20, the functor $v$ above classifies a coCartesian fibration commonly denoted by $p : \mathcal{V}^{\otimes} \to \mathrm{Fin}_*$. The fiber of $p$ over $\langle n \rangle \in \mathrm{Fin}_*$ is by definition equivalent to $\mathcal{V}^{\times n}$, and the parallel transports along the $\rho_i$ induce the projections out of this product.

*Remark* ([Lur17, 4.1.2.5]). Similarly, a *monoidal* $\infty$*-category* with underlying $\infty$-category $\mathcal{V}$ can be defined as a functor $v : \Delta^{op} \to \mathcal{C}at_\infty$, such that the images of the boundary maps $\rho_i : [1] \to [n]$ sending $0 \mapsto i-1$ and $1 \mapsto i$, for $i = 1, \ldots, n$, are the projection maps exhibiting $v([n]) \cong \mathcal{V}^{\times n}$.

**Construction 1.6.4** ([Lur17, 4.1.2.10])**.** There is a canonical *cut functor* $c : \Delta^{op} \to \mathrm{Fin}_*$ that sends $[n]$ to $\langle n \rangle$, and a monotone map $\alpha : [n] \to [m]$ to the map $\langle m \rangle \to \langle n \rangle$ that sends $i \in \langle m \rangle - \{*\}$ to $\min \alpha^{-1}(\{i, i+1, \ldots, m\})$ if this set is non-empty, and $*$ otherwise. Precomposition with this functor sends a symmetric monoidal $\infty$-category to the underlying monoidal $\infty$-category.

**Example 1.6.5.**

- The *trivial symmetric monoidal* $\infty$*-category* is determined by the functor $\underline{\Delta^0}$ : $\mathrm{Fin}_* \to \mathcal{C}at_\infty$ that is constant on $\Delta^0$. Its underlying $\infty$-category is clearly $\Delta^0$, and it is classified by the coCartesian fibration $\mathrm{id}_{\mathrm{Fin}_*} : \mathrm{Fin}_* \to \mathrm{Fin}_*$. Similarly, we define the *trivial monoidal* $\infty$*-category.*
- For $R$ a ring, the $\infty$-category of chain complexes $\mathcal{C}\mathrm{h}(R)$ is symmetric monoidal with respect to the tensor product of chain complexes. Similarly for the derived $\infty$-category $D(R)$ and the derived tensor product.
- $\mathcal{C}\mathrm{h}(R)^{op}$ is also symmetric monoidal with respect to $\otimes$, similarly for any symmetric monoidal $\infty$-category.

- The $\infty$-category of $\mathcal{S}p$ is symmetric monoidal with respect to the smash product $\wedge$, which is the unique (by 1.5.26) tensor product preserving colimits in both variables with the sphere spectrum $\mathbb{S}$ as the unit. On the homotopy category of symmetric spectra, this agrees with the usual definition of the smash product; we will also see a very abstract construction in a moment.
- Any $\infty$-category with finite products is symmetric monoidal with respect to the product, similarly for coproducts.
- For $\mathrm{Pr}^{\mathrm{L}}$ the non-full subcategory of $\mathcal{C}at_\infty$ on presentable $\infty$-categories and colimit-preserving functors (i.e. by the Adjoint Functor Theorem 1.2.19, left adjoint functors), the tensor product $\mathcal{C} \otimes \mathcal{D} := \mathrm{Fun}^{\lim}(\mathcal{C}^{op}, \mathcal{D})$ we have introduced in 1.3.16 induces a symmetric monoidal structure, with unit object $\mathcal{S}$ as by the $\infty$-categorical analogon to 1.1.6.
- If we denote by $\mathrm{Pr}^{\mathrm{st}}$ the full subcategory of $\mathrm{Pr}^{\mathrm{L}}$ on presentable stable $\infty$-categories, the fact that functors into a stable $\infty$-category form a stable $\infty$-category themselves 1.5.8 tells us that the above symmetric monoidal structure restricts to $\mathrm{Pr}^{\mathrm{st}}$. Its unit is the $\infty$-category of spectra $\mathcal{S}p$, since it satisfies the universal property 1.5.26.

**Definition 1.6.6.** A morphism $\alpha : \langle n \rangle \to \langle m \rangle$ in $\mathrm{Fin}_*$ is called *inert* if every $i \in \langle m \rangle - \{*\}$ that is not the pointing $*$ has a unique preimage in $\langle n \rangle$. In other words, $\alpha^{-1} : \langle m \rangle - \{*\} \to \langle n \rangle - \{*\}$ is well-defined and automatically injective, determining $\alpha$. Conversely, a morphism $\alpha$ as above is called *active* if $\alpha^{-1}(\{*\}) = \{*\}$. Any morphism in $\mathrm{Fin}_*$ can uniquely be factored as the composition of an active after an inert morphism.

**Definition 1.6.7.** A morphism $\alpha : [n] \to [m]$ in $\Delta$ is called *inert* if it is injective and embeds $[n]$ into $[m]$ as an interval, i.e. $\alpha(i) = \alpha(0) + i$. On the other hand, it is called *active* if $\alpha(0) = 0$ and $\alpha(n) = m$. Again, any morphism in $\Delta$ can be factored as an inert after an active morphism (so in $\Delta^{op}$, we obtain a factorization as above).

*Remark.* Intuitively, inert morphism encode trivial operations in symmetric monoidal $\infty$-categories, for example the map $\langle 2 \rangle \to \langle 1 \rangle$ sending $*, 2 \mapsto *$ and $1 \mapsto 1$ induces the map $\mathcal{V}^{\times 2} \to \mathcal{V}$ projecting on the first component. On the other hand, active morphisms encode the interesting operations, like the tensor product of $n$ objects, the identity or the braiding above. Similarly in the monoidal case, or for general $\infty$-operads.

**Definition 1.6.8.** Let $p : \mathcal{V}^{\otimes} \to \mathrm{Fin}_*$ be a coCartesian fibration classifying a symmetric monoidal $\infty$-category. A morphism $f$ in $\mathcal{V}^{\otimes}$ is called *inert* if $p(f)$ is inert and $f$ is $p$-coCartesian (i.e. it encodes a non-trivial operation in $\mathcal{V}$), and *active* if $p(f)$ is active. These again form a factorization system by [Lur17, 2.1.2.4].

**Definition 1.6.9.** For $(\mathcal{V}_1, \otimes), (\mathcal{V}_2, \otimes)$ symmetric monoidal $\infty$-categories, a *symmetric monoidal functor* $F : \mathcal{V}_1 \to \mathcal{V}_2$ is a natural transformation between the corresponding functors $v_1, v_2 : \mathrm{Fin}_* \to \mathcal{C}at_\infty$. Equivalently, it is a functor of the underlying $\infty$-categories that preserves unit, tensor product and its braiding up to coherent isomorphism, as well as the higher coherences encoded by higher compositions.

A *lax monoidal functor* $F : \mathcal{V}_1 \to \mathcal{V}_2$ is a morphism $F$ between the classifying coCartesian fibrations $\mathcal{V}_1^\otimes \to \mathrm{Fin}_*$ and $\mathcal{V}_2^\otimes \to \mathrm{Fin}_*$ in the slice category $\mathcal{C}at_{\infty\,/\,\mathrm{Fin}_*}$ that sends inert morphisms in $\mathcal{V}_1$ to inert morphisms in $\mathcal{V}_2$. Using 1.2.20, we see that this induces a symmetric monoidal functor iff $F$ preserves all coCartesian edges, not just those over inert morphisms in $\mathrm{Fin}_*$.

*Technical Remark.* We could also define lax monoidal functors without resorting to the Grothendieck construction, they are given by (lax) natural transformations of the composites $v_1, v_2 : \mathrm{Fin}_* \to \mathcal{C}at_\infty \subseteq \boldsymbol{\mathcal{C}at}_\infty$ regarded as functors of $(\infty, 2)$-categories.

**Definition 1.6.10.** Let us define an $\infty$-category

$$\mathrm{Fun}^\otimes(\mathcal{V}_1, \mathcal{V}_2) := \mathrm{Map}_{\mathrm{Fun}(\mathrm{Fin}_*, \mathcal{C}at_\infty)}(v_1, v_2) \tag{1.45}$$

of symmetric monoidal functors between $\mathcal{V}_1$ and $\mathcal{V}_2$. Similarly, denote by

$$\mathrm{Fun}^{\mathrm{lax}}(\mathcal{V}_1, \mathcal{V}_2) \subseteq \mathrm{Map}_{\mathcal{C}at_{\infty\,/\,\mathrm{Fin}_*}}(\mathcal{V}_1^\otimes, \mathcal{V}_2^\otimes) \tag{1.46}$$

the full subcategory on lax monoidal functors (those preserving inert morphism), which by the discussion above contains the full subcategory of functors preserving coCartesian edges which is equivalent to $\mathrm{Fun}^\otimes(\mathcal{V}_1, \mathcal{V}_2)$.

*Remark.* The intuition for these classes of functors is the same as in the classical case. A functor $F : \mathcal{V}_1^\otimes \to \mathcal{V}_2^\otimes$ is symmetric monoidal if (among higher relations) it preserves the tensor product up to isomorphism,

$$F(X \otimes Y) \cong F(X) \otimes F(Y)\ . \tag{1.47}$$

It is lax monoidal if there is a natural map $F(X) \otimes F(Y) \to F(X \otimes Y)$, which does not have to be an isomorphism and should be regarded as part of the data contained in $F$ (among higher relations). A lax monoidal functor is monoidal iff these maps are always isomorphisms.

**Definition 1.6.11.** Similarly, we define monoidal functors between monoidal $\infty$-categories $v_1, v_2 : \Delta^{op} \to \mathcal{C}at_\infty$ as natural transformations; and lax monoidal functors as morphisms between the classifying coCartesian fibrations in the slice category $\mathcal{C}at_{\infty\,/\,\mathrm{Fin}_*}$ preserving inert morphisms. Again, a lax monoidal functor is monoidal iff it preserves all coCartesian edges.

**Definition 1.6.12.** For $(\mathcal{V}, \otimes)$ a symmetric monoidal $\infty$-category with symmetric monoidal structure determined by $p : \mathcal{V}^{\otimes} \to \mathrm{Fin}_*$, a *commutative algebra* in $\mathcal{V}$ is a lax monoidal functor $A : \mathrm{id}_{\mathrm{Fin}_*} \to p$ in $\mathcal{C}at_{\infty/\mathrm{Fin}_*}$, from the trivial commutative algebra in $\mathcal{C}at_\infty$ to $\mathcal{V}$. In other words, it is a section $A : \mathrm{Fin}_* \to \mathcal{V}^{\otimes}$ of $p$ preserving inert morphisms, in particular $A(\rho_i) : A(\langle n \rangle) \to A(\langle 1 \rangle)$ is $p$-coCartesian for $i = 1, \dots, n$. But the definition of a symmetric monoidal $\infty$-category entails that the coCartesian transports along the $\rho_i$ are the projections exhibiting $v(\langle n \rangle) = v(\langle 1 \rangle)^{\times n}$; and we have seen that these the $A(\rho_i)$ are coCartesian, so they exhibit $A(\langle n \rangle) = (A(\langle 1 \rangle), \dots, A(\langle 1 \rangle)) \in \mathcal{V}^{\times n}$. If we abuse notation by denoting $A(\langle 1 \rangle) =: A$, we obtain composition maps

$$A \otimes \cdots \otimes A = A^{\otimes k} \to A \tag{1.48}$$

for each $k \geq 1$, inducing a unit and an algebra multiplication on $A$ satisfying higher coherences. We denote the $\infty$-category of commutative algebra objects in $\mathcal{V}^{\otimes}$ by $\mathrm{CAlg}(\mathcal{V}) := \mathrm{Fun}^{\mathrm{lax}}(\Delta^0, \mathcal{V})$.

**Definition 1.6.13.** Similarly, we define *(associative) algebra objects* in a monoidal $\infty$-category $(\mathcal{V}, \otimes)$ as lax monoidal functors from the trivial monoidal $\infty$-category to $\mathcal{V}$. Spelling this out yields, for each $k \in \mathbb{N}_0$ and ever total order on $\{1, \dots, k\}$, a morphism

$$A \otimes A \otimes \cdots \otimes A \to A \tag{1.49}$$

with $k$ factors of $A := A([0])$ on the left, together with compatibility conditions among them. The $\infty$-category of algebra objects in $\mathcal{V}^{\otimes}$ will be denoted by $\mathrm{Alg}(\mathcal{V})$.

**Definition 1.6.14.** For $A$ an algebra object in $\mathcal{V}$, we define the *opposite algebra object* with reversed multiplication by conjugating the defining functor $A : \Delta^{op} \to \mathcal{V}$ with the functor $\mathrm{rev} : \Delta \to \Delta$ sending $[n] \mapsto [n]$ and $\alpha : [n] \to [m]$ to $\mathrm{rev}(\alpha) : [n] \to [m]$ with $\mathrm{rev}(\alpha)(i) := \alpha(n - i)$. The functor $A^{op} := \mathrm{rev} \circ A \circ \mathrm{rev}$ is still an algebra object since $\mathrm{rev}^2 = \mathrm{Id}_{\Delta}$ and it preserves inert morphisms in $\Delta^{op}$ (which agree with inert morphisms in $\Delta^{op} = (\Delta^0)^{\otimes}$ regarded as the trivial monoidal $\infty$-category). Explicitly, the composition map

$$A \otimes A \otimes \cdots \otimes A = A^{\otimes k} \to A \tag{1.50}$$

in $A^{op}$, associated to a fixed total order on $\{1, \dots, k\}$, is the composition map in $A$ associated to the reversed total order.

**Definition 1.6.15.** A *ring spectrum* is an associative algebra object in the symmetric monoidal category of spectra $(\mathcal{S}p, \wedge)$, regarded as a monoidal $\infty$-category using the cut functor. A *commutative ring spectrum* is a commutative algebra object in $(\mathcal{S}p, \wedge)$.

**Example 1.6.16.**

- For $(\mathcal{V}, \otimes)$ an ordinary symmetric monoidal category, algebra objects in its nerve regarded as a symmetric monoidal $\infty$-category coincide with the usual definition of algebra objects, namely objects of $\mathcal{V}$ with unit and associative multiplication. Similarly in the commutative case.

- For $\mathcal{C}$ any $\infty$-category and $C \in \mathcal{C}$, the *endomorphism space* $\mathrm{End}(C) := \mathrm{Map}_{\mathcal{C}}(C, C)$ is an associative algebra object in $(\mathcal{S}, \times)$, with multiplication determined by the composition of endomorphisms.
- If $\mathcal{C}$ is even stable, this can be lifted to an *endomorphism spectrum* $\mathrm{end}(C) := \mathrm{map}_{\mathcal{C}}(C, C)$ which is an associative algebra object in $(\mathcal{S}p, \wedge)$, in other words a ring spectrum.
- For $(\mathcal{V}, \otimes)$ a monoidal $\infty$-category, the unit $1_{\mathcal{V}}$ comes equipped with a multiplication $1_{\mathcal{V}} \otimes 1_{\mathcal{V}} \cong 1_{\mathcal{V}}$ and similar higher multiplications, making it into the *initial algebra object*. Similarly, the unit in a symmetric monoidal $\infty$-category is the *initial commutative algebra object*.
- The category of algebra objects in the $\infty$-category $(\mathcal{C}at_\infty, \times)$ of $\infty$-categories equipped with the Cartesian product is the category of monoidal $\infty$-categories, with monoidal functors as morphisms. Similarly, commutative algebras in $(\mathcal{C}at_\infty, \times)$ are precisely the symmetric monoidal $\infty$-categories.
- Since we have seen that $\mathcal{S}p$ is the unit in $\mathrm{Pr}^{\mathrm{st}}$, it is naturally equipped with an algebra structure as explained above. By the last point, this makes $\mathcal{S}p$ into a presentable stable symmetric monoidal $\infty$-category itself, with multiplication given by the *smash product* of spectra $\wedge$ that preserves colimits in both variables. This is of course very inexplicit, but agrees with the usual definition.

**Proposition 1.6.17** ([Lur17, 4.1.2.10])**.** If $A : \mathrm{Fin}_* \to \mathcal{V}^{\otimes}$ is a commutative algebra object in a symmetric monoidal $\infty$-category $\mathcal{V}$ defined by $v : \mathrm{Fin}_* \to \mathcal{C}at_\infty$, then $A \circ c : \Delta^{op} \to \mathcal{V}^{\otimes}$ is an algebra object in the underlying monoidal $\infty$-category of $\mathcal{V}$ define by $v \circ c : \Delta^{op} \to \mathcal{C}at_\infty$.

Now, let us define modules over algebra objects. Just as the multiplication in an algebra object $A$ had to be defined via giving all maps of the form $A \otimes A \otimes \cdots \otimes A \to A \otimes \cdots \otimes A$ over morphisms of $\Delta^{op}$, not just the multiplication $A \otimes A \to A$ and unit $1_{\mathcal{V}} \to A$; we have to specify not only the action $M \times A \to M$ to give a left module $M$, but maps of the forms

$$A \otimes A \otimes \cdots \otimes A \to A \otimes \cdots \otimes A \,,$$
$$M \otimes A \otimes A \otimes \cdots \otimes A \to M \otimes A \otimes \cdots \otimes A \,,$$
$$M \otimes A \otimes A \otimes \cdots \otimes A \to A \otimes \cdots \otimes A \,.$$

This are precisely the edges of the simplicial set $\mathrm{Fin}_* \times \Delta^1$ or $\Delta^{op} \times \Delta^1$, depending on whether we put an ordering on the multiplications in $A$ or not.

**Definition 1.6.18** ([Lur17, 4.2.2.2])**.** Given an algebra object $A : \Delta^{op} \to \mathcal{V}$ in a symmetric monoidal $\infty$-category $\mathcal{V}^{\otimes}$, a *left module object* over $A$ is a functor $M : \Delta^{op} \times \Delta^1 \to \mathcal{V}$ such that

- $M|_{\Delta^{op} \times \{1\}}$ agrees with $A$
- For each $[n] \in \Delta^{op}$, the map $M([n], 0) \to M([n], 1)$ induced by the identity on $[n]$ and the map $M([n], 0) \to M([0], 0)$ induced by $[0] \to [n]$ sending $0 \mapsto n$ exhibit $M([n], 0) \simeq M([n], 1) \times M([0], 0)$.

Similarly, one can define *right module objects* over $A$ by using the map $[0] \to n$ sending $0 \mapsto 0$ instead. Denote the full subcategory of $\mathrm{Fun}(\Delta^{op} \times \Delta^1, \mathcal{V})$ on left modules over $A$ by $\mathrm{LMod}_A(\mathcal{V})$, and the full subcategory on right modules by $\mathrm{RMod}_A(\mathcal{V})$, omitting $\mathcal{V}$ if it is clear.

**Definition 1.6.19.** Given a commutative algebra object $A : \mathrm{Fin}_* \to \mathcal{V}$ in $\mathcal{V}$, a *module object* over it is a functor $M : \mathrm{Fin}_* \times \Delta^1 \to \mathcal{V}$ such that

- $M|_{\mathrm{Fin}_* \times \{1\}}$ agrees with $A$
- For each $\langle n \rangle \in \Delta^{op}$, the map $M(\langle n \rangle, 0) \to M(\langle n \rangle, 1)$ induced by the identity on $[n]$ and the map $M(\langle n \rangle, 0) \to M(\langle 0 \rangle, 0)$ induced by the unique map $\langle n \rangle \to \langle 0 \rangle$ exhibit $M(\langle n \rangle, 0) \simeq M(\langle n \rangle, 1) \times M(\langle 0 \rangle, 0)$.

There is no directionality involved in this definition since the map $\langle n \rangle \to \langle 0 \rangle$ is unique, so left and right modules do not have to be distinguished. Denote the full subcategory on $A$-modules by $\mathrm{Mod}_A(\mathcal{V}) \subseteq \mathrm{Fun}(\mathrm{Fin}_* \times \Delta^1, \mathcal{V})$.

*Remark* ([Lur17, 4.5.1.6])*.* If we regard a commutative algebra $A$ as an associative algebra object in the underlying monoidal $\infty$-category to $\mathcal{V}$, then $\mathrm{LMod}_A(\mathcal{V}) \simeq \mathrm{Mod}_A(\mathcal{V}) \simeq \mathrm{RMod}_A(\mathcal{V})$.

**Definition 1.6.20.** Given a ring spectrum $R$, we define the $\infty$-category of left module spectra $\mathrm{LMod}_R$ as the category of module objects over it in $(\mathcal{S}p, \wedge)$, and similarly the category of left module spectra $\mathrm{RMod}_R$.

**Proposition 1.6.21** ([Lur17, 7.1.1.5])**.** The $\infty$-categories $\mathrm{LMod}_R$ and $\mathrm{RMod}_R$ are again stable, and the forgetful functor $\mathrm{LMod}_R \to \mathcal{S}p$ sending $M : \Delta^{op} \times \Delta^1 \to \mathcal{S}p$ to the image of $(\langle 0 \rangle, 0)$ in $\mathcal{S}p$ is exact.

**Example 1.6.22.** Every object $V$ of a monoidal $\infty$-category $\mathcal{V}$ is both a left and a right module over the unit $1_{\mathcal{V}}$, with module actions induced by the isomorphisms $1_{\mathcal{V}} \otimes V \cong V \cong V \otimes 1_{\mathcal{V}}$. In particular, any spectrum is a module over the sphere spectrum $\mathbb{S}$; and every ring spectrum $R$ is both a left and right module over itself, with module action determined by the ring multiplication.

*Technical Remark.* In particular, every presentable stable $\infty$-category is a module over the unit object $(\mathcal{S}p, \wedge)$, we say that it is *tensored* over spectra. If all involved $\infty$-categories are presentable and all functors colimit-preserving, a tensoring is equivalent to an enrichment by the Adjoint-Functor-Theorem, compare [GH15, Chapter 7]. Generally, this is however false; a general stable $\infty$-category is enriched over spectra as explained in 1.5.29, but only tensored over finite spectra (see [CDH$^+$20a, 5.1.1]) since the colimits involved in tensoring with infinite spectra may not exist.

**Definition 1.6.23.** We define the stable $\infty$-category of *finitely presented* $R$-module spectra $\mathrm{LMod}_R^{\mathrm{fp}}$ as the smallest stable subcategory (as in 1.5.7) containing $R$. In other words, it contains all module spectra that can be generated by $R$ using fibers and shifts (and consequently also direct sums).

Similarly, the stable $\infty$-category of *perfect* $R$-module spectra $\mathrm{LMod}_R^{\mathrm{perf}}$ is the smallest full subcategory containing $R$ that is closed under fibers, shifts and direct summands (i.e. if $M = N \oplus P \in \mathrm{LMod}_R^{\mathrm{perf}}$, then $N$ and $P$ as well).

**Definition 1.6.24.** For $R, S$ ring spectra, a $R$-$S$-bimodule is a spectrum $M$ equipped with both a left module structure over $R$, and a right module structure over $S$. In particular, there are multiplication maps of the form

$$R \wedge \cdots \wedge R \wedge M \wedge S \wedge \cdots \wedge S \ .$$

Denote the $\infty$-category of $R$-$S$-bimodules by

$${}_R\mathrm{BiMod}_S \subseteq \mathrm{Fun}(\Delta^{op} \times \Delta^1 \times \Delta^{op}, \mathcal{S}p) \ .$$

*Remark.* We could by [Lur17, 4.3.2.7] also define ${}_R\mathrm{BiMod}_S \simeq \mathrm{RMod}_S(\mathrm{LMod}_R(\mathcal{S}p)) \simeq \mathrm{LMod}_R(\mathrm{RMod}_S(\mathcal{S}p))$, and all of these characterizations work similarly in general monoidal $\infty$-categories.

*Technical Remark.* What we call ring spectra are also referred to as $\mathbb{E}_1$- or $A_\infty$-ring spectra in the literature, as they are modules over the $\mathbb{E}_1$-operad in $(\mathcal{S}p, \wedge)$. Similarly, commutative ring spectra are algebras over the $\mathbb{E}_\infty$-operad.

The abstract definitions of (commutative) algebra objects and modules over them obtain a nice geometric interpretation employing the theory of $\mathbb{E}_M$-algebras (also called locally constant factorization algebras): To any manifold $M$ with boundary we can associate an $\infty$-operad $\mathbb{E}_M$ describing how disjoint unions of charts in $M$ may be embedded into each other. In the case $M = \mathbb{R}$ this recovers the $\mathbb{E}_1$-operad, where the ordering of factors in a tensor product corresponds to the ordering induced on disjoint open intervals in $\mathbb{R}$; in the case $M = \mathbb{R}^n$ with $n \to \infty$ we recover $\mathbb{E}_\infty$ since there is enough space to move charts around each other almost freely.

For $\mathbb{R}^n$ with $1 < n < \infty$ we obtain $\mathbb{E}_n$-algebras where multiplication is symmetric, but higher coherences break down, this allows e.g. for the definition of *braided monoidal $\infty$-categories* $\mathrm{Alg}_{\mathbb{E}_2}(\mathcal{C}at_\infty)$. For $M = \mathbb{R}_{\geq 0}$, we obtain pairs of $\mathbb{E}_1$-algebras and left modules over them. See [Zet23] for more.

## 1.7 Brave New Algebra

In many aspects, ring spectra behave similarly to ordinary rings – many constructions and statements from commutative algebra still hold, which is why their theory is called *brave new algebra.* Both settings can be compared using:

**Construction 1.7.1.** Recall from 1.5.22 that the stable $\infty$-category of spectra $\mathcal{S}p$ contains the ordinary category of abelian groups Ab as a full subcategory (its heart), by sending each $A \in \mathrm{Ab}$ to the associated Eilenberg-MacLane spectrum

$$[K(A,0), K(A,1), K(A,2), \dots] =: HA \tag{1.51}$$

with homotopy groups concentrated in degree 0. Using the universal property of the derived $\infty$-category and its $t$-structure [Lur17, 1.3.3.2], this extends to a functor $D(\mathbb{Z}) \to \mathcal{S}p$. For an arbitrary commutative ring $R$, compose it with the forgetful functor $D(R) \to D(\mathbb{Z})$ (since forgetting the $R$-multiplication preserves quasi-isomorphisms) to obtain the *Eilenberg-MacLane functor*

$$H : D(R) \to \mathcal{S}p \,. \tag{1.52}$$

In particular, $H(R[0])$ is the Eilenberg-MacLane spectrum $HR$ on the underlying abelian group of $R$, and the module structure of elements of $D(R)$ over $R[0]$ translates into a module structure over $HR$ on the right since it turns out that $H$ is lax monoidal.

*Remark.* An alternative, more explicit way to construct $H$ would be to use the (Ab-enriched) nerve-realization paradigm 1.1.7 to extend $H : \mathrm{Ab} \to \mathcal{S}p$ to simplicial abelian groups, which by the Dold-Kan Correspondence are equivalent to $\mathrm{Ch}^+(\mathbb{Z})$. Since the homotopy groups $\pi_n HC$ agree with the homology groups of the complex $C$ as the simplicial sphere is identified with the complex $\mathbb{Z}[n]$, we see by Whitehead that $H$ sends quasi-isomorphisms to homotopy equivalences and thus factors through $D^+(\mathbb{Z})$. Extend to all of $D(\mathbb{Z})$ using shifts.

**Theorem 1.7.2** (Stable Dold-Kan Correspondence, [Lur17, 7.1.2.13])**.** For $R$ an ordinary commutative ring, the Eilenberg-MacLane functor induces an equivalence of $\infty$-categories

$$D(R) \simeq \mathrm{LMod}_{HR} \tag{1.53}$$

that restricts to equivalences of the stable subcategories $D^{\mathrm{perf}}(R) \simeq \mathrm{LMod}_{HR}^{\mathrm{perf}}$ and $D^{\mathrm{fp}}(R) \simeq \mathrm{LMod}_{HR}^{fp}$.

This statement may be seen as a conceptual reason for why derived categories are interesting in the first place. Its proof relies on the following recognition criterion for $\infty$-categories of module spectra:

**Definition 1.7.3.** For $\mathcal{C}$ a stable $\infty$-category, an object $C \in \mathcal{C}$ is a *compact generator* if it is *compact*, i.e. the functor

$$\mathrm{map}_{\mathcal{C}}(C, -) : \mathcal{C} \to \mathcal{S}p \tag{1.54}$$

it corepresents preserves filtered colimits, and it generates $\mathcal{C}$ in the sense that for each $D \in \mathcal{C}$, if $\mathrm{map}_{\mathcal{C}}(C, D) \cong 0 \in \mathcal{S}p$, then $D \cong 0$.

**Theorem 1.7.4** (Schwede-Shipley recognition criterion, see [Lur17, 7.1.2.1])**.** If $\mathcal{C}$ is a presentable stable $\infty$-category with a compact generator $X$, and $\mathrm{end}(X)$ is the associative ring spectrum of endomorphisms of $X$, then there is a canonical equivalence of categories

$$\mathcal{C} \simeq \mathrm{LMod}_{\mathrm{end}(X)} \ . \tag{1.55}$$

*Proof of 1.7.2.* We already know that $D(R)$ is stable and presentable by 1.5.16. The complex $R[0]$ concentrated in degree 0 is a compact generator since the it represents the identity functor

$$\mathrm{Map}_{D(R)}(R[0], -) = \mathrm{RHom}(R[0], -) = \mathrm{Id}_{D(R)} \tag{1.56}$$

which is in particular faithful and preserves filtered colimits. By the recognition criterion, we thus know that $D(R) \simeq \mathrm{LMod}_{\mathrm{end}(R)}$. But

$$\pi_n \,\mathrm{end}(R) = \mathrm{Ext}_R^{-n}(R, R) = \begin{cases} R & \text{for } n = 0, \\ 0 & \text{otherwise} \end{cases} \tag{1.57}$$

so we can identify $\mathrm{end}(R) \cong HR$, with multiplication given by the Yoneda product on Ext groups, which in this case is just the product in $R$.

The perfect and finitely presented cases follow from the very definitions of the respective subcategories as the smallest ones generated by $R[0]$ or $HR$ respectively via shifts, fibers and potentially direct summands. □

This result explains one of the main ideas behind brave new algebra: Instead of trying to understand derived categories, we could do algebra over general ring spectra which is formally very similar to algebra over ordinary rings, but when restricted to Eilenberg-MacLane spectra automatically lives in the derived world. Many definitions and results from commutative algebra and algebraic geometry carry over to the theory of ring spectra, for example localizations, étale maps, Kähler differentials, Henselian rings, schemes and stacks. As long as we are working over a base (ordinary) ring containing $\mathbb{Q}$, the resulting geometry, called *spectral algebraic geometry*, is by [Lur17, 7.1.4.11] equivalent to derived algebraic geometry over commutative differential graded algebras or simplicial rings, but in non-zero characteristic, over general discrete rings or even ring spectra it is different and superior for some applications, like chromatic homotopy theory or elliptic cohomology.

Let us develop some more elementary results on ring and module spectra.

**Definition 1.7.5** ([Lur17, Section 4.4])**.** If $A, B, C$ are ring spectra, we define the *relative tensor product*

$$\otimes_B : {}_A\mathrm{BiMod}_B \times {}_B\mathrm{BiMod}_C \to {}_A\mathrm{BiMod}_C \tag{1.58}$$

using the two-sided bar construction

$$M \otimes_B N := \operatorname*{colim}_{\Delta} \left( \dots M \wedge B \wedge B \wedge N \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; M \wedge B \wedge N \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; M \wedge N \right) \tag{1.59}$$

where maps from the right to the left are induced by multiplication, and from the left to the right by the unit in $R$. This operation is associative, with unit $B$ regarded as a $B$-$B$-bimodule.

*Remark.* Compare with the definition of the tensor product of ordinary bimodules as the coequalizer $\mathrm{coequ}\left( M \otimes B \otimes N \rightrightarrows M \otimes N \right)$, or with the Hochschild complex that arises via the similar cyclic bar construction. One can also define the relative tensor product as a representing object for bilinear maps, just like the ordinary tensor product of rings.

**Proposition 1.7.6.** The relative tensor product is equivariant with respect to the functor $(-)^{op} : {}_A\mathrm{BiMod}_B \xrightarrow{\simeq} {}_{B^{op}}\mathrm{BiMod}_{A^{op}}$ applied to both arguments (and exchanging them) or the target, in the sense that $(M \otimes_B N)^{op} \cong N^{op} \otimes_{B^{op}} M^{op}$. In particular, if $R$ is a commutative ring spectrum so that we can regard any $R$-module as a left or right module, i.e. as a bimodule, the relative tensor product

$$- \otimes_R - : \mathrm{Mod}_R \times \mathrm{Mod}_R \to \mathrm{Mod}_R \tag{1.60}$$

is symmetric in its arguments.

**Definition 1.7.7.** The relative tensor product has adjoint internal Hom functors

$$\begin{aligned} \underline{\mathrm{Hom}}_C &: {}_B\mathrm{BiMod}_C \times {}_A\mathrm{BiMod}_C \to {}_A\mathrm{BiMod}_B \\ {}_A\underline{\mathrm{Hom}} &: {}_A\mathrm{BiMod}_B \times {}_A\mathrm{BiMod}_C \to {}_B\mathrm{BiMod}_C \end{aligned}$$

equipping the mapping spectra $\mathrm{map}_{\mathrm{RMod}_C}(-,-)$ and $\mathrm{map}_{\mathrm{LMod}_A}(-,-)$ with bimodule structures.

*Remark.* While we distinguish between ${}_A\underline{\mathrm{Hom}}$ and $\underline{\mathrm{Hom}}_A$ in this statement for clarity, in the next chapters we will always use the latter, more common notation.

*Proof.* Since the smash product preserves colimits separately in each variable, and the relative tensor product is defined as a colimit, it preserves colimits in each variable as well (compare [Lur17, 4.4.2.15] for a more general statement). Hence, by the adjoint functor theorem, the functors $M \otimes_B -$ and $- \otimes_B N$ admit right adjoints ${}_A\underline{\mathrm{Hom}}(M, -)$ and

$\underline{\mathrm{Hom}}_C(N,-)$ respectively, which are also functorial in $M$. In particular, ${}_B\underline{\mathrm{Hom}}(B,-)$ is adjoint to the identity functor for $A = B$, so it is isomorphic to the identity functor itself. Similarly for $\underline{\mathrm{Hom}}_B(B,-)$.

For $M$ an $A$-$C$- and $N$ a $B$-$C$-bimodule, the spectrum underlying $\underline{\mathrm{Hom}}_C(N,M)$ is

$$\mathrm{map}_{\mathrm{RMod}_B}(B, \underline{\mathrm{Hom}}_C(N,M)) \simeq \mathrm{map}_{\mathrm{RMod}_C}(B \otimes_B N, M) \simeq \mathrm{map}_{\mathrm{RMod}_C}(N, M)$$

and similarly for ${}_A\underline{\mathrm{Hom}}$. □

**Definition 1.7.8.** For $f : A \to B$ a map of commutative ring spectra, precomposition with $f$ induced the *change-of-coefficients* or *restriction-of-scalars* functor

$$(-)_A : \mathrm{Mod}_B \to \mathrm{Mod}_A \tag{1.61}$$

which by [Lur17, 4.5.3.1] has a left adjoint, the *extension-of-scalars* functor

$$B \otimes_A - : \mathrm{Mod}_A \to \mathrm{Mod}_B \tag{1.62}$$

which is symmetric monoidal.

*Remark.* Explicitly, if $M : \Delta^{op} \times \Delta^1 \to \mathcal{S}p$ is a $B$-module, then $M_A : \Delta^{op} \times \Delta^1 \to \mathcal{S}p$ is determined by $M_A|_{\Delta^{op}\times\{0\}} = M|_{\Delta^{op}\times\{0\}}$ and $M_A|_{\Delta^{op}\times\{1\}} = B$ together with the natural transformation $M|_{\Delta^{op}\times\{0\}} \xrightarrow{M} A \xrightarrow{f} B$.

Similarly, for $N$ an $A$-module, the $B$-module $B \otimes_A N$ is indeed defined as a relative tensor product, where we view $N$ as an $A$-$\mathbb{S}$-bimodule and $B$ as a $B$-$A$-module with module structure

$$B \otimes B \otimes \dots B \otimes A \otimes \dots \otimes A \to B \tag{1.63}$$

defined by applying $f$ to the factors of $A$ and then multiplying in $B$.

# 2 Algebraic L-Theory

In this chapter, we introduce the notion of Poincaré $\infty$-categories, which are stable $\infty$-categories equipped with a notion of quadratic forms. We associate L-groups and L-spectra to them, generalizing several variations of classical L-theory as developed for example in [Ran92]. Our constructions are originality due to Lurie's Lecture notes [Lur11], and have been refined in the series of papers [CDH+20a], [CDH+20b], [CDH+21] by several authors that we use as a main reference.

## 2.1 Bilinear and Quadratic Functors

As algebraic L-theory is classically all about classifying quadratic or symmetric forms modulo algebraic bordism (i.e. dividing by Lagrangian subspaces), its formulation via $\infty$-categories will rely on categorifying these concepts. We follow a very straightforward analogy: For us, stable $\infty$-categories should be regarded as a categorification of vector spaces or modules.

Remember that functor $F : \mathcal{C} \to \mathcal{D}$ between stable $\infty$-categories is called

- *reduced* if it preserves the zero object,
- *excisive* if it sends pushout squares to pullback squares,
- *exact* if it preserves finite limits (or equivalently, finite colimits).

**Proposition 2.1.1.** A functor $F$ as above is exact iff it is pointed and excisive.

*Proof.* First of all, a square in a stable $\infty$-category is a pushout iff it is a pullback. Since terminal object and pullbacks are special cases of finite limits, the *if* direction is clear. For the *only if* direction, every finite limit can be written using the terminal object and pullbacks by dualizing [Lur09a, 4.4.2.5]. □

Under above comparison of stable $\infty$-categories with vector spaces, reduced functors are zero-preserving maps of vector spaces, excisive functors are affine maps, while exact functors are linear maps. More generally, any smooth function between finite-dimensional real vector spaces can be Taylor expanded as the sum of a constant, a linear function, a quadratic function and so on. Similarly, a functor between stable (and even slightly more general) $\infty$-categories can be Taylor expanded:

**Definition 2.1.2.** Let $p : (\Delta^1)^n \to \mathcal{C}$ be an $n$-cubical diagram in an $\infty$-category $\mathcal{C}$ that possesses finite limits, and let $K_n$ be the simplicial set obtained from $(\Delta^1)^n$ by removing the point $(0, \dots, 0)$ and all simplices containing it. $p$ is called

- *Cartesian* if, identifying $(\Delta^1)^n \cong K_n^{\triangleleft}$, it is a limit cone,
- *strongly Cartesian* if for every 2-dimensional cubical face

  $$f : \{i_1\} \times \cdots \times \{i_{k-1}\} \times \Delta^1 \times \{i_{k+1}\} \times \cdots \times \{i_{l-1}\} \times \Delta^1 \times \{i_{l+1}\} \times \cdots \times \{i_n\} \subseteq (\Delta^1)^n$$

  with $1 \leq k < l \leq n$ and $i_j \in \{0, 1\}$ for $j \in \{1, \dots, \hat{k}, \dots, \hat{l}, \dots, n\}$, the restriction $p \circ f$ is a Cartesian square.

In particular (by a cofinality argument), strongly Cartesian squares are Cartesian. Similarly, one can define *(strongly) coCartesian cubical diagrams*.

**Definition 2.1.3.** A functor $F : \mathcal{C} \to \mathcal{D}$ between $\infty$-categories $\mathcal{C}$ with finite colimits and $\mathcal{D}$ with finite limits is called *$n$-excisive* if it sends strongly coCartesian $(n+1)$-cubes in $\mathcal{C}$ to Cartesian $(n+1)$-cubes in $\mathcal{D}$. Again, by a cofinality argument, an $n$-excisive functor is automatically $(n+1)$-excisive.

*Remark.* In stable $\infty$-categories, strongly coCartesian and strongly Cartesian cubes agree; however one can also implement this definition and the *Goodwillie calculus* that builds on it more generally, where this distinction is important.

**Example 2.1.4.**

- A functor $F : \mathcal{C} \to \mathcal{D}$ is 0-excisive iff it sends every morphism to an isomorphism, since for a 1-cube being strongly (co)Cartesian is a vacuous conditions, while being Cartesian is equivalent to being an isomorphism.
- $F$ is 1-excisive iff it is excisive.

These are the functors that replace polynomials of degree $n$ for Taylor expansion. Of interest for us will be a characterization of 2-excisive functors.

Let us first take a slight detour into the theory of quadratic forms: Fix $R$ be a commutative ring and $M$ a (projective) $R$-module.

**Definition 2.1.5.** A map $b : M \times M \to R$ is *bilinear* if for any $m \in M$, the induced maps $b(m, -) : M \to R$ and $b(-, m) : M \to R$ are linear. We say that $b$ is *symmetric* if $b(m, m') = b(m', m)$ for all $m, m' \in M$.

If we denote the dual of $M$ by $M^\vee = \mathrm{Hom}_R(M, R)$, then a symmetric bilinear map $b$ induces an adjoint map $b^\sharp : M \to M^\vee$ sending $m \mapsto b(m, -)$, and $b$ is called *non-degenerate* if $b^\sharp$ is an isomorphism.

**Definition 2.1.6.** An *inhomogeneous quadratic form* on $M$ is a map $q : M \to R$ such that $q(0) = 0$, and for any $m, m' \in M$ the *polarization* $b_q(m, m') := q(m+m') - q(m) - q(m')$ is symmetric bilinear. It is a *quadratic form* if for any $n \in N$ and $m \in M$, we have $q(nm) = n^2 q(m)$.

**Proposition 2.1.7.** If $2 \in R$ is invertible, then the map $q \mapsto b_q$ from quadratic forms to symmetric bilinear forms is a bijection.

*Proof.* To a symmetric bilinear form $b : M \times M \to R$, we may conversely associate a quadratic form $q_b(m) := \frac{1}{2} b(m, m)$ with polarization

$$b_{q_b}(m, m') = q_b(m + m') - q_b(m) - q_b(m') = \frac{b(m + m', m + m') - b(m, m) - b(m', m')}{2} =$$
$$= \frac{1}{2}(b(m, m') + b(m', m)) = b(m, m') \ .$$

In particular, this expression is bilinear; also $q_b(0) = 0$ and $q(nm) = n^2 q(m)$ hold so $q_b$ is indeed quadratic. Finally,

$$q_{b_q}(m) = \frac{1}{2} \left(q(2m) - q(m) - q(m)\right) = \frac{1}{2}(4 - 1 - 1)q(b) = q(b) \ . \qquad \square$$

**Proposition 2.1.8.** Let $2 \in R$ still be invertible and $q : M \to R$ be an inhomogeneous quadratic form with polarization $b$, then the difference $l(m) := q(m) - q_b(m)$ is a $\mathbb{Z}$-linear map and $q$ is a quadratic form iff $l$ vanishes.

*Remark.* This means that an inhomogeneous quadratic form can uniquely be decomposed into a quadratic and a $\mathbb{Z}$-linear form, and conversely it is clear that any sum of a quadratic and $\mathbb{Z}$-linear form is inhomogeneous quadratic.

*Proof.* Additivity of $l$ follows from the calculation

$$l(m + m') = q(m + m') - \frac{1}{2} b(m + m', m + m') =$$
$$= q(m) + q(m') + b(m, m') - \frac{1}{2}(b(m, m) + b(m', m') + 2b(m, m')) =$$
$$= q(m) - \frac{1}{2} b(m, m) + q(m') - \frac{1}{2} b(m', m') = l(m) + l(m') \ .$$

Further, $l$ vanishes iff $q = q_b$, but in this case $q_b(nm) = \frac{1}{2} b(nm, nm) = n^2 q_b(m)$ making $q$ quadratic. Conversely, if $q$ is quadratic, the difference $l = q - q_b$ satisfies $l(nm) = n^2 l(m)$ as well, but for $n = 2$ this means that $l(n) + l(n) = l(n+n) = l(2n) = 2^2 l(n)$ so $2l(n) = 0$, implying that $l$ must vanish. $\square$

We will categorify these statements in the course of this section, allowing us to drop the invertibility requirement on 2.

**Proposition 2.1.9** ([Nik20] 5.7)**.** For $P$ a finitely generated projective $R$-module, let $B(P,P)$ be the $R$-module of bilinear forms $b : P \times P \to R$, and equip it with the $S_2$-action determined by sending $b \mapsto b \circ \tau$, where $\tau : P \times P \to P \times P$ exchanges the components. Then, the space of orbits (i.e. the coinvariants of this action)

$$B(P,P)_{S_2} = \frac{B(P,P)}{R\langle b - b \circ \tau | b \in B(P,P)\rangle} \tag{2.1}$$

is isomorphic to the $R$-module $\mathcal{Q}(R)$ of quadratic forms.

*Proof.* Given an orbit $[b] \in B(P,P)_{S_2}$ with representative $b : P \otimes_R P \to R$, we can associate to it the map $q^b : P \to R$ that sends $p \mapsto q^b(p) := b(p,p)$. This clearly satisfies $q_b(0) = 0$ and $q^b(np) = n^2 q^b(p)$, also its polarization $b + b \circ \tau$ is bilinear. We need to show that the construction $b \mapsto q^b$ is injective and surjective. First, let us restrict to the case where $P$ is finitely free with basis $(e_i)_{i=1}^N$, so $b$ is determined by a quadratic matrix.

*Injectivity*: If $q^b(p) = b(p,p) = 0$ for all $p \in P$, then for $p' \in P$ we have $b(p,p') + b(p',p) = b(p+p', p+p') - b(p,p) - b(p',p') = 0$ so that $b$ is antisymmetric. But this means that we can write $b = b^{\text{ut}} - b^{\text{ut}} \circ \tau$ where $b^{\text{ut}}$ is the upper triangular part of $b$ regarded as a matrix, so $[b] = 0$ in the coinvariants.

*Surjectivity*: Given any quadratic form $q$, we define $b^q$ via the upper triangular matrix with diagonal entry on the basis element $e_i$ given by $q(e_i, e_i)$, and upper non-diagonal entries determined by the polarization $b_q$ of $q$. Then,

$$q(\sum_i \lambda_i e_i) = q(\lambda_1 e_1) + q\left(\sum_{i=2}^N \lambda_i e_i\right) + b_q\left(\lambda_1 e_1, \sum_{i=2}^N \lambda_i e_i\right) = \cdots =$$
$$= \sum_i q(\lambda_i e_i) + \sum_{i<j} b_q(\lambda_i e_i, \lambda_j e_j) = \sum_{i,j} b^q(\lambda_i e_i, \lambda_j e_j) = q^{b^q}\left(\sum_i \lambda_i e_i\right)$$

Now, for $P$ finitely generated, we may write it as the quotient $\pi : P' \to P \cong {}^{P}\!/\!_{N}$ of a finitely free $R$-module $P'$ to which we can apply the above argument. Since $P$ is projective, the short exact sequence

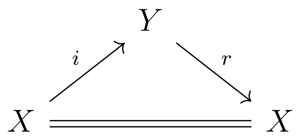
$$0 \to N \to P' \to P \to 0 \tag{2.2}$$

splits, so $P' \cong P \oplus N$. By definition, quadratic forms on $P$ are the same thing as quadratic forms $P' \to R$ that factor through $P$; and similarly bilinear forms on $P$ are precisely those homomorphisms $P' \otimes P' \to R$ that send $P \otimes P' \oplus P' \otimes P$ to zero. Such a form $b \in B(P,P)$ can be written as $b_0 - b_0 \circ \tau$ for some $b \in B(P,P)$ iff $b = b_1 - b_1 \circ \tau$ for $b_1 \in B(P', P')$, since we can just subtract from $b_1$ its projection to the direct summand $P' \otimes P \oplus P \otimes P'$. Our isomorphism constructed for the free $P'$ thus identifies the respective subspaces of forms on $P$ on both sides. $\square$

*Remark.* Similarly, invariants of this action are precisely the symmetric bilinear forms, and the canonical map $B(M,M)^{S_2} \to B(M,M)_{S_2}$ sending a symmetric bilinear form to its orbit is an equivalence if 2 is invertible, as we have seen.

Our next step will be to categorify these definitions.

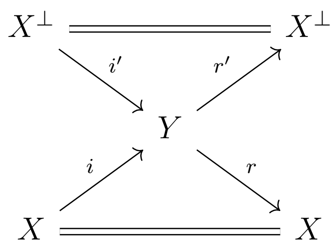
**Definition 2.1.10.** Let $\mathcal{C}$ be an arbitrary $\infty$-category, and $X, Y \in \mathcal{C}$. We call $X$ a *retract* of $Y$ if there is a retraction diagram of the following form, exhibiting $r \circ i = \mathrm{id}_X$:



**Proposition 2.1.11.** If $\mathcal{C}$ is a stable $\infty$-category with $X, Y \in \mathcal{C}$, then $X$ is a retract of $Y$ iff there is an object $X^\perp \in \mathcal{C}$ such that we can write $Y \cong X \oplus X^\perp$. In other words, retracts and direct summands are the same thing in the stable case.

*Proof.* If $Y \cong X \oplus X^\perp$, then the canonical inclusion and projection maps $X \xrightarrow{i_1} X \oplus X^\perp \xrightarrow{p_1} X$ associated to a biproduct exhibit $X$ as a retract of $Y$.

Conversely, given maps $X \xrightarrow{i} Y \xrightarrow{r} X$ that compose to the identity, one can set $X^\perp := \mathrm{fib}(r)$ to obtain the commutative diagram



where the map $r'$ is obtained by applying the universal property of $\mathrm{fib}(r)$ to factor the morphism $(1 - ir) : Y \to Y$ through $X^\perp$. By the universal property of the biproduct, these maps combine to a composition $X \oplus X^\perp \to Y \to X \oplus X^\perp$. We need to show that it agrees with the identity, and that the inverted composition $Y \to X \oplus X^\perp \to Y$ does so as well.

The first claim follows from $r \circ i = \mathrm{id}_X$ and $r' \circ i' = \mathrm{id}_{X^\perp}$, where the latter equality holds since $r' \circ i'$ is obtained by uniquely factoring $(1 - ir)i'$ through $X^\perp$, but $ri' = 0$ be definition.

For the second claim, it suffices to show that $i \circ r + i' \circ r' = \mathrm{id}_Y$ by the definition of the sum of morphisms in a stable $\infty$-category. But $i'r' = 1 - ir$ by definition of $r'$. $\square$

**Proposition 2.1.12.** In the situation of 2.1.11, the complementing direct summand $X^\perp$ is essentially unique.

*Proof.* Assume that we have an isomorphism $\alpha : X \oplus Z \to X \oplus Z'$, that induces the identity on $X$. To show that in this case $Z \cong Z'$, we write down the diagram

$$\begin{array}{ccccc} X & \xrightarrow{i_1} & X \oplus Z & \longrightarrow & \mathrm{cofib}(i_1) \\ \| & & \downarrow \alpha & & \downarrow \\ X & \xrightarrow{i_1'} & X \oplus Z' & \longrightarrow & \mathrm{cofib}(i_1') \end{array}$$

where the left square commutes by assumption, and the right square by functoriality of the cofiber. Using its universal property or a pasting argument, it is clear that $\mathrm{cofib}(i_1) \cong Z$ and $\mathrm{cofib}(i_1') \cong Z'$, and these are isomorphic since we have seen $i_1 \cong i_1'$. □

Let in the following $\mathcal{C}$ and $\mathcal{D}$ be stable $\infty$-categories.

**Definition 2.1.13.** Given a reduced functor $\tilde{B} : \mathcal{C} \times \mathcal{C} \to \mathcal{D}$ and $X, Y \in \mathcal{C}$, the inclusions and projections of the biproduct $X \oplus Y$ induce morphisms

$$\tilde{B}(X,X) \oplus \tilde{B}(Y,Y) \longrightarrow \tilde{B}(X \oplus Y, X \oplus Y) \longrightarrow \tilde{B}(X,X) \oplus \tilde{B}(Y,Y) \tag{2.3}$$

that compose to the identity. This exhibits $\tilde{B}(X,X) \oplus \tilde{B}(Y,Y)$ as a direct summand of $\tilde{B}(X \oplus Y, X \oplus Y)$, and we call its complement $B(X,Y)$ the *polarization* of $\tilde{B}$.

**Proposition 2.1.14.** The polarization $B : \mathcal{C} \times \mathcal{C} \to \mathcal{D}$ of a reduced functor $\tilde{B}$ is *bireduced*, i.e. $B(0,-)$ and $B(-,0)$ are identical to the 0-functor. In fact, the construction $\tilde{B} \mapsto B$ is left and right adjoint to the inclusion of bireduced into reduced functors, in particular it is functorial itself.

*Proof.* By definition, $\tilde{B}(X \oplus Y, X \oplus Y) = \tilde{B}(X,X) \oplus \tilde{B}(Y,Y) \oplus B(X,Y)$. If $X = 0$, this means $\tilde{B}(Y,Y) = 0 \oplus \tilde{B}(Y,Y) \oplus B(0,Y)$ so by uniqueness of the complement, $B(0,Y) \cong 0$. Conversely for $Y = 0$. The second claim is [CDH+20a, 1.1.3], in particular the (co-)units of these adjunctions are given by the identity transformation on bireduced functors, and the transformations $B(X,Y) \to \tilde{B}(X,Y) \to B(X,Y)$ which are induced by the inclusions and projections out of the direct sums in $\tilde{B}(X \oplus Y, X \oplus Y)$. □

**Definition 2.1.15.** For $\Omega : \mathcal{C}^{op} \to \mathcal{S}p$ a reduced functor, the functor $\tilde{B} : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathcal{S}p$ determined by $\tilde{B}(X,Y) := \Omega(X \oplus Y)$ is reduced. We can use 2.1.13 to construct its polarization $B_\Omega(X \oplus Y)$, which is also called the *polarization* of $\Omega$.

From this construction, it is automatic that there is an isomorphism $\eta_{X,Y} : B_{\mathfrak{Q}}(X,Y) \cong B_{\mathfrak{Q}}(Y,X)$ for all $X, Y \in \mathcal{C}$ since the direct sum $\oplus$ is symmetric in its arguments. In fact, this isomorphism is natural in $X, Y$ and there is a 2-simplex $\sigma$ witnessing $\eta_{X,Y} \circ \eta_{Y,X} \cong \mathrm{id}$. Also, the whiskerings $\sigma \circ \eta_{X,Y}$ and $\eta_{X,Y} \circ \sigma$ are isomorphic 2-simplices, and there are infinitely many higher coherence relations of similar forms. A great advantage of our $\infty$-categorical approach is the following elegant way to phrase this:

**Definition 2.1.16.** For $\mathcal{C}$ an $\infty$-category and $G$ a group (or a monoid), an *object with $G$-action* in $\mathcal{C}$ is a functor $f : BG \to \mathcal{C}$, and the image $X = f(*)$ of the unique object of $BG$ in $\mathcal{C}$ is its *underlying object*.

The *homotopy invariants* $X^{hG}$ are the limit over the diagram $f$ if it exists, and the *homotopy coinvariants* $X_{hG}$ are the colimit over $f$.

**Example 2.1.17.** Let $\mathcal{C}$ be the derived category $D(R)$ of a commutative ring $R$, and $M$ a (projective) $R$-module equipped with a $G$-action $\tau_g : M \to M$ for $g \in G$. Then, using 1.5.18 the homotopy coinvariants of this action on $M[0] \in D(R)$ can be calculated as

$$M_{hG} = \left( \cdots \to \bigoplus_{g,h \in G} M \xrightarrow{d_2} \bigoplus_{g \in G} M \xrightarrow{d_1} M \to 0 \to \dots \right) \tag{2.4}$$

with e.g. $d_1(m_g) := \sum_g (g-1) m_g$, so its zeroth homology yields precisely the ordinary coinvariants ${}^{M}\!/_{G}$. Note that the above is precisely the complex computing the group homology of the $G$-module $M$. Similarly, the homotopy invariants $M^{hG}$ compute group cohomology.

**Construction 2.1.18.** For $G$ finite and $\mathcal{C}$ a stable $\infty$-category, there is a canonical *norm map* $X_{hG} \to X^{hG}$, and the cofiber

$$X^{tG} := \mathrm{cofib}\left(X_{hG} \to X^{hG}\right) \tag{2.5}$$

is the *Tate spectrum* of $X$. We refer to [Lur17, 6.1.6] for a precise construction of this map, since we will only need it in a special case where it simplifies a lot (see the next construction). Intuitively, an orbit in $X_{hG}$ is sent to the sum over all of its elements, which is an invariant in $X^{hG}$. In the case of a finite group action on an $R$-module as above, this agrees with the ordinary norm map, and the mapping cone construction for the cofiber shows that the complex $M[0]^{tG}$ calculates Tate cohomology.

*Remark.* We will mainly deal with $S_2$-actions and their (co)invariants, for $S_2$ the symmetric group on two objects.

**Construction 2.1.19** ([CDH+20a] 1.1.10)**.** If $B_{\Qoppa}$ is the polarization of the reduced functor $\Qoppa : \mathcal{C}^{op} \to \mathcal{S}p$, there are canonical natural transformations

$$B_{\Qoppa}(X,X)_{hS_2} \to \Qoppa(X) \to B_{\Qoppa}(X,X)^{hS_2} \tag{2.6}$$

in $X$ whose composition is the norm map. They are induced by the codiagonal and diagonal maps $\nabla : X \oplus X \to X$ and $\Delta : X \to X \oplus X$, inducing

$$B_{\Qoppa}(X,X) \to \Qoppa(X \oplus X) \to \Qoppa(X) \to \Qoppa(X,X) \to B_{\Qoppa}(X,X) \tag{2.7}$$

where the first and last map are inclusion of and projection onto a direct summand, which factor through the homotopy (co)invariants with respect to the $S_2$-action exchanging the summands of $X$. Alternatively, they are induced by the (co)units of the adjunctions in 2.1.14 when setting $X = Y$ and taking homotopy (co)invariants.

**Definition 2.1.20.** A functor $B : \mathcal{C} \times \mathcal{C} \to \mathcal{D}$ is called *bilinear* if for each $C \in \mathcal{C}$, the functors $B(C,-)$ and $B(-,C)$ are exact. In particular, it has to be bireduced.

Additionally, $B$ is called *symmetric bilinear* it is a homotopy fixed point under the $S_2$-action on the $\infty$-category of bilinear functors that exchanges both arguments. Explicitly, $B(X,Y) \cong B(Y,X)$ for all $X, Y \in \mathcal{C}$, and this natural isomorphism satisfies higher coherence relations.

**Proposition 2.1.21** ([CDH+20a] 1.1.13)**.** For $\mathcal{C}$ a stable $\infty$-category, a reduced functor $\Qoppa : \mathcal{C}^{op} \to \mathcal{S}p$ and $B_{\Qoppa}$ its polarization, the following conditions are equivalent:

- $\Qoppa$ is 2-excisive.
- The functor $\Lambda_{\Qoppa} : \mathcal{C}^{op} \to \mathcal{S}p$ mapping $X$ to the fiber of the canonical map

$$\Lambda_{\Qoppa}(X) := \mathrm{fib}\left(\Qoppa(X) \to B_{\Qoppa}(X,X)^{hS_2}\right) \tag{2.8}$$

  is exact, and $B_{\Qoppa}$ is bilinear.
- The functor $X \mapsto \mathrm{cofib}\,(B_{\Qoppa}(X,X)_{hS_2} \to \Qoppa(X))$ is exact, and $B_{\Qoppa}$ is bilinear.

If any of those conditions hold, $\Qoppa$ is called a *quadratic functor*, and the pair $(\mathcal{C}, \Qoppa)$ is then called a *hermitian* $\infty$*-category*. In this case, $B_{\Qoppa}$ is automatically symmetric bilinear, as it is defined as the polarization of $\Qoppa(X \oplus Y)$ which is of course symmetric, compare [CDH+20a, 1.1.9].

*Remark.* The letter $\Qoppa$ ("Qoppa", pronounced "Koppa") stems from the early greek alphabet.

**Definition 2.1.22** ([CDH+20a, 1.2.1])**.** A *hermitian functor* $F : (\mathcal{C}, \Qoppa) \to (\mathcal{D}, \Phi)$ between hermitian $\infty$-categories consists is an exact functor $F : \mathcal{C} \to \mathcal{D}$, equipped with a natural transformation $\Qoppa \Rightarrow F^*\Phi := \Phi \circ F^{op}$. Formally, we can construct an $\infty$-category $\mathcal{C}at^h_\infty$ of hermitian $\infty$-categories and hermitian functors as the (cartesian) Grothendieck construction of the functor

$$(\mathcal{C}at^{ex}_\infty)^{op} \to \widehat{\mathcal{C}at_\infty} \tag{2.9}$$

that associates to each stable $\infty$-category $\mathcal{C}$ its category of quadratic forms, defined as a full subcategory on $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}p)$. We need to work with large $\infty$-categories $\widehat{\mathcal{C}at_\infty}$ since $\mathcal{C}at^{ex}_\infty$ is large.

*Remark.* The natural transformation $\eta : \Qoppa \Rightarrow \Phi \circ F^{op}$ induces transformations $B_\Qoppa \Rightarrow B_\Phi(F-, F-)$ and hence

$$\mathrm{Map}_\mathcal{C}(C, D_\Qoppa C') \to \mathrm{Map}_\mathcal{D}(FC, D_\Phi FC')$$

natural in $C, C'$. Setting $C = D_\Qoppa C'$ and inserting $\mathrm{id}_{D_\Qoppa C'}$, we obtain $\tau_\eta : FD_\Qoppa \Rightarrow D_\Phi F^{op}$.

One can also go in the inverse direction:

**Proposition 2.1.23** ([CDH+20a, 1.1.17, 1.3.5])**.** A symmetric bilinear functor $B : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathcal{S}p$ induces two quadratic functors $\mathcal{C}^{op} \to \mathcal{S}p$:

- $\Qoppa^q_B(X) := B(X, X)_{hS_2}$ is the spectrum of *quadratic forms* on $X$,
- $\Qoppa^s_B(X) := B(X, X)^{hS_2}$ is the spectrum of *symmetric forms* on $X$.

Given any quadratic functor $\Qoppa : \mathcal{C}^{op} \to \mathcal{S}p$ with polarization $B_\Qoppa$, the natural transformations

$$\Qoppa^q_{B_\Qoppa} \Rightarrow \Qoppa \Rightarrow \Qoppa^s_{B_\Qoppa} \tag{2.10}$$

from 2.1.19 are (co-)units exhibiting these constructions as the left and right adjoint of the functor sending $\Qoppa \mapsto B_\Qoppa$.

*Remark.* Usually, there will be several quadratic functors with the same polarization $B$. The above proposition may be interpreted as saying that $\Qoppa^q_B$ and $\Qoppa^s_B$ are the left and right extremes in this set, while other quadratic functors lie between them in a way.

We will be interested in a more refined situation:

**Definition 2.1.24.** A bilinear functor $B : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathcal{S}p$ is called *right non-degenerate* if for any $Y \in \mathcal{C}$, the functor $B(-, Y) : \mathcal{C}^{op} \to \mathcal{S}p$ is representable by an object $D_B Y \in \mathcal{C}$ in the sense that

$$B(X, Y) \cong \mathrm{map}_\mathcal{C}(X, D_B Y) \tag{2.11}$$

The representing objects assemble (using the Yoneda Lemma) into an exact functor $D_B : \mathcal{C}^{op} \to \mathcal{C}$, called the *duality functor* associated to $B$. Dually, $B$ is *left non-degenerate* if $B(X, -)$ is always representable.

**Definition 2.1.25.** A symmetric bilinear functor $B : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathcal{S}p$ is called *non-degenerate* if the underlying bilinear functor is right (or equivalently left) non-degenerate. Also, a quadratic functor $\Omega$ is called non-degenerate if $B_\Omega$ is, and we denote $D_{B_\Omega}$ by $D_\Omega$.

**Definition 2.1.26.** For $B$ a symmetric bilinear functor as above, the composition

$$\begin{aligned} \operatorname{map}_{\mathcal{C}}(Y, X) = \operatorname{map}_{\mathcal{C}^{op}}(X, Y) \to \operatorname{map}_{\mathcal{C}}(D_B^{op}X, D_B Y) \cong B(D_B^{op}X, Y) \cong \\ \cong B(Y, D_B^{op}X) \cong \operatorname{map}_{\mathcal{C}}(Y, D_B D_B^{op}X) \end{aligned}$$

induces a natural transformation $\mathrm{id} \Rightarrow D_B D_B^{op}$, called the *evaluation map.* If this is a natural isomorphism, we call $D_B$ and $B$ *perfect.* Similarly, a quadratic functor $\Omega$ is perfect if $B_\Omega$ is.

**Definition 2.1.27.** A hermitian $\infty$-category $(\mathcal{C}, \Omega)$ is called *Poincaré $\infty$-category* if $\Omega$ is perfect. A hermitian functor between Poincaré $\infty$-categories $F : (\mathcal{C}, \Omega) \to (\mathcal{D}, \Phi)$ with associated natural transformation $\eta : \Omega \Rightarrow \Phi \circ F^{op}$ is called *duality-preserving* if the natural transformation $\tau_\eta : F \circ D_\Omega \Rightarrow D_\Phi \circ F^{op}$ canonically induced by $\eta$ is a natural isomorphism.

The *$\infty$-category of Poincaré $\infty$-categories* $\mathcal{C}at_\infty^p \subseteq \mathcal{C}at_\infty^h$ is the non-full subcategory spanned by Poincaré $\infty$-categories and duality-preserving functors.

*Technical Remark.* The non-fullness might seem strange at first glance, but it is a shadow of the fact that $\mathcal{C}at_\infty^h$ should actually be an $(\infty, 2)$-category, while $\mathcal{C}at_\infty^p$ gets rid of some lax information.

While we have used $\Omega$ to construct $B_\Omega$ and $D_\Omega$, one could also go the other way around:

**Definition 2.1.28.** A *stable $\infty$-category with duality* $(\mathcal{C}, D)$ is a stable $\infty$-category $\mathcal{C}$ equipped with an exact anti-autoequivalence $D : \mathcal{C}^{op} \to \mathcal{C}$ such that $\mathrm{Id}_{\mathcal{C}} \cong D \circ D^{op}$ holds, as well as higher coherences. To be precise, there is an action of $S_2$ on $\mathcal{C}at_\infty^{ex}$ sending $\mathcal{C} \mapsto \mathcal{C}^{op}$, and we require $(\mathcal{C}, D)$ to be a homotopy fixed point of this action.

**Construction 2.1.29.** Starting from a stable $\infty$-category with duality $(\mathcal{C}, D)$, one can construct an associated symmetric bilinear functor $B(-,-) := \operatorname{map}_{\mathcal{C}}(-, D(-)) : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathcal{S}p$ which is automatically perfect. Via 2.1.23 one obtains associated quadratic functors

$$\begin{aligned} \Omega_D^q &: \mathcal{C}^{op} \to \mathcal{S}p\,,\ C \mapsto \operatorname{map}_{\mathcal{C}}(C, DC)_{hS_2}\,, \\ \Omega_D^s &: \mathcal{C}^{op} \to \mathcal{S}p\,,\ C \mapsto \operatorname{map}_{\mathcal{C}}(C, DC)^{hS_2}\,. \end{aligned}$$

**Example 2.1.30** ([Nik20, 5.7])**.** For $R$ a commutative ring, equip the stable $\infty$-category $D^{\mathrm{perf}}(R)$ with the functor

$$Ϙ^s_R : D^{\mathrm{perf}}(R) \to \mathcal{S}p\,, \quad M \mapsto \mathrm{map}_{D(R)}(M \otimes^L M, R[0])^{hS_2} \tag{2.12}$$

and also define $Ϙ^q_R$ by replacing homotopy invariants with coinvariants. For $M = P[0]$ with $P$ a finitely generated projective $R$-module, $\mathrm{map}_{D(R)}(P[0] \otimes^L P[0], R[0]) \cong \mathrm{Hom}_R(P \otimes P, R)[0]$ since we are already working with a projective resolution, so that

$$\pi_0 Ϙ^s_R(M) = H_0(\mathrm{Hom}_R(P \otimes P, R)[0]^{hS^2}) = \mathrm{Hom}_R(P \otimes P, R)^{S^2} \tag{2.13}$$

is the $R$-module of symmetric bilinear forms on $M$. Similarly, using 2.1.9 we see that $\pi_0 Ϙ^q_R(M)$ agrees with the $R$-module of quadratic forms on $M$.

To show that $Ϙ^s_R$ and $Ϙ^q_R$ are quadratic functors equipping $D^{\mathrm{perf}}(R)$ with the structure of a Poincaré $\infty$-category, we calculate their polarization

$$\begin{aligned} B_{Ϙ^s_R}(M,N) \oplus Ϙ^s_R(M) \oplus Ϙ^s_R(N) &\cong Ϙ^s_R(M \oplus N) \cong \mathrm{map}_{D(R)}(M \otimes^L M, R[0])^{hS_2} \oplus \\ &\oplus \left(\mathrm{map}_{D(R)}(M \otimes^L N, R[0]) \oplus \mathrm{map}_{D(R)}(N \otimes^L M, R[0])\right)^{hS_2} \oplus \mathrm{map}_{D(R)}(N \otimes^L N, R[0])^{hS_2} \end{aligned}$$

Since $M \otimes^L N \cong N \otimes^L M$, and the non-cross terms cancel, we are left with

$$B_{Ϙ^s_R}(M,N) \cong \mathrm{map}_{D(R)}(M \otimes^L N, R[0]) \cong B_{Ϙ^q_R}(M,N) \tag{2.14}$$

after performing the same calculation in the quadratic case. In particular, we can read off $D_{Ϙ^s_R}(N) \cong \underline{\mathrm{RHom}}(N, R[0]) \cong D_{Ϙ^q_R}(N)$ where $\underline{\mathrm{RHom}}$ is the internal Hom right adjoint to $- \otimes^L N$, in other words the ordinary derived Hom functor. Clearly $D_{Ϙ^q_R}$ is exact, and it is known (compare 2.4.7) that $D_{Ϙ^s_R} \circ D_{Ϙ^s_R} \simeq \mathrm{Id}_{D^{\mathrm{perf}}(R)}$ for perfect complexes. Since $Ϙ^s_R$ and $Ϙ^q_R$ are clearly the two universal quadratic functors associated to $D_{Ϙ^s_R}$, which is a duality functor by its construction, we are finished.

*Remark.* This also works for $D^{\mathrm{fp}}(R)$ and several other subcategories of the derived category, and for non-commutative rings. For more, see the section on module spectra and apply the stable Dold-Kan correspondence.

## 2.2 L-Groups of a Poincaré $\infty$-category

Let $(\mathcal{C}, Ϙ)$ be a Poincaré $\infty$-category.

**Proposition 2.2.1.** For $n \in \mathbb{N}$, the $n$-shifted quadratic form $Ϙ^{[n]} := \Sigma^n \circ Ϙ$ makes $(\mathcal{C}, Ϙ^{[n]})$ into a Poincaré $\infty$-category as well.

*Proof.* We can calculate $B_{\mathfrak{P}^{[n]}} = \Sigma^n \circ B_{\mathfrak{P}}$ and $D_{\mathfrak{P}^{[n]}} = \Sigma^n \circ D_{\mathfrak{P}}$ from their definitions. As $\Sigma^n$ is exact, it preserves reducedness of $\mathfrak{P}$, bilinearity of $B_{\mathfrak{P}}$ and exactness of $\Lambda_{\mathfrak{P}}$, making $\mathfrak{P}^{[n]}$ quadratic. Also since $\mathbb{D}_{\mathfrak{P}}$ is exact, $\Sigma^n_{\mathcal{C}} \circ D_{\mathfrak{P}} = D_{\mathfrak{P}} \circ \Sigma^n_{\mathcal{C}^{op}} = D_{\mathfrak{P}} \circ (\Omega^n_{\mathcal{C}})^{op}$ and therefore $D_{\mathfrak{P}^{[n]}} \circ D^{op}_{\mathfrak{P}^{[n]}} \cong D_{\mathfrak{P}} \circ \Omega^n \circ \Sigma^n \circ D^{op}_{\mathfrak{P}} \circ \mathrm{Id}_{\mathcal{C}}$ so $\mathfrak{P}$ is perfect. □

**Definition 2.2.2.** A *quadratic object* $(C, q)$ in $\mathcal{C}$ is an object $C \in \mathcal{C}$ equipped with a point $q \in \Omega^\infty \mathfrak{P}(C)$. We call $q$ a *quadratic form* on $C$, and identify it with a map of spectra $\mathbb{S} \to \mathfrak{P}(C)$. Similarly, an *n-dimensional quadratic object* $(C, q)$ of $\mathcal{C}$ shall be defined as a quadratic object in $(\mathcal{C}, \mathfrak{P}^{[-n]})$.

**Definition 2.2.3.** A *quadratic object* $(C, q)$ in $(\mathcal{C}, \mathfrak{P})$ induces a point in

$$q \in \Omega^\infty \mathfrak{P}(C) \to \Omega^\infty B(C, C) \cong \Omega^\infty \operatorname{map}(C, D_{\mathfrak{P}}C) = \operatorname{Map}_{\mathcal{C}}(C, D_{\mathfrak{P}}C)\ , \tag{2.15}$$

where the arrow $\mathfrak{P}(C) \to \mathfrak{P}(C \oplus C) \to B(C, C)$ is again induced by the diagonal map. If the corresponding map $q_\sharp : C \to D_{\mathfrak{P}}C$ is an isomorphism, we call $(C, q)$ a *Poincaré object*. Similarly, we define *n-dimensional Poincaré objects*.

**Example 2.2.4.** Let $M$ be a compact topological $n$-manifold; then for $R$ a commutative ring, the singular complex $C^*(X; R) \in D^{\mathrm{perf}}(R)$ is an $n$-dimensional Poincaré object, with quadratic form induced by the Kronecker pairing making use of its fundamental class. It is difficult to show that $C^*(X; R)$ is perfect; this is equivalent to the cohomology groups of $M$ being bounded and finitely generated, which follows from the fact that as an absolute neighborhood retract $M$ is homotopy equivalent to a finite CW complex. Being a Poincaré object is then just restatement of Poincaré duality; the map

$$- \cap [M] : C^*(M; R) \to D_{\mathfrak{P}^{[-n]}} C^*(M; R) = \operatorname{RHom}(C^*(M; R), R)[-n] = C_{n-*}(M; R) \tag{2.16}$$

representing the Kronecker pairing is a quasi-isomorphism. Note that RHom is the usual internal Hom since $C^*(M; R)$ is free in each degree.

**Definition 2.2.5.** The functor $\Omega^\infty \mathfrak{P} : \mathcal{C}^{op} \to \mathcal{S}$ sends $P \in \mathcal{C}$ to the space of quadratic objects with underlying object $P$. Applying 1.2.25, this functor classifies a right fibration $\mathrm{He}(\mathcal{C}, \mathfrak{P}) \to \mathcal{C}$ with fiber over $P$ given by $\Omega^\infty \mathfrak{P}(P)$. We call the total space the *$\infty$-category of quadratic forms* in $\mathcal{C}$, and its largest subgroupoid $\mathrm{Fm}(\mathcal{C}, \mathfrak{P}) := \mathrm{He}(\mathcal{C}, \mathfrak{P})^{\simeq}$ the *space of quadratic forms* in $\mathcal{C}$.

Similarly, the full subgroupoid of this space spanned by the Poincaré objects is the *space of Poincaré objects* $\mathrm{Pn}(\mathcal{C}, \mathfrak{P})$. By construction and functoriality of the Grothendieck construction, $\mathrm{Pn}, \mathrm{Fm} : \mathcal{C}at^p_\infty \to \mathcal{S}$ are functorial with respect to duality-preserving functors.

*Remark.* In particular, it follows from the definition of $\mathrm{Pn}(\mathcal{C}, \mathfrak{P})$ that an isomorphism between two quadratic objects $(P, q)$ and $(P', q')$ is an isomorphism $f : P \to P'$, together with a path from $f^*q'$ to $q$ in $\Omega^\infty \mathfrak{P}(P)$. Here and in the following, by $f^*$ we mean the map $\mathfrak{P}(f) : \mathfrak{P}(P') \to \mathfrak{P}(P)$.

**Definition 2.2.6.** Let $(P, q)$ be a Poincaré object in $\mathcal{C}$. A *Lagrangian* of it is a morphism $f : L \to P$ in $\mathcal{C}$, together with a path $\eta : f^*q \to 0$ in $\Omega^\infty Ϙ(L)$, where $f^* := \Omega^\infty Ϙ(f)$; such that the sequence

$$L \xrightarrow{f} P \overset{q}{\cong} D_Ϙ P \xrightarrow{D_Ϙ f} D_Ϙ L \tag{2.17}$$

is a fiber sequence in $\mathcal{C}$, where we identify $P$ and $D_Ϙ P$ using the fact that $(P, q)$ is a Poincaré object. This involves in particular the vanishing of the composition of the above maps, which we require to be witnessed by the path in $\mathrm{Map}_{\mathcal{C}}(L, D_Ϙ L)$ induced by the path $\eta$ in $\Omega^\infty Ϙ(L)$. If $(P, q)$ admits a Lagrangian, it is called *metabolic*. We also extend this definition to $n$-dimensional Poincaré-objects replacing $Ϙ$ by $Ϙ^{[-n]}$.

*Remark.* We may rewrite this as requiring that $\eta$ induces an isomorphism

$$L \overset{\eta}{\cong} \mathrm{fib}(D_Ϙ P \to D_Ϙ L) \cong D_Ϙ \,\mathrm{cofib}(L \to P) \tag{2.18}$$

**Example 2.2.7.** For $M$ a compact oriented topological $n$-manifold with boundary $i : \partial M \hookrightarrow M$, the pullback

$$i^* : C^*(M; R) \to C^*(\partial M; R) \tag{2.19}$$

is a Lagrangian of the $(n-1)$-dimensional Poincaré object $C^*(\partial M; R) \in D^{\mathrm{perf}}(R)$, with $\eta$ induced by capping with the relative fundamental class. This is a restatement of Poincaré-Lefschetz duality, as we require the map

$$D_Ϙ C^*(\partial M; R)[-n] = C_{n-*}(\partial M; R) \simeq \mathrm{cofib}(i^*) = C^*(M, \partial M; R) \tag{2.20}$$

induced by $\eta$ to be a quasi-isomorphism.

**Example 2.2.8.** Equip the zero object $0 \in \mathcal{C}$ with its unique quadratic form, making it into an $n$-dimensional Poincaré object for arbitrary $n$. A terminal map $f : L \to 0$ for $L \in \mathcal{C}$ together with a path $\eta : f^*0 = 0 \to 0$ is a Lagrangian iff the sequence

$$L \longrightarrow 0 \longrightarrow D_Ϙ L[-n] \tag{2.21}$$

is a fiber sequence, in other words the loop $\eta \in \Omega\Omega^\infty Ϙ^{[-n]}(L) = \Omega^\infty Ϙ^{[-n-1]}(L) \to \mathrm{Map}_{\mathcal{C}}(L, D_Ϙ L[-n-1])$ induces an isomorphism

$$\eta : L \xrightarrow{\cong} D_Ϙ L[-n-1] \tag{2.22}$$

making $L$ into an $(n+1)$-dimensional Poincaré object. Compare this with the last example: A null-bordism of the empty $n$-manifold is the same thing as an $(n+1)$-manifold.

**Proposition 2.2.9.** The set of isomorphism classes of Poincaré objects $\pi_0 \,\mathrm{Pn}(\mathcal{C}, Ϙ)$ forms a commutative monoid under the operation

$$[(P, q)] \oplus [(P', q')] := [(P \oplus P', q \oplus q')] \,. \tag{2.23}$$

where the *orthogonal sum* $q \oplus q' \in \Omega^\infty Ϙ(P \oplus P') \cong \Omega^\infty Ϙ(P) \oplus \Omega^\infty Ϙ(P') \oplus \Omega^\infty B_Ϙ(P, P')$ corresponds to $(q, q', 0)$. The classes of metabolic Poincaré objects form a commutative submonoid $\pi_0 \,\mathrm{Pn}^\partial(\mathcal{C}, Ϙ)$.

*Proof.* This operation well-defined on isomorphism classes as it is clearly functorial, also it is associative since the direct sum is. The zero Poincaré object 0 equipped with its unique quadratic form acts as a unit since $P \oplus 0 \cong P$, and we know $\oplus$ is commutative. Finally, the metabolic objects form a submonoid since 0 is metabolic, and the orthogonal sum of two metabolic objects admits the direct sum of the respective Lagrangians as a Lagrangian. ☐

**Definition 2.2.10.** Let $(\mathcal{C}, \Qoppa)$ be a Poincaré $\infty$-category, then we define its $n$-th L-group as the quotient of commutative monoids

$$L_n(\mathcal{C}, \Qoppa) := \frac{\pi_0 \operatorname{Pn}(\mathcal{C}, \Qoppa^{[-n]})}{\pi_0 \operatorname{Pn}^{\partial}(\mathcal{C}, \Qoppa^{[-n]})}\ . \tag{2.24}$$

**Lemma 2.2.11.** For $\mathcal{C}$ a stable $\infty$-category and $C \in \mathcal{C}$, the sequence

$$C \xrightarrow{\Delta} C \oplus C \xrightarrow{(\mathrm{id}_C, -\mathrm{id}_C)} C \tag{2.25}$$

is always a fiber sequence.

*Proof.* Via matrix multiplication $(\mathrm{id}_C, \mathrm{id}_C) \circ (\mathrm{id}_C, -\mathrm{id}_C)^T = 0$, we see that this sequence composes to the zero map (witnessed by a canonical homotopy). We have to show that the left square in the diagram

$$\begin{array}{ccccc} C & \xrightarrow{\Delta} & C \oplus C & \xrightarrow{\pi_1} & C \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{id}_C, -\mathrm{id}_C)} & & \downarrow \\ 0 & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

is a pushout, which by the pasting lemma and the fact that pushout and pullback squares agree is equivalent to the right square being a pushout. Since the middle vertical map agrees with $\pi_1 - \pi_2$, by definition of addition and subtraction of morphisms we may shift $\pi_2$ to the right horizontal arrow, obtaining $\nabla = \pi_1 + \pi_2$. We reduce to showing that the lower right square in the diagram below is a pushout, which again follows by iteratively applying the pasting lemma.

$$\begin{array}{ccccc} 0 & \longrightarrow & C & & \\ \downarrow & & \downarrow{\scriptstyle i_1} & & \\ C & \xrightarrow{i_2} & C \oplus C & \xrightarrow{\nabla} & C \\ \downarrow & & \downarrow{\scriptstyle \pi_1} & & \downarrow \\ 0 & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

☐

**Proposition 2.2.12.** The commutative monoids $L_n(\mathcal{C}, \Qoppa)$ are actually abelian groups.

*Proof.* The inverse to $[(P, q)]$ is given by $[(P, -q)]$ where $-q$ arises from $\Omega^\infty \Qoppa(P)$ being an infinite loop space, in particular we can invert loops. To see this, use the Lagrangian $\Delta : P \to P \oplus P$ given by the diagonal map. In the induced sequence

$$P \xrightarrow{\Delta} P \oplus P \overset{q\oplus(-q)}{\cong} D_\Qoppa(P \oplus P) \xrightarrow{D_\Qoppa \Delta} D_\Qoppa P \ , \tag{2.26}$$

the right map $(D_\Qoppa \Delta) \circ (q \oplus (-q)) : P \oplus P \to D_\Qoppa P$ is isomorphic to $q \circ (\mathrm{id}_P, -\mathrm{id}_P)$ since the diagonal map in $\mathcal{C}^{op}$ is the codiagonal in $\mathcal{C}$, so the square

$$\begin{array}{ccc} P \oplus P & \xrightarrow{\nabla} & P \\ \big\downarrow{\scriptstyle q\oplus q} & & \big\downarrow{\scriptstyle q} \\ D_\Qoppa(P \oplus P) & \xrightarrow{D_\Qoppa \Delta} & D_\Qoppa P \end{array}$$

commutes by definition of $q \oplus q$. Therefore, we are finished after we apply the previous Lemma and the fact that $q : P \to D_\Qoppa P$ is an isomorphism. □

## 2.3 Tensor and Cotensor Poincaré Structures

**Construction 2.3.1** ([CDH+20a, 6.3.2]). Given an arbitrary $\infty$-category $\mathcal{J}$ and a hermitian $\infty$-category $(\mathcal{C}, \Qoppa)$, let us define the *cotensor hermitian $\infty$-category* $(\mathcal{C}, \Qoppa)^\mathcal{J}$ with underlying stable $\infty$-category given by $\mathrm{Fun}(\mathcal{J}, \mathcal{C})$. The corresponding quadratic functor $\Qoppa^\mathcal{J} : \mathrm{Fun}(\mathcal{J}, \mathcal{C})^{op} \to \mathcal{S}p$ is defined as

$$\Qoppa^\mathcal{J}(F) := \lim_{i \in \mathcal{J}^{op}} \Qoppa(F(i)) \tag{2.27}$$

which means that

$$B^\mathcal{J}(F, F') = \lim_{i \in \mathcal{J}^{op}} B(F(i), F'(i)) \ . \tag{2.28}$$

If $\Qoppa$ is non-degenerate, admitting a duality functor $D : \mathcal{C}^{op} \to \mathcal{C}$, then

$$D^\mathcal{J}(F)(i) := \lim_{(i \to j) \in (\mathcal{J}_{i/})^{op}} DF(j) \tag{2.29}$$

is a duality functor for $\Qoppa^\mathcal{J}$, if all involved limits exist in $\mathcal{C}$.

*Proof.* Seeing that $B^\mathcal{J}$ is the polarization of $\Qoppa^\mathcal{J}$ is straightforward since limits commute with direct sums (as those are also limits). Clearly, $\Qoppa^\mathcal{J}$ is reduced and $B^\mathcal{J}$ is bilinear. Similarly, we calculate

$$\Lambda^\mathcal{J}(F) = \mathrm{fib}(\Qoppa(F) \to B_\Qoppa(F, F)^{hS_2}) = \lim_{i \in \mathcal{J}^{op}} \Lambda_\Qoppa(F(i)) \tag{2.30}$$

which is also exact. For the duality functor, rewrite

$$\mathrm{nat}(F, D^{\mathcal{J}}(F')) \simeq \int_{i\in\mathcal{J}} \mathrm{map}\left(F(i), \lim_{(i\to j)\in(\mathcal{J}_{i/})^{op}} DF(j)\right) \simeq \int_{i\in\mathcal{J}} \lim_{(i\to j)\in(\mathcal{J}_{i/})^{op}} B_{\Qoppa}(F(i), F'(j))$$

It remains to show that this limit diagram (regarding the end as a limit) just calculates the limit over $\mathcal{J}^{op}$. As explained in the reference, one can rewrite it as a limit over the double twisted arrow category $\mathrm{TW}(\mathrm{TW}(\mathcal{C}))$, and using Quillens Theorem A 1.2.15 we could show that:

- The projection $\mathrm{TW}(\mathrm{TW}(\mathcal{J})) \to \mathrm{Fun}(\Delta^1, \mathcal{J})$ to the ordinary arrow category, induced by sending an object

$$\begin{array}{ccc} i & \longrightarrow & l \\ \downarrow & & \uparrow \\ j & \longrightarrow & k \end{array}$$

  to the diagonal $i \to k$ which is covariant in $i$ and $k$, is left cofinal.

- The diagonal $\mathcal{J} \to \mathrm{Fun}(\Delta^1, \mathcal{J})$ sending an object to its identity morphism is left cofinal.

From the explicit expression for $D^{\mathcal{J}}$, and the fact that limits in a functor category are calculated pointwise, it is clear that the duality functor is exact, so we are finished. □

**Example 2.3.2.** Even if $\mathcal{J}$ is finite and $(\mathcal{C}, \Qoppa)$ is a Poincaré $\infty$-category, the cotensor $(\mathcal{C}, \Qoppa)^{\mathcal{J}}$ need not be Poincaré. As an example, set $\mathcal{J} = (* \to * \leftarrow *)$ and let $F : \mathcal{J} \to \mathcal{C}$ be given by the diagram $(A \to C \leftarrow B)$. Then,

$$D^{\mathcal{J}}F = (\lim(A \to C) \to \lim(F) \leftarrow \lim(B \to C)) = (C \overset{\mathrm{id}_C}{\to} C \overset{\mathrm{id}_C}{\leftarrow} C) \tag{2.31}$$

since all involved limit diagrams contain $C$ as a final object, and $(D^{\mathcal{J}})^2F \simeq (C \overset{\mathrm{id}_C}{\to} C \overset{\mathrm{id}_C}{\leftarrow} C)$ for the same reason. This is clearly not isomorphic to $F$.

**Example 2.3.3.** If $\mathcal{J}$ is the ordinary category $(* \leftarrow * \to *)$ and $(\mathcal{C}, \Qoppa)$ is Poincaré, then $\mathrm{Fun}(\mathcal{J}, \mathcal{C})$ consists of spans

$$\begin{array}{ccccc} & & L & & \\ & \swarrow^{f} & & \searrow^{f'} & \\ P & & & & P' \end{array}$$

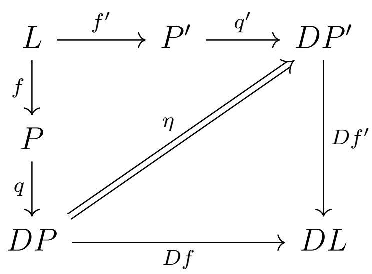
and the dual is $D^{\mathcal{J}}(P \leftarrow L \to P') = (DP \leftarrow DP \times_{DL} DP' \to DP')$. In particular,

$$\begin{aligned}(D^{\mathcal{J}})^2(P \leftarrow L \to P') &= (D^2P \leftarrow D^2P \times_{D(DP\times_{DL}DP')} D^2P' \to D^2P') \cong \\ &\cong (P \leftarrow P \times_{P\amalg_L P'} P' \to P') \cong (P \leftarrow L \to P')\end{aligned}$$

since pushout squares and pullback squares agree, so $(\mathcal{C}, \Omega)^{\partial}$ is again Poincaré. A span as above is a Poincaré object iff it is equipped with a quadratic form $q \in \Omega^\infty(\Omega(P) \times_{\Omega(L)} \Omega(P')) = \Omega^\infty \Omega(P) \times_{\Omega(L)} \Omega(P')$ that induces compatible isomorphisms

$$P \cong DP,\ P' \cong DP',\ L \cong DP \times_{DL} DP'\ . \tag{2.32}$$

More explicitly, a Poincaré object of this cotensor consists of two Poincaré objects $(P, q), (P', q')$ in $\mathcal{C}$ and a span $P \xleftarrow{f} L \xrightarrow{f'} P'$ together with a path $\eta : f^*q \to f'^*q'$ in $\Omega^\infty \Omega(L)$ such that the induced map $n : L \to DP \times_{DL} DP' \cong D(P \amalg_L P')$ from the square below is an isomorphism, where $\eta$ witnesses commutativity.




Equivalently, we could have required the map

$$\begin{aligned} \mathrm{fib}(L \to P) \cong \mathrm{fib}(P' \to P \amalg_L P') &\xrightarrow{Dn \circ -} \mathrm{fib}(P' \to DL) \cong \\ \cong \mathrm{fib}(DP' \to DL) \cong D\,\mathrm{cofib}(L \to P') &\cong (D\,\mathrm{fib}(L \to P'))[-1] \end{aligned} \tag{2.33}$$

to be an isomorphism.

**Definition 2.3.4.** A Poincaré object $(P \leftarrow L \to P')$ as in the example above is called a *Lagrangian correspondence* or *algebraic bordism* between $P$ and $P'$.

**Lemma 2.3.5.** A diagram $(P \leftarrow L \to P')$ together with a path $\eta : f^*q \to f'^*q'$ as above is a Lagrangian correspondence iff the map $-f \oplus f' : L \to P \oplus P'$ equipped with an induced path $\eta' : (-f \oplus f')^*(q \oplus q') \to 0$ is a Lagrangian.

*Proof.* Rewrite the defining property of a Lagrangian correspondence as

$$L \cong DP \times_{DL} DP' = (DP \oplus DP') \times_{DL} (DP \oplus DP') = \mathrm{fib}(f \oplus f' : DP \oplus DP' \to DL)$$

where we switch the side of $DP$ in the pullback, inducing a minus sign. Identify $\eta$ with a path $-f^*q + f'^*q' \to 0$, and note that the left agrees with $(-f \oplus f')^*(q \oplus q')$ by definition of the orthogonal sum $q \oplus q'$. Going through the above calculation, the induced path $\eta'$ exhibits $-f \oplus f$ as isotropic. $\square$

**Example 2.3.6.**

- By this Lemma, a Lagrangian correspondence $(0 \leftarrow L \to P)$ is a Lagrangian of $P$.

- A Lagrangian correspondence from the zero Poincaré object to itself is a Lagrangian of 0, i.e. a 1-dimensional Poincaré object by 2.2.8.
- Let $W$ be a compact oriented topological $(n+1)$-manifold with boundary $\partial W = -M \sqcup N$ where $M, N$ are closed oriented topological $n$-manifolds, in other words $W$ is a bordism from $M$ to $N$. Then, for $R$ a commutative ring, the restrictions

$$C^*(M;R) \leftarrow C^*(W;R) \rightarrow C^*(N;R) \tag{2.34}$$

of singular cochain complexes form a Lagrangian correspondence between $n$-dimensional Poincaré objects in $D^{\mathrm{perf}}(R)$ by virtue of Poincaré-Lefschetz duality: Capping with the relative fundamental class induces a quasi-isomorphism

$$C^*(W;R)/C^*(M;R) \simeq D\left(C^*(W;R)/C^*(N;R)\right)[1-n] \tag{2.35}$$

where the right agrees with the relative singular chain complex $C^{n+1-*}(W,N;R)$.

*Remark.* If $\mathcal{J}$ is the poset $\mathrm{sd}(\Delta^n)^{op} = (\{S \subseteq \{1,\dots,n\} | S \neq \emptyset\}, \supseteq)$ of simplices in $\Delta^n$ ordered by containment (not inclusion), we will see in a moment that $(\mathcal{C}, \Omega)^{\mathcal{J}}$ is also Poincaré, retaining the above example as a special case for $n=1$. We can regard the Poincaré objects in $(\mathcal{C}, \Omega)^{\mathcal{J}}$ as higher algebraic bordisms in $\mathcal{C}$, i.e. bordisms between bordisms and so on. This will be exploited in the $\rho$-construction 2.5.2 to define the L-spectrum.

**Construction 2.3.7** ([CDH+20a, 6.5.8]). Let $\mathcal{J}$ be a strongly finite $\infty$-category, i.e. with a finite set of objects and finite mapping spaces, and $(\mathcal{C}, \Omega)$ a Poincaré $\infty$-category. We then dually define the *tensor hermitian $\infty$-category* $(\mathcal{C}, \Omega)_{\mathcal{J}}$ by equipping $\mathrm{Fun}(\mathcal{J}^{op}, \mathcal{C})$ with the quadratic functor

$$\Omega_{\mathcal{J}}(S) := \operatorname*{colim}_{i \in \mathcal{J}} \Omega(S(i)) \tag{2.36}$$

inducing the bilinear functor

$$B_{\mathcal{J}}(S, S') \simeq \operatorname*{colim}_{i \in \mathcal{J}} B(S(i), S'(i)) \tag{2.37}$$

for $S, S' : \mathcal{J}^{op} \to \mathcal{C}$. If $\Omega$ is non-degenerate with duality functor $D : \mathcal{C}^{op} \to \mathcal{C}$, then

$$D_{\mathcal{J}}S(i) := \operatorname*{colim}_{j \in \mathcal{J}} D(S(j))^{\mathrm{Map}(j,i)} \tag{2.38}$$

is a duality functor for $\Omega_{\mathcal{J}}$, where the involved finite colimits always exist in a stable $\infty$-category. We have also used the cotensoring of any stable $\infty$-category over finite spaces, defined as $C^{\mathrm{Map}(j,i)} := \lim_{f \in \mathrm{Map}(j,i)} \underline{C}$ over the constant functor. This formula becomes particularly simple if $\mathcal{J}$ is finite poset, since the mapping spaces are either empty or contractible in this case:

$$D_{\mathcal{J}}S(i) = \operatorname*{colim}_{j \in \mathcal{J}} \begin{cases} D(S(j)) \text{ if } j \leq i \\ 0 \quad \text{otherwise} \end{cases} \tag{2.39}$$

*Proof.* Analogous to the tensor hermitian ∞-category, once we check that $D_{\mathcal{J}}$ is the correct duality functor. We calculate

$$\mathrm{nat}\,(S, D_{\mathcal{J}}S') \cong \int_{i\in\mathcal{J}} \mathrm{map}\left(S(i), \operatorname*{colim}_{j\in\mathcal{J}} DS(j)^{\mathrm{Map}(j,i)}\right)$$

The functor $\mathrm{map}(S(i), -)$ preserves limits, hence also finite colimits by 1.5.6; similarly for the end. We therefore may pull the colimit out, obtaining

$$\operatorname*{colim}_{j\in\mathcal{J}} \int_{i\in\mathcal{J}} \mathrm{map}(S(i), DS(j)^{\mathrm{Map}(j,i)}) \cong \operatorname*{colim}_{j\in\mathcal{J}} \lim_{[i\to i']\in\mathrm{TW}(\mathcal{J})^{op}} \mathrm{map}(S(i), DS(j))^{\mathrm{Map}(j,i)}$$

by universal property of the cotensoring. But $(-)^{\mathrm{Map}(j,i)} = \lim_{f\in\mathrm{Map}(j,i)}(-)$, and we can combine both limits into a limit over $\mathrm{TW}(\mathcal{J}_{j/})^{op}$. The functor $\mathrm{TW}(\mathcal{J}_{j/})^{op} \to \mathcal{J}_{j/}$ sending a morphism to its target is right cofinal, and $\mathcal{J}_{j/}$ has an initial object $\mathrm{id}_j$, so we are finished. □

*Remark.* This can be extended to ∞-categories $\mathcal{J}$ that are not strongly finite as done in [CDH⁺20a, 6.4.1], which is however fairly complicated and we will not need it.

**Proposition 2.3.8** ([Lur11, Lecture 19, Proposition 3])**.** Let $\mathcal{J}$ be the poset $\mathcal{I}_K^{op}$ of faces of a finite simplicial complex as defined in 4.1.1, ordered by containment (not inclusion), and $(\mathcal{C}, Ϙ)$ a Poincaré ∞-category. Then, the cotensor hermitian ∞-category $(\mathcal{C}, Ϙ)^{\mathcal{J}}$ and the tensor hermitian ∞-category $(\mathcal{C}, Ϙ)_{\mathcal{J}}$ are both Poincaré; we then call them *(co)tensor Poincaré ∞-categories.*

We will give a proof in the tensor case in 4.1.19, the cotensor case follows formally as explained in the reference.

*Remark.* In this case, as indicated in the examples above, the cotensor hermitian ∞-category describes data on simplicial complexes satisfying a Poincaré-Lefschetz-type duality the boundary of each simplex, in a compatible way. In particular, the value $D^{\mathcal{J}}F(\sigma)$ on any simplex $\sigma$ is obtained by "dividing out" the values of $F$ at its boundary faces. Dually, in the tensor hermitian ∞-category, $D_{\mathcal{J}}S(\sigma)$ depends on the simplices that $\sigma$ is a face of, it is related to the relative homology $H^*(|K|, |K| - |\sigma|)$. Therefore, it can be used to model Verdier duality on simplicial complexes and PL spaces, see 4.1.18.

## 2.4 L-Groups of a Ring Spectrum

For this section, fix a commutative ring spectrum $k \in \mathrm{CAlg}(\mathcal{S}p)$ that we use as a ground ring, and a ring spectrum $R \in \mathrm{Alg}(\mathrm{Mod}_k)$ that is a $k$-algebra. For $k = \mathbb{S}$ this just means that $R$ is a ring spectrum, but e.g. for $k = H\mathbb{Z}$ this allows us to make statements about differential graded algebras using the stable Dold-Kan correspondence 1.7.2.

**Definition 2.4.1.** The $\infty$-category $\mathrm{LMod}_{R\otimes_k R}$ of spectra with two compatible left $R$-module structures admits an $S_2$-action exchanging the two factors of $R$. An *$R$-module with involution* is a homotopy fixed point of this action; in other words an $(R \otimes_k R)$-module $M$ together with an isomorphism of spectra $\sigma : M \to M$ that is linear over the exchange isomorphism $\tau : R \otimes_k R \to R \otimes_k R$ in the sense that $\sigma : M \cong \tau_* M$ is an isomorphism of $R$-modules. Further, $\sigma^2 \cong \mathrm{id}_M$ together with higher coherence relations on this isomorphism.

**Example 2.4.2.** If $R$ is an ordinary ring and $M$ an $(R \otimes^L_{\mathbb{Z}} R)$-module, equipped with an isomorphism $\sigma : M \to \tau_* M$ interchanging the two $R$-module structures, then $M[0] \in D(R \otimes^L_{\mathbb{Z}} R) \simeq \mathrm{LMod}_{HR\otimes_{H\mathbb{Z}} HR}$ becomes a $HR$-module with involution, where higher coherence relations are trivial.

**Definition 2.4.3.** An $R$-module $M$ with involution is called an *invertible $R$-module* if

- It is perfect with respect to either of the $A$-module structures (applying the involution, it is then automatically perfect with respect to the other).
- If we equip $M$ with the first $R$-module structure, the second can be rewritten as an action of $R$ on $M$, i.e. a morphism $R \to \underline{\mathrm{Hom}}_R(M, M)$. We require this to be an isomorphism. Equivalently, applying the involution, we could have exchanged the roles of both module structures.

**Example 2.4.4.** If $R$ is a commutative ring spectrum, we may regard it as an $(R\otimes_k R)$-module over itself by restricting the scalars of its canonical $R$-left-module structure along the multiplication map $R\otimes_k R \to R$, in other words we define the module multiplication as $(r_1 \otimes r_2) \cdot r := r_1 \cdot r_2 \cdot r$. Since the multiplication is symmetric in its arguments, this canonically makes $R$ into a homotopy invariant with respect to the $S_2$ action exchanging the factors of $R$, so it becomes a module with involution. In fact, the map $R \to \underline{\mathrm{Hom}}_R(R, R)$ is the identity be definition, making $R$ into an invertible $R$-module.

In nature, invertible modules often arise via the following construction:

**Definition 2.4.5.** The category $\mathrm{Alg}(\mathrm{Mod}_k)$ possesses an $S_2$-action sending a $k$-algebra $A$ to its opposite, in the sense of 1.6.14. A *$k$-algebra with anti-involution* is a homotopy fixed point of this action, i.e. a $k$-algebra $A$ together with an isomorphism $\tau : A \to A^{op}$ satisfying higher coherence conditions.

**Proposition 2.4.6** ([CDH+20a, 3.1.9])**.** If $(A, \tau)$ is a $k$-algebra with anti-involution, then it comes naturally equipped with the structure of a $(A \otimes_k A)$-module by applying $\tau$ to the second component, so the module action looks like

$$(A \otimes_k A) \otimes_k A \to A\,, \quad (a \otimes a') \cdot b := a \cdot b \cdot \tau(a')\,. \tag{2.40}$$

This naturally makes $A$ into an invertible $A$-module.

*Proof.* Since $\tau$ is a morphism of $k$-algebras this does define a module action, restricting coefficients of the canonical $A$-$A^{op}$-bimodule structure on $A$. Also, $A$ is by definition a finitely presented $A$-module, so it suffices to check that the map $A \to \underline{\mathrm{Hom}}_A(A, A) \cong A^{op}$ is an isomorphism. The *op* appears since the $A$-module structure on the right side is induced by the second module structure of $A$, which was multiplication precomposed with $\tau$. Hence, the map agrees with $\tau$, which is an isomorphism by definition. □

Now, let us use invertible modules to define Poincaré structures on $R$-modules.

**Proposition 2.4.7** ([Lur17, 7.2.4.4])**.** For $P \in \mathrm{LMod}^{\mathrm{perf}}(R)$ a perfect $R$-module, the canonical biduality morphism

$$P \to \underline{\mathrm{Hom}}_R(\underline{\mathrm{Hom}}_R(P, R), R) \tag{2.41}$$

is an isomorphism.

*Remark.* This generalizes the statement that a perfect complex $P$ over an ordinary ring $R$ is quasi-isomorphic to its bidual $\mathrm{RHom}(\mathrm{RHom}(P, R), R)$, by the stable Dold-Kan equivalence 1.7.2. In particular, any finite-dimensional vector space is isomorphic to its bidual space.

**Proposition 2.4.8.** Given an invertible $R$-module $M$, we can define the functors

$$\Omega^s_M : \mathrm{LMod}_R^{op} \to \mathcal{S}p \ , \quad P \mapsto \mathrm{map}_{R \otimes_k R}(P \otimes_R P, M)^{hS_2} \tag{2.42}$$

and $\Omega^q_M$ involving coinvariants on the stable $\infty$-category $\mathrm{LMod}_R^{\mathrm{perf}}$ of $R$-module spectra. The associated bilinear functor is in both cases given by

$$B_M(P, P') = \mathrm{map}_{R \otimes_k R}(P \otimes_R P', M) \tag{2.43}$$

so we can identify the duality functor as

$$D_M(P) = \underline{\mathrm{Hom}}_R(P, M) \in \mathrm{LMod}_R \tag{2.44}$$

where we may equip $M$ with either of its $R$-module structures. The pairs $(\mathrm{LMod}_R^{\mathrm{perf}}, \Omega^s_M)$ and $(\mathrm{LMod}_R^{\mathrm{perf}}, \Omega^q_M)$ are Poincaré $\infty$-categories, and if $M$ is finitely presented with respect to either of its module structures the same holds in the finitely presented case.

*Proof.* This is a generalization of, and works precisely like the example of derived categories 2.1.30. We have to check that for $P$ a perfect $R$-module, $\underline{\mathrm{Hom}}_R(P, M)$ is perfect as well, and similarly if $P, M$ are finitely presented. Also, we need to show biduality:

$$D_M D_M(P) = \underline{\mathrm{Hom}}_R(\underline{\mathrm{Hom}}_R(P, M), M) \tag{2.45}$$

- If $P = R$, by assumption $D_M(R) = \underline{\mathrm{Hom}}_R(R, M) \cong M$ is perfect/ finitely presented. Also, $D_M D_M(R) \cong D_M(M) = \underline{\mathrm{Hom}}_R(M, M) \cong R$ as $M$ is invertible.

- Any finitely presented $R$-module $P$ is generated by $R$ under cofibers and shifts, and both of the above properties are preserved under these operations.
- If $P$ is a direct summand of $P' = P \oplus Q$ and $D_M(P') = \underline{\mathrm{Hom}}_R(P', M) = D_M(P) \oplus D_M(Q)$ is perfect, then its direct summand $D_M(P)$ is also perfect since perfect modules are by definition closed under direct summands.
- Similarly if $P' = P \oplus Q$ satisfies biduality, then $D_M D_M(P)$ is the direct summand of $D_M D_M(P') \cong P'$ on homomorphisms that send $\underline{\mathrm{Hom}}_R(Q, M)$ to zero, which itself is the summand of homomorphisms in $D_M(P')$ sending $P$ to zero. By construction of the evaluation map $P' \to D_M D_M(P')$ we can hence identify it with $P$. $\square$

**Definition 2.4.9.** For $R$ an associative ring spectrum and $M$ an invertible $R$-module, we define the *projective quadratic* and *projective symmetric L-groups*

$$L_n^{pq}(R, M) := L_n(\mathrm{LMod}_R^{\mathrm{perf}}, Ϙ_M^q)\,, \quad L_n^{ps}(R, M) := L_n(\mathrm{LMod}_R^{\mathrm{perf}}, Ϙ_M^s)\,. \tag{2.46}$$

Similarly, if $M$ is additionally finitely presented with respect to either of its $R$-module structures, we define the *quadratic and symmetric L-groups* $L_n^q(R, M)$ and $L_n^s(R, M)$ by replacing perfect by finitely presented $R$-modules.

*Remark.* If $M = R$ with both $R$-module structures induced by multiplication in $R$, we simply denote the respective L-groups by $L_n^q(R), L_n^s(R)$ etc., without specifying $M$.

*Remark.* By their construction, the stable Dold-Kan correspondence 1.7.2 identifies the quadratic functors $Ϙ_R^q$ and $Ϙ_R^s$ with those on derived categories from 2.1.30.

**Theorem 2.4.10** (Ranicki periodicity, [CDH+20a, 3.5.14.(i), 3.5.16])**.**
Let $k_0$ be an ordinary commutative ring and $k = Hk_0$. Then, the (projective) symmetric and quadratic L-groups of any ring spectrum $R$ over $k$ with respect to an invertible $R$-module $M$ are always 4-periodic, in the sense that

$$L_{n+4}^q(R, M) \cong L_n^q(R, M) \tag{2.47}$$

and similarly in the other cases. If we further denote by $-M$ the invertible $R$-module obtained by replacing the involution $\sigma : M \cong \tau_* M$ by $-\sigma$, then in all cases

$$L_{n+2}^q(R, M) \cong L_n^q(R, -M)\,. \tag{2.48}$$

*Proof Sketch.* Let $(P, q)$ be an $n$-dimensional Poincaré object representing a class of $L_n^s(R, M)$, the other cases are analogous. This means that the quadratic form

$$q \in \Sigma^{-n} \operatorname{map}(P \otimes_R P, M)^{hS_2} \tag{2.49}$$

induces an isomorphism $P \cong \Sigma^{-n}\underline{\mathrm{Hom}}_R(P, M)$. But then $\Sigma P \cong \Sigma^{-n-1}\underline{\mathrm{Hom}}_R(P, M)$, so it appears that $q$ makes $\Sigma P$ into an $(n + 2)$-dimensional Poincaré object. This is not completely true:

Since $M, P$ are modules over the $k$-algebra $R$, they themselves are modules over $k = Hk_0$ as well, so by the stable Dold-Kan correspondence 1.7.2 we may represent them by (finitely presented) chain complexes $M, P \in D(k_0)$. Here, $q$ corresponds to a symmetric bilinear form $b : P \otimes P \to M$ with respect to the involution $\sigma$ of $M$, however by the Koszul sign rule $\Sigma b : \Sigma P \otimes \Sigma P \to \Sigma^2 M$ is antisymmetric. This can be remedied by changing the sign of $\sigma$, because it is involved in the $S_2$-action making sense of which forms are symmetric, and which antisymmetric.

The functor $\Sigma$ thus associates to an $n$-dimensional Poincaré object with respect to $M$ and $(n+2)$-dimensional Poincaré object with respect to $-M$, and is clearly invertible by applying $\Omega$. We obtain the second claimed isomorphism of L-groups; and applying it twice yields the first isomorphism as $-(-M) = M$. □

**Example 2.4.11** ([Lur11, Lectures 13, 15, 16])**.** If we regard the integers $\mathbb{Z}$ as a commutative ring spectrum $H\mathbb{Z}$ with trivial involution, its L-groups are given by

$$L_n^q(\mathbb{Z}) = \begin{cases} 8\mathbb{Z}, & \text{for } n = 0 \operatorname{mod} 4 \\ 0, & \text{for } n = 1 \operatorname{mod} 4 \\ \mathbb{Z}/2\mathbb{Z}, & \text{for } n = 2 \operatorname{mod} 4 \\ 0, & \text{for } n = 3 \operatorname{mod} 4 \end{cases}, \quad L_n^s(\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{for } n = 0 \bmod 4 \\ \mathbb{Z}/2\mathbb{Z}, & \text{for } n = 1 \operatorname{mod} 4 \\ 0, & \text{for } n = 2 \operatorname{mod} 4 \\ 0, & \text{for } n = 3 \operatorname{mod} 4 \end{cases}$$

where the component in degree 0 corresponds to the *signature* of a quadratic form, while the $\mathbb{Z}/2\mathbb{Z}$ component is called the *Arf-Kervaire* and *deRham invariant* respectively.

For $k$ a field of characteristic $\neq 2$, the L-groups $L_n^q(k) = L_n^s(k)$ vanish unless $n = 0$ modulo 4, where they agree with the classical *Witt group* of quadratic spaces $W(k)$. If $k$ is algebraically closed the $W(k) \cong \mathbb{Z}/2\mathbb{Z}$, while for any real-closed field (in particular the real numbers) $W(k) \cong \mathbb{Z}$ via the signature. The Witt group $W(\mathbb{Q})$ is however infinite and complicated. See the referenced lectures for proofs of our statements, or [CDH$^+$21].

*Warning.* If $R$ is an ordinary ring, there are several non-equivalent ways to extend the notion of quadratic and symmetric forms to $\mathrm{LMod}_{HR}$. Recall from 2.1.9 that the space of quadratic forms on a finitely generated projective $R$-module $P$ is given by

$$\mathrm{Quad}_R(P) = \mathrm{Hom}_R(P \otimes P, R)_{S_2} \tag{2.50}$$

while the space of symmetric forms $\mathrm{Symm}_R(M)$ is obtained by replacing coinvariants with invariants. Extending to $D(R) \simeq \mathrm{LMod}_{HR}$ seems straightforward: We need to derive the functors $\mathrm{Quad}_R$ and $\mathrm{Symm}_R$. However, they are not additive!

Still, this can be done using the concept of a *non-abelian derived category* or *animation*, where we resolve by simplicial objects instead of chain complexes. The derived functor $\Qoppa^{gq} : \mathrm{LMod}_{HR} \to \mathcal{S}p$ of $\mathrm{Quad}_R$ does *not* agree with $\Qoppa^q$; similarly for symmetric forms $\Qoppa^{gs}$ does not agree with $\Qoppa^s$. Their associated L-groups are non-periodic and called *genuine quadratic* and *genuine symmetric L-groups.* We will not need them, but want the reader to be aware that L-groups depend not only on $D_\Qoppa$, but also $\Qoppa$ itself in a subtle way.

## 2.5 $\rho$-construction and L-spectra

We give an alternative, more geometric definition for the L-groups of a Poincaré $\infty$-category $(\mathcal{C}, \mathfrak{P})$ that mimics the classical definition of a *Quinn-spectrum* or an *ad-theory*.

As a motivation, we remind the reader that the singular chain complex $C^*(M;\mathbb{Z})$ of a closed oriented $n$-manifold $M^n$ equipped with the Kronecker pairing determines a class in $L^s_n(\mathbb{Z})$, and for a bordant manifold $N^n$, the associated class agrees. In fact, one should regard algebraic L-theory as an algebraic analogue of bordism theory, and many geometric constructions and results should have analogues in the algebraic world. Particularly interesting is the following example:

**Theorem 2.5.1.** Let $(P,q),(P',q')$ and $(P'',q'')$ be $n$-dimensional Poincaré objects and $P \xleftarrow{f} L \xrightarrow{f'} P'$ as well as $P' \xleftarrow{g} L' \xrightarrow{g'} P''$ Lagrangian correspondences with associated paths $\eta : f^*q \simeq f'^*q'$ and $\eta' : g^*q' \to g^*q''$. Then, the span $P \leftarrow L \times_{P'} L' \to P''$ induced by the diagram

$$
\begin{array}{ccccccc}
 & & & L \times_{P'} L' & & & \\
 & & \swarrow^{\pi_1} & & \searrow^{\pi_2} & & \\
 & L & & & & L' & \\
 {}^{f}\swarrow & & \searrow^{f'} & & {}^{g}\swarrow & & \searrow^{g'} \\
P & & & P' & & & P''
\end{array}
$$

is also a Lagrangian correspondence, with associated path $\pi_2^*\eta' \circ \pi_1^*\eta$.

*Proof.* We know that $P \cong P^\vee[-n]$ induced by $q$, and similarly for $P'$ and $P''$. Also, $L, L'$ being Lagrangian correspondences amounts to isomorphisms

$$
\begin{aligned}
L &\cong P^\vee[-n] \times_{L^\vee[-n]} P'^\vee[-n] \, , \\
L' &\cong P'^\vee[-n] \times_{L'^\vee[-n]} P''^\vee[-n]
\end{aligned}
$$

induced by $\eta, \eta'$. Consequently we can use the Pasting Lemma to dualize and extend the above commutative diagram to

$$
\begin{array}{ccccc}
(L \times_{P'} L')^\vee[-n] & \longleftarrow & L^\vee[-n] & \longleftarrow & P^\vee[-n] \\
\uparrow & & \uparrow & & \uparrow \\
L'^\vee[-n] & \longleftarrow & P'^\vee[-n] & \longleftarrow & L \\
\uparrow & & \uparrow & & \uparrow \\
P''^\vee[-n] & \longleftarrow & L' & \longleftarrow & L \times_{P'^\vee[-n]} L'
\end{array}
$$

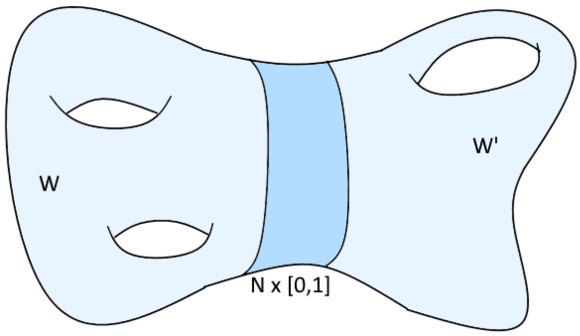
where every square is a pullback (and by stability also a pushout). This implies, using $P' \cong P'^{\vee}[-n]$ and pasting, that $L \times_{P'} L' \cong P^{\vee}[-n] \times_{(L \times_{P'} L')^{\vee}[-n]} P''^{\vee}[-n]$. Compatibility of the required witnessing 2-morphisms can be verified by chasing them around the diagram. □

*Remark.* Compare this with the observation that a bordism $W$ between closed oriented manifolds $M$ and $M'$, and a bordism $W'$ from $M'$ to $M''$, can be glued along a collar to a bordism from $M$ to $M''$. In particular, if $(P, q)$ is an $n$-dimensional Poincaré object and $f : L \to P$, $f' : L' \to P$ equipped with $\eta, \eta'$ are two distinct Lagrangians, then the pullback $L \times_P L'$ comes equipped with the structure of an $(n+1)$-dimensional Poincaré object, with quadratic form $\pi_1^* \eta \circ \pi_2^* \eta' : 0 \to \pi_1^* f^* q = \pi_2^* f'^* q' \to 0$ being a path from 0 to 0 in $\Omega^\infty \Omega^{[-n]}(L \times_P L')$, i.e. a point in the loop space

$$\Omega^1 \Omega^\infty \Omega^{[-n]}(L \times_P L') = \Omega^\infty \Omega^{[-n-1]}(L \times_P L') \ .$$

This is analogous to how two null-bordisms of an $n$-manifold can be glued to an $(n+1)$-manifold.

Figure 2.1: Gluing two null-bordisms $W, W'$ of $N$ to a closed manifold




This tells us that classes in $L_{n+1}$, being $(n+1)$-dimensional Poincaré objects, can be constructed from Poincaré objects of dimension $n$ by gluing two Lagrangians, just as for example the circle $S^1$ can be constructed from $S^0$ by gluing two intervals (i.e. nullbordisms). Iteratively, we could also glue $S^2$ from the obtained $S^1$ by gluing two disks that we regard null-bordisms, and so on. Similarly, Poincaré objects of higher degree can be obtained from ordinary Poincaré objects by gluing of Lagrangians and “higher Lagrangians". The underlying combinatorics are captured by the following construction:

**Construction 2.5.2** ($\rho$-construction)**.** For $n \in \mathbb{N}_0$ and $[n] = \{0, \dots, n\}$, let $\mathrm{sd}([n])^{op}$ be the power set of $[n]$ equipped with the opposite ordering to the one given by inclusion. The cotensor hermitian $\infty$-categories

$$\rho_n(\mathcal{C}, \Omega) := (\mathcal{C}, \Omega)^{\mathrm{sd}([n])^{op}} = (\mathrm{Fun}(\mathrm{sd}([n])^{op}, \mathcal{C}), \Omega^{\mathrm{sd}([n])^{op}}) \qquad (2.51)$$

are Poincaré $\infty$-categories by 2.3.8 and are functorial in the sense that they naturally fit into a simplicial object $\rho_\bullet(\mathcal{C}, Ϙ) \in \mathrm{Fun}(\Delta^{op}, \mathrm{Cat}_\infty^p)$, as can be deduced from [CDH+20a, 6.6.1, 6.6.2]. Composing this with any functor $\mathcal{F} : \mathrm{Cat}_\infty^p \to \mathcal{S}$ consequently yields a simplicial space $\mathcal{F}\rho_\bullet(\mathcal{C}, Ϙ) : \Delta^{op} \to \mathcal{S}$.

**Definition 2.5.3.** A *simplicial space* is a functor $X : \Delta^{op} \to \mathcal{S}$, where we denote $X_n := X([n])$. The *geometric realization* $|X| \in \mathcal{S}$ is defined as the quotient topological space

$$|X| := \bigsqcup_n X_n \times |\Delta^n| \Big/ \sim \tag{2.52}$$

by the relations introduced by the face maps in $X$, which we regard as a Kan complex. This is an analogue of the geometric realization of a simplicial set, since it is formally obtained by applying the nerve-realization paradigm to the functor $r_{\text{space}} : \Delta \to \mathcal{S}$ sending $[n]$ to a Quillen-fibrant replacement of $\Delta^n$ like $\mathrm{Sing}\,|\Delta^n|$:

$$|X| := \mathrm{Lan}_h\, r_{\text{space}}(X) = \operatorname*{colim}_{h([n]) \to X} r_{\text{space}}([n]) = \int^{n \in \Delta} X_n \times \mathrm{Sing}\,|\Delta^n| \tag{2.53}$$

The colimit or coend introduces the mentioned gluing.

**Definition 2.5.4.** For $(\mathcal{C}, Ϙ)$ a Poincaré $\infty$-category, its *L-theory space* $\mathcal{L}(\mathcal{C}, Ϙ)$ is defined as the geometric realization of the simplicial space determined by Poincaré objects in the $\rho$-construction:

$$\mathcal{L}(\mathcal{C}, Ϙ) := |\,\mathrm{Pn}\,\rho(\mathcal{C}, Ϙ)| \tag{2.54}$$

This yields a functor $\mathcal{L} : \mathcal{C}at_\infty^p \to \mathcal{S}$ since the geometric realization, Pn and the $\rho$-construction are functorial.

**Example 2.5.5.** For $X$ a simplicial space, let us calculate $\pi_0|X|$. This only depends on the 1-skeleton, and can be identified with the set of vertices divided by the relation identifying vertices connected by an edge in $|X|$. But since $|X|$ is a quotient of $\bigsqcup_n X_n \times |\Delta^n|$, its edges either come from edges of $X_0$, or vertices of $X_1$. This means that $\pi_0|X|$ agrees with the quotient of $\pi_0 X$ by the relation generated by $[x] \simeq [y]$ for $x, y \in X_0$ if there is a $z \in X_1$ such that $x = X(d_0)(z)$ and $y = X(d_1)(z)$.

In particular, this means that $\mathcal{L}_0(\mathcal{C}, Ϙ)$ is the quotient of $\pi_0\,\mathrm{Pn}(\mathcal{C}, Ϙ)$ by the relation generated by Lagrangian correspondences, agreeing with our definition of $L_0(\mathcal{C}, Ϙ)$ because we may equivalently divide by metabolic objects by 2.3.5. Let us try to generalize this to the higher L-groups and homotopy groups.

**Proposition 2.5.6** ([CDH+20b, 3.5.8])**.** In the above situation, there is a natural homotopy equivalence

$$\Omega\mathcal{L}(\mathcal{C}, Ϙ) \xrightarrow{\simeq} \mathcal{L}(\mathcal{C}, Ϙ^{[-1]})\ . \tag{2.55}$$

*Proof Idea.* This is difficult to show without further preparations, let us first explain what happens on objects. On the left, those are loops in $\mathcal{L}(\mathcal{C}, \text{Ϙ})$, but we have seen that the edges in this space consist of isomorphisms of Poincaré objects and of Lagrangian correspondences. The former are a special case of the latter, so such a loop at the zero objects is equivalent to a Lagrangian correspondence from the zero Poincaré object to itself. By 2.3.6, this is the same thing as a 1-dimensional Poincaré object. Hence, the difficulty lies in extending this to higher simplices.

To do this, we need to make the set of Lagrangian correspondences into a space. We have already done this in 2.3.3, where we had noticed that Lagrangian correspondences are precisely the Poincaré objects in the cotensor Poincaré $\infty$-category $\mathcal{C}^{(*\leftarrow*\rightarrow*)}$. Let us look at the stable subcategory on diagrams of the form $(0 \leftarrow L \rightarrow P)$ on correspondences beginning at the zero object; one can show [CDH+20a, 2.3.5] that the cotensor quadratic functor restricts to a Poincaré structure on it. We call this to *metabolic Poincaré $\infty$-category* $\mathrm{Met}(\mathcal{C}, \text{Ϙ})$ associated to $(\mathcal{C}, \text{Ϙ})$, and its Poincaré objects are pairs of Poincaré objects in $\mathcal{C}$ and associated Lagrangians.

There is a canonical duality-preserving functor $(\mathcal{C}, \text{Ϙ}^{[-1]}) \rightarrow \mathrm{Met}(\mathcal{C}, \text{Ϙ})$ sending $C \mapsto (0 \leftarrow 0 \rightarrow C)$; such that the associated functor on Poincaré objects maps a 1-dimensional Poincaré object in $\mathcal{C}$ to the associated Lagrangian of the zero object by the above discussion. Together with the duality-preserving functor $(0 \leftarrow L \rightarrow C) \mapsto C$ sending a Lagrangian to the underlying Poincaré-object, we obtain by [CDH+20b, 1.2.5] a split Poincaré-Verdier sequence (see the next section)

$$(\mathcal{C}, \text{Ϙ}^{[-1]}) \longrightarrow (\mathrm{Met}(\mathcal{C}, \text{Ϙ}), \text{Ϙ}^{\mathrm{met}}) \longrightarrow (\mathcal{C}, \text{Ϙ}) \ . \tag{2.56}$$

We are finished after noting the following:

- The functor $\mathcal{L} : \mathcal{C}at_\infty^P \rightarrow \mathcal{S}$ sending a Poincaré $\infty$-category to its L-theory space is *additive* as defined in [CDH+20b, 1.5.4], which means in particular that it sends Poincaré-Verdier sequences to fiber sequences.
- The functor $\mathcal{L}$ is also *bordism-invariant*, so by [CDH+20b, 3.5.4] it sends metabolic Poincaré $\infty$-categories to the contractible space $\mathcal{L}(\mathrm{Met}(\mathcal{C}, \text{Ϙ}^{\mathrm{met}})) = *$. This is intuitively clear, since in the L-theory space there is a path joining any metabolic object to zero and similarly for higher simplices, but in $\mathrm{Met}(\mathcal{C}, \text{Ϙ}^{\mathrm{met}})$ everything is metabolic by definition – alternatively, show that the homotopy groups $\pi_n\mathcal{L}(\mathrm{Met}(\mathcal{C}, \text{Ϙ}^{\mathrm{met}}))$ are trivial as in [CDH+20b, 3.5.5].

□

**Definition 2.5.7.** For $(\mathcal{C}, \text{Ϙ})$ a Poincaré $\infty$-category, its *L-theory spectrum* $\mathbb{L}(\mathcal{C}, \text{Ϙ})$ is the infinite loop space

$$\mathbb{L}(\mathcal{C}, \text{Ϙ}) := [\mathcal{L}(\mathcal{C}, \text{Ϙ}), \mathcal{L}(\mathcal{C}, \text{Ϙ}^{[1]}), \mathcal{L}(\mathcal{C}, \text{Ϙ}^{[2]}), \ldots] \in \mathcal{S}p \tag{2.57}$$

where the transition maps are induced by the last proposition. By its construction from $\mathcal{L}$, the association $\mathbb{L} : \mathcal{C}at_\infty^p \rightarrow \mathcal{S}p$ is functorial.

**Theorem 2.5.8.** The homotopy groups $\pi_n\mathbb{L}(\mathcal{C},\Upsilon)$ agree with the L-groups $L_n(\mathcal{C},\Upsilon)$ we have defined in 2.2.10. For $n \geq 0$, they also agree with the homotopy groups $\pi_n\mathcal{L}(\mathcal{C},\Upsilon)$.

*Proof.* By 2.5.6, we know that the structure maps $\mathcal{L}(\mathcal{C},\Upsilon^{[m]}) \to \Omega\mathcal{L}(\mathcal{C},\Upsilon^{[m+1]})$ are homotopy equivalences (classically, spectra with this property are called $\Omega$-spectra), so

$$\pi_n\mathcal{L}(\mathcal{C},\Upsilon^{[m]}) \cong \pi_{n-1}\Omega\mathcal{L}(\mathcal{C},\Upsilon^{[m]}) \cong \pi_{n-1}\mathcal{L}(\mathcal{C},\Upsilon^{[m-1]}) \cong \pi_{n-i}\mathcal{L}(\mathcal{C},\Upsilon^{[m-i]})$$

for all $m \in \mathbb{Z}, n \in \mathbb{N}_0$ and $i = 0, \ldots, n$. This implies for $i = n, m = 0$ that

$$\pi_n\mathcal{L}(\mathcal{C},\Upsilon) \cong \pi_0\mathcal{L}(\mathcal{C},\Upsilon^{[-n]}) = L_n(\mathcal{C},\Upsilon)$$

proving one of the equivalences. Also, it follows that for $n \geq 0$ the colimits calculating its stable homotopy groups are essentiall constant:

$$\pi_n\mathbb{L}(\mathcal{C},\Upsilon) = \operatorname*{colim}_{m\in\mathbb{N}} \pi_{n+m}\mathcal{L}(\mathcal{C},\Upsilon^{[m]}) = \pi_0\mathcal{L}(\mathcal{C},\Upsilon^{[-n]}) = L_n(\mathcal{C},\Upsilon)$$

by the same argument as above. This also works for $n < 0$ if we ignore the terms in the colimit where $n + m < 0$. $\square$

While calculating the L-groups, or the L-spectrum, of a given ring spectrum $R$ equipped with an invertible module $M$ as in the last section seems like a very daunting task, a powerful tool called *algebraic surgery theory* can be used for this purpose. While it would take a while to properly introduce (see [CDH+21, Section 1] or [Lur11, Lecture 11-16]), we close this section by stating an important result that can be derived in this manner:

**Definition 2.5.9.** A module $M$ over a ring spectrum $R$ is called *projective* if it is a direct summand of $R^n$ for some $n \in \mathbb{N}$. In particular, this implies that $M$ is perfect.

**Theorem 2.5.10** (Algebraic $\pi$-$\pi$-theorem, [CDH+21, 1.2.33])**.** Let $M$ be a projective invertible module over a connective ring spectrum $R$, then there is a canonical equivalence

$$\mathbb{L}^q(R,M) \cong \mathbb{L}^q(\pi_0(R),\pi_0(M)) \ . \tag{2.58}$$

# 3 A Zoology of Decompositions

Just as a topological space can be decomposed into an open subspace and its closed complement, or how an $R$-module may be written as a direct sum or more general extension of two other $R$-modules, there several different ways to define a decomposition of a stable or Poincaré $\infty$-category. We follow [CDH$^+$20b] in introducing (split) Verdier and (split) Poincaré-Verdier sequences as well as several useful results about them, and draw a comparison with the notion of recollement in [Lur17, A.8] as well as semiorthogonal decompositions of triangulated categories. After a brief digression on the much stricter orthogonal decompositions, we generalize some of the above construction to the case where we decompose our $\infty$-category in more than two parts.

## 3.1 Recollements

This section summarizes parts of the discussion of recollements of categories with finite limits in [Lur17], and specifies to the stable case.

**Definition 3.1.1** ([Lur17, A.8.1])**.** An $\infty$-category $\mathcal{C}$ with finite limits is called the *recollement* of two full subcategories $\mathcal{C}_0, \mathcal{C}_1 \subseteq \mathcal{C}$ if:

- The inclusions $i_0, i_1 : \mathcal{C}_0, \mathcal{C}_1 \hookrightarrow \mathcal{C}$ are reflective, i.e. they admit left adjoints $L_0, L_1$.
- The reflections $L_0, L_1$ are left exact.
- If $X \in \mathcal{C}_0$, then $L_1(X) = *$ is the terminal object.
- $L_0$ and $L_1$ are *jointly conservative*: If $\alpha$ is a morphism in $\mathcal{C}$ such that $L_0(\alpha)$ and $L_1(\alpha)$ are isomorphisms, then $\alpha$ is an isomorphism.

*Remark.* Many authors also assume that $\mathcal{C}_0, \mathcal{C}_1 \subseteq \mathcal{C}$ are replete, i.e. closed under isomorphism. We do not assume this as, being an evil[1] notion, it is ultimately irrelevant; however we sometimes abuse notation in the sense that if for $C \in \mathcal{C}$ we say $C \in \mathcal{C}_0$, we actually mean that $C$ is isomorphic to an object in $\mathcal{C}_0$. Should confusion arise, be assured that we always use non-evil notions.

[1]A property in category theory is called *evil* if it is not invariant under equivalences of categories.

**Lemma 3.1.2.** Conversely to the third point, if $X \in \mathcal{C}$ and $L_1(X) = *$, then the unit $\eta_0 : X \to L_0X$ is an isomorphism since $L_1\eta_0 : * \to *$ and $L_0\eta_0 : L_0X \to L_0^2X$ are, so $X$ (essentially) lies in $\mathcal{C}_0$. In other words, $\mathcal{C}_0$ agrees with the kernel of $L_1$.

**Lemma 3.1.3.** If in the above situation $C_0 \in \mathcal{C}_0$ and $C_1 \in \mathcal{C}_1$, then

$$\mathrm{Map}_{\mathcal{C}}(C_0, C_1) \cong \mathrm{Map}_{\mathcal{C}_1}(L_1C_0, C_1) \cong \mathrm{Map}_{\mathcal{C}_1}(*, C_1) \simeq \Delta^0$$

is contractible. Conversely, if $C \in \mathcal{C}$ and $\mathrm{Map}_{\mathcal{C}}(C, C_1) \simeq \Delta^0$ for all $C_1 \in \mathcal{C}_1$, then by the Yoneda Lemma $L_1C = 0$ so by the last lemma, $C \in \mathcal{C}_0$.

**Proposition 3.1.4** ([Lur17, A.8.16])**.** Let $X$ be a topological space, $j : U \hookrightarrow X$ an open subset, and $i : X - U =: Z \hookrightarrow X$ the complementary closed subset. Then, the fully faithful pushforward functors $i_* : \mathcal{S}h(Z) \hookrightarrow \mathcal{S}h(X)$ and $j_* : \mathcal{S}h(U) \hookrightarrow \mathcal{S}h(X)$ exhibit $\mathcal{S}h(X)$ as a recollement of $\mathcal{S}h(Z)$ and $\mathcal{S}h(U)$, with reflections the pullbacks $i^*, j^*$.

*Remark.* By virtue of this example, we call $\mathcal{C}_0$ the *closed* and $\mathcal{C}_1$ the *open subcategory* of the above recollement datum. Also, we will often denote $i_0, i_1$ by $i_*$ and $j_*$ and $L_0, L_1$ by $i^*, j^*$ following this analogy.

*Remark* ([Lur17, A.8.5], [Lur17, A.8.13])*.* In a similar manner to the later proof of 3.2.7, one can show that

- If $\mathcal{C}_0$ has an initial object, the functor $j^*$ has a fully faithful left adjoint $j_+ : \mathcal{C}_1 \to \mathcal{C}$.
- If $\mathcal{C}$ has a zero object meaning that its terminal object is also initial, the inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ admits a right adjoint $i^- : C \mapsto \mathrm{fib}(C \to j_*j^*(C))$, the fiber of the unit of $j^* \dashv j_*$.

**Proposition 3.1.5** ([HPT20, 5.20])**.** Let the $\infty$-category $\mathcal{V}$ be

- presentable and stable, or
- the tensor product of a compactly generated $\infty$-category and an $\infty$-topos.

Then, tensoring with $\mathcal{V}$ preserves recollements of presentable $\infty$-categories. In particular in the situation of 3.1.4, the $\infty$-category $\mathcal{S}h(X; \mathcal{V})$ of $\mathcal{V}$-valued sheaves is the recollement of $\mathcal{S}h(Z; \mathcal{V})$ and $\mathcal{S}h(U; \mathcal{V})$.

*Remark.* The case where $\mathcal{V}$ itself is compactly generated is particularly well-behaved, compare 3.5.9.

**Proposition 3.1.6** ([Lur17, A.8.17])**.** Let $\mathcal{C}$ be the recollement of $\mathcal{C}_0$ and $\mathcal{C}_1$. Then, $\mathcal{C}$ is stable iff both $\mathcal{C}_0$ and $\mathcal{C}_1$ are stable and $L_0|_{\mathcal{C}_1} : \mathcal{C}_0 \to \mathcal{C}_1$ is exact. We call this situation a *stable recollement.*

*Proof.* For the *only if* direction, $\mathcal{C}_0$ and $\mathcal{C}_1$ are the essential images of the left exact localization functors $L_0$ and $L_1$ and therefore, as subcategories of $\mathcal{C}$, closed under finite limits. In particular, they contain the zero object and are closed under fibers and finite products, and thus form stable subcategories by 1.5.7. Further, $L_0|_{\mathcal{C}_1} = L_0 \circ i_1$ is a composition of left exact functors (since $i_1$ is a right adjoint), but being left exact is equivalent to being exact in the stable case.

The *if* direction is more involved; since $\mathcal{C}$ has finite limits it suffices to show that

- The terminal object $* \cong L_0(0) \cong L_1(0)$ in $\mathcal{C}$ is initial,
- A sequence $C' \to C \to C''$ in $\mathcal{C}$ is a (co)fiber sequence iff its images under $L_0, L_1$ are (co)fiber sequences both in $\mathcal{C}_0$ and $\mathcal{C}_1$.

The first point is clear since $L_0, L_1$ preserve colimits; note that the indicated sequence of isomorphisms follows from joint conservativity. Similarly, the second claim follows from the fact that $L_0, L_1$ preserve finite limits and colimits and are jointly conservative, for example $C' \cong \mathrm{fib}(C \to C'')$ iff the analogous statements for $L_0(C')$ and $L_1(C')$ hold. For a more abstract argument, see the reference. □

*Remark.* In the stable case, the extra adjoints $j_+$ and $i^-$ always exist.

## 3.2 Verdier Sequences

**Definition 3.2.1.** Given an exact functor $F : \mathcal{C} \to \mathcal{D}$ between stable $\infty$-categories, we call a morphism $f$ in $\mathcal{D}$ an *equivalence modulo* $\mathcal{C}$ if its fiber $\mathrm{fib}(f)$ lies in the smallest stable subcategory of $\mathcal{D}$ spanned by the essential image of $F$. Note that if $F$ is the inclusion of a stable subcategory, this just means $\mathrm{fib}(f) \in \mathcal{C} \subseteq \mathcal{D}$. The *Verdier quotient* $\mathcal{D}/\mathcal{C}$ is defined as the localization (see 1.2.3) of $\mathcal{D}$ with respect to this class of morphisms.

**Proposition 3.2.2** ([CDH+20b] A.1.5, A.1.6)**.** In the above situation, $\mathcal{D}/\mathcal{C}$ is stable and the localization functor $\mathcal{D} \to \mathcal{D}/\mathcal{C}$ is exact. Conversely, every localization of a stable $\infty$-category with these properties is a Verdier quotient.

**Definition 3.2.3** ([CDH+20b] A.1.10)**.** A sequence $\mathcal{C} \xrightarrow{i_*} \mathcal{D} \xrightarrow{j^*} \mathcal{E}$ of stable $\infty$-categories and exact functors is called a *Verdier sequence* if:

- The composition $j^* \circ i_*$ is the zero functor,
- $j^*$ exhibits $\mathcal{E}$ as the Verdier quotient $\mathcal{D}/\mathcal{C}$,
- $i_*$ is fully faithful, embedding $\mathcal{C}$ as the full subcategory spanned by the objects $D \in \mathcal{D}$ satisfying $j^*(C) = 0$

Verdier sequences of stable $\infty$-categories are the analogue of short exact sequences, in our earlier analogy between stable $\infty$-categories and $R$-modules. However, the following proposition does not hold any more:

**Proposition 3.2.4** (Splitting Lemma)**.**
A short exact sequence $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ of $R$-modules *splits* if it is isomorphic to the trivial extension $0 \to M' \xrightarrow{i_1} M' \oplus M'' \xrightarrow{p_2} M'' \to 0$ in the sense that there is an isomorphism $M \cong M' \oplus M''$ making both involved squares commute. This is equivalent to any of the following statements:

- There is an $R$-module-homomorphism $s : M \to M'$ such that $s \circ f = \mathrm{id}_{M'}$.
- There is an $R$-module-homomorphism $t : M'' \to M$ such that $g \circ t = \mathrm{id}_{M''}$.

To develop a partial analogue to these splitting criteria for stable $\infty$-categories, we need some background on localizations.

**Definition 3.2.5.** Let $\mathcal{C}$ be an $\infty$-category, $W$ a class of morphisms in $\mathcal{C}$ and $L : \mathcal{C} \to \mathcal{C}[W^{-1}]$ the associated localization. We call $L$ a *reflection* or *left Bousfield localization* if it admits a fully faithful right adjoint, exhibiting $\mathcal{C}[W^{-1}]$ as a reflective subcategory of $\mathcal{C}$. Dually, $L$ is a *coreflection* or *right Bousfield localization* if it admits a fully faithful left adjoint.

**Lemma 3.2.6** ([CDH+20b, A.2.1])**.** A localization functor $L : \mathcal{C} \to \mathcal{C}[W^{-1}]$ is a reflection iff for every $C \in \mathcal{C}$, there exists and object $C' \in \mathcal{C}$ together with an isomorphism $LC \cong LC'$ such that the functor

$$\mathrm{Map}_{\mathcal{C}}(-, C') : \mathcal{C}^{op} \to \mathcal{S} \tag{3.1}$$

sends all morphisms in $W$ to isomorphisms. Dually, $L$ is a coreflection iff for each $C \in \mathcal{C}$ there is a $C' \in \mathcal{C}$ such that $LC \cong LC'$ and $\mathrm{Map}_{\mathcal{C}}(C', -)$ sends morphisms in $W$ to isomorphisms.

*Proof.* We only prove the case of reflections, the other case is dual. For the *only if* direction, let $i : \mathcal{C}[W^{-1}] \hookrightarrow \mathcal{C}$ be the inclusion right adjoint to $L$ and set $C' := iL(C)$, then $LC' = LiL(C) \cong LC$ since the counit $Li \Rightarrow \mathrm{Id}_{\mathcal{C}[W^{-1}]}$ is an isomorphism as $i$ is fully faithful. Further, $\mathrm{Map}(C', -) \cong \mathrm{Map}(LC, L-)$ sends $W$ to isomorphisms since $L$ does.

Conversely, *if* an object $C'$ with these properties always exists, it suffices to show that it represents the functor $\mathrm{Map}_{\mathcal{C}}(L-, LC)$ since $L$ is essentially surjective by construction of the localization so $LC$ covers all objects of the localization, and the representing objects $C' =: iLC$ assemble into the desired left adjoint. Note that $i$ is fully faithful since the counit $LC' \cong LC$ is an isomorphism.

By assumption, $\mathrm{Map}(-,C')$ factors as $M \circ L$ with $M : \mathcal{C}[W^{-1}]^{op} \to \mathcal{S}$, and we may calculate that for any other $G : \mathcal{C}[W^{-1}]^{op} \to \mathcal{S}$ we have

$$\mathrm{Nat}(M \circ L, G \circ L) \cong \mathrm{Nat}(\mathrm{Map}(-,C'), G \circ L) \cong G \circ L(C') \cong G(LC)$$

by the Yoneda Lemma. The last expression agrees with $\mathrm{Nat}(\mathrm{Map}(-,LC), G)$ while the first, by the universal property of a localization exhibiting $\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{S}^{op}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{S}^{op})$ as a full subcategory, agrees with $\mathrm{Nat}(M, G)$. Hence, $\mathrm{Map}(-,LC) \cong M$ so $\mathrm{Map}(-,LC) \circ L \cong M \circ L = \mathrm{Map}(-,C')$, as claimed. □

*Remark* ([CDH+20b, A.2.2]). In fact, if $i : \mathcal{C}' \hookrightarrow \mathcal{C}$ is fully faithful and admits a left (or right) adjoint $L$, then by [Lur09a, 5.2.7.12] the functor $L$ is a localization at those morphisms that are sent to isomorphisms by the functor (co)represented by any object in $\mathcal{C}'$.

**Proposition 3.2.7** ([CDH+20b, A.2.5])**.** Let $\mathcal{C}_0 \xrightarrow{i_*} \mathcal{C} \xrightarrow{j^*} \mathcal{C}_1$ be a sequence of stable $\infty$-categories and exact functors such that $j^* i_* = 0$. Then, the following are equivalent:

- The functor $i_*$ is fully faithful and exhibits $\mathcal{C}_0$ as the full subcategory of $\mathcal{C}$ on objects $C$ with $j^*C = 0$, and $j^*$ possesses a fully faithful left adjoint $j_+$ (or fully faithful right adjoint $j_*$)
- The functor $j^*$ exhibits $\mathcal{C}_1$ as the Verdier quotient of $i_* : \mathcal{C}_0 \hookrightarrow \mathcal{C}$, and $i_*$ is fully faithful and admits a left adjoint $i^*$ (or right adjoint $i^-$)

*Proof.* If a left adjoint $j_+ \dashv j^*$ exists, we may define $\boxed{i^* := \mathrm{cofib}(j_+ j^* \to \mathrm{id}_{\mathcal{C}})}$ as the cofiber of the counit, which takes values in $\mathcal{C}_0 = \ker(j^*)$ since

$$j^* i^* C \cong \mathrm{cofib}(j^* j_+ j^* C \to j^* C) \cong \mathrm{cofib}(\mathrm{id}_{j^*C}) = 0 \; .$$

This defines a left adjoint to $i_*$ since

$$\mathrm{Map}(i^*C, C_0) \simeq \mathrm{fib}\left(\mathrm{Map}(C, C_0) \to \mathrm{Map}(j_+ j^* C, C_0)\right) = \mathrm{Map}(C, i_* C_0)$$

as the second mapping space is contractible because $j_+ \dashv j^*$ and $j^* C_0 = 0$. Since $j^*$ by assumption admits a fully faithful left adjoint, it is a localization and in fact even the Verdier quotient by $\mathcal{C}_0$ as a morphism $\alpha$ is sent to an isomorphism iff $\mathrm{fib}(\alpha) \in \ker(j^*) = \mathcal{C}_0$. Also, $i^*$ is fully faithful by assumption.

Conversely, given $i^*$, let us define $\boxed{j_+ := \mathrm{fib}(\mathrm{id}_{\mathcal{C}} \to i_* i^*)}$ as the fiber of the unit; while this is a priori a functor $\mathcal{C} \to \mathcal{C}$ it factors through $\mathcal{C}_1 \to \mathcal{C}$ as for $\alpha$ is a morphism in $\mathcal{C}$ with fiber $F$ in $\mathcal{C}_0$, we have

$$\mathrm{fib}\, j_+(\alpha) = j_+(F) = \mathrm{fib}\left(F \to i_* i^* F\right) = 0$$

so $j_+(\alpha)$ is an isomorphism. If $C_1 \in \mathcal{C}_1$ is represented as the localization $j^*\hat{C}_1$,

$$\mathrm{Map}(j_+C_1, C) = \mathrm{Map}\left(\mathrm{fib}(\hat{C}_1 \to i_*i^*\hat{C}_1), C\right) \simeq \mathrm{cofib}\left(\mathrm{Map}(i_*i^*\hat{C}_1, C) \to \mathrm{Map}(\hat{C}_1, C)\right) .$$

In the case that $C = i_*C_0$ is in the essential image of $\mathcal{C}_0$, this evaluates to $\mathrm{cofib}\,(\mathrm{Map}(i^*C', C_0) \to \mathrm{Map}(C', i_*C_0)) = 0$ using that $i_*$ is fully faithful. Hence, by the same argument as above, $\mathrm{Map}(j_+C_1, -)$ factors through $\mathcal{C}_1$. But $j^*j_+ = \mathrm{fib}(j^* \to j^*i_*i^*) = \mathrm{id}_{\mathcal{C}_1}$ regarded as a functor on $\mathcal{C}_1$. This verifies both conditions of 3.2.6, so we are finished.

For the case of right adjoints, given $j_*$ we construct $\boxed{i^- := \mathrm{fib}(C \to j_*j^*C)}$ which lies in $\mathcal{C}_0$ since $j^*i^-C = \mathrm{fib}(j^*C \to j^*j_*j^*C) = 0$ as $j^*$ is a left-exact reflection. Then,

$$\mathrm{Map}(C_0, i^-C) \simeq \mathrm{fib}(\mathrm{Map}(C_0, C), \mathrm{Map}(C_0, j_*j^*C)) \cong \mathrm{Map}(C, C_0)$$

again by 3.1.3. Conversely, $\boxed{j_* := \mathrm{cofib}(i_*i^- \to \mathrm{id}_{\mathcal{C}})}$ also uses 3.2.6. □

*Remark.* As indicated by the boxes, behold the canonical fiber sequences

$$i_*i^- \Rightarrow \mathrm{Id} \Rightarrow j_*j^*, \quad j_+j^* \Rightarrow \mathrm{Id} \Rightarrow i_*i^* . \tag{3.2}$$

**Definition 3.2.8.** A Verdier sequence $\mathcal{C}_0 \overset{i_*}{\hookrightarrow} \mathcal{C} \overset{j^*}{\to} \mathcal{C}_1$ is called *split* if, equivalently (by above Lemma),

- $j^*$ admits both a left and a right adjoint,
- $i_*$ admits both a left and a right adjoint.

It is called *left* or *right split* if only the left or right adjoint exist. The adjoints of $p$ are automatically fully faithful.

*Warning.* Not every split Verdier sequence is equivalent to an *orthogonal decomposition* $\mathcal{C} \hookrightarrow \mathcal{C} \oplus \mathcal{D} \to \mathcal{D}$. More on this in 3.4.

**Theorem 3.2.9.** The following data are equivalent:

- A split Verdier sequence $\mathcal{C}_0 \overset{i_*}{\to} \mathcal{C} \overset{j^*}{\to} \mathcal{C}_1$
- A stable $\infty$-category $\mathcal{C}$ that is a (stable) recollement of $\mathcal{C}_0$ and $\mathcal{C}_1$, with inclusions $i_*, j_*$ and reflections $i^*, j^*$.

*Proof.* Given a split Verdier sequence as above, we have seen that $i_*$ and the right adjoint $j_*$ of $j^*$ are fully faithful, so they exhibit $\mathcal{C}_0$ and $\mathcal{C}_1$ as full subcategories of $\mathcal{C}$. Also, $i_*$ possesses a left adjoint $i^*$, and $j^*i_* = 0$ by definition, so we only need to show that $i^*, j^*$ are jointly conservative.

If $\alpha$ is a morphism in $\mathcal{C}$ such that $i^*\alpha, j^*\alpha$ are isomorphisms, then $j^* \operatorname{fib}(\alpha) = \operatorname{fib}(j^*\alpha) = 0$ so $\operatorname{fib}(\alpha) \in \ker(j^*) \simeq \mathcal{C}_0$. But this means that $\operatorname{fib}(\alpha) \cong i_*i^* \operatorname{fib}(\alpha) = i_* \operatorname{fib}(i^*\alpha) = 0$, so $\alpha$ is an isomorphism.

Conversely, let $\mathcal{C}$ be a stable recollement of $\mathcal{C}_0$ and $\mathcal{C}_1$ as above, then $\mathcal{C}_0 = \ker(j^*)$ by 3.1.2. Also, the localization $j^* : \mathcal{C} \to \mathcal{C}_1$ sends a morphism $\alpha$ in $\mathcal{C}$ to an isomorphism iff $\operatorname{fib}(j^*\alpha) \cong j^* \operatorname{fib}(\alpha) = 0$, i.e. $\operatorname{fib}(\alpha) \in \ker(j^*) = \mathcal{C}_0$. This means that $j^*$ exhibits $\mathcal{C}_1$ as a Verdier quotient of $\mathcal{C}$ by $\mathcal{C}_0$ as claimed. Existence of the remaining adjoint follows from 3.2.7 □

**Proposition 3.2.10.** If the middle sequence in the diagram

$$\begin{array}{ccccc} & \xleftarrow{\;i^*\;} & & \xleftarrow{\;j_+\;} & \\ \mathcal{C} & \xrightarrow{\;i_*\;} & \mathcal{D} & \xrightarrow{\;j^*\;} & \mathcal{E} \\ & \xleftarrow{\;i^-\;} & & \xleftarrow{\;j_*\;} & \end{array}$$

is a split Verdier sequence with indicated adjoints, then the upper and lower sequences are right and left split Verdier sequences.

*Proof.* We already know from 3.2.7 that $j_*, j_+$ are fully faithful; also by construction $i^-, i^*$ have fully faithful left/ right adjoints making them (co)reflections. Finally, by the proof of the mentioned proposition $\ker(i^*)$ consists of precisely those $D \in \mathcal{D}$ with $j_+j^*D \cong D$, i.e. the essential image of $j_+$. Similarly for $i^-$. □

**Definition 3.2.11.** If $\mathcal{C}_0 \subseteq \mathcal{C}$ is a stable subcategory of a stable $\infty$-category, denote by $\mathcal{C}_0^\perp$ the full subcategory spanned by those $C \in \mathcal{C}$ such that for each $C_0 \in \mathcal{C}_0$, the mapping space $\operatorname{Map}_{\mathcal{C}}(C_0, C)$ is contractible. Dually, define ${}^\perp\mathcal{C}_0$ as the full subcategory on those $C \in \mathcal{C}$ with $\operatorname{Map}_{\mathcal{C}}(C, C_0)$ contractible for $C_0 \in \mathcal{C}_0$.

**Proposition 3.2.12.** If $\mathcal{C}_0 \subseteq \mathcal{C}$ is a reflective and coreflective stable subcategory of a stable $\infty$-category, then $\mathcal{C}$ is a stable recollement of $\mathcal{C}_0$ and $\mathcal{C}_0^\perp$.

Dually, if $\mathcal{C}_1 \subseteq \mathcal{C}$ is a reflective stable subcategory such that the reflection $j^* : \mathcal{C} \to \mathcal{C}_1$ has an additional left adjoint $j^+$, then $\mathcal{C}$ is a stable recollement of ${}^\perp\mathcal{C}_1$ and $\mathcal{C}_1$.

*Proof.* For the first claim, we need to show that the inclusion $j_* : \mathcal{C}_0^\perp \hookrightarrow \mathcal{C}$ has a left adjoint $j^*$ such that $\ker(j^*) = \mathcal{C}_0$, since as a reflection it is then automatically a localization at the morphisms with fiber in $\mathcal{C}_0$ and all required adjoints exist. Define

$j^*(C) := \mathrm{cofib}(i_*i^-C \to C)$ where $i^* \dashv i_* \dashv i^-$ and $i_* : \mathcal{C}_0 \hookrightarrow \mathcal{C}$ is the inclusion, then for $C_0 \in \mathcal{C}_0$ and $\mathcal{C}_1 \in \mathcal{C}_0^\perp$,

$$\begin{aligned}\mathrm{Map}(C_0, j^*C) &= \mathrm{cofib}(\mathrm{Map}(i_*C_0, i_*i^-C) \to \mathrm{Map}(i_*C_0, C)) \cong \\ &\cong \mathrm{cofib}(\mathrm{Map}(C_0, i^-C) \to \mathrm{Map}(i_*C_0, C)) \cong 0 \\ \mathrm{Map}(j^*C, C_1) &= \mathrm{fib}(\mathrm{Map}(C, i_*C_1) \to \mathrm{Map}(i_*i^!C, i_*C_1)) = \mathrm{Map}(C, i_*C_1)\end{aligned}$$

so $j^* : \mathcal{C} \to \mathcal{C}_0^\perp$ is well-defined and left adjoint to $j_*$. Finally, $j^*(C) = 0$ iff $i_*i^-C \cong C$ iff $C \in \mathcal{C}_0$, so we are done.

The second claim is easier; we only need to show that ${}^\perp\mathcal{C}_1 = \ker(j^*)$ since $j^*$ is a reflection and all required adjoints exist. For $C \in \mathcal{C}$, by Yoneda $j^*(C) = 0$ iff for all $C_1 \in \mathcal{C}_1$ we have $\mathrm{Map}(j^*(C), C_1) \simeq \mathrm{Map}(C, j_*C_1) = 0$, which is equivalent to $C \in {}^\perp\mathcal{C}_1$. □

**Proposition 3.2.13.** If $\mathcal{C}$ is a stable recollement of $\mathcal{C}_0$ and $\mathcal{C}_1$, then $\mathcal{C}_1 = \mathcal{C}_0^\perp$ and $\mathcal{C}_0 = {}^\perp\mathcal{C}_1$ (as always, up to completing under isomorphisms).

*Proof.* First of all, for $C_0 \in \mathcal{C}_0$ and $C_1 \in \mathcal{C}_1$, we have

$$\mathrm{Map}_{\mathcal{C}}(C_0, C_1) = \mathrm{Map}_{\mathcal{C}_1}(L_1 i_0 C_0, C_1) = \mathrm{Map}_{\mathcal{C}_1}(0, C_1) = 0$$

which shows one inclusion of each identity.

Conversely, if $C \in \mathcal{C}$ such that $\mathrm{Map}(C_0, C) = 0$ for all $C_0 \in \mathcal{C}_0$, then applying $\mathrm{Map}(C_0, -)$ to the counit map

$$\mathrm{Map}(C_0, C \to j_*j^*C) = (\mathrm{Map}(C_0, C, C) \to \mathrm{Map}(j^*i_*C_0, j^*C)) = 0$$

yields an isomorphism so by the Yoneda-Lemma $i^*(C \to j_*j^*C)$ is an isomorphism, hence by joint conservativity it is enough to show $j^*C \to j^*j_*j^*C$ is an isomorphism as well, which is clear since $j^*j_* = \mathrm{id}_{\mathcal{C}_1}$.

Finally, if $C \in \mathcal{C}$ satisfies $\mathrm{Map}_{\mathcal{C}}(C, C_1) = 0$ for all $C_1 \in \mathcal{C}_1$, then

$$\mathrm{Map}_{\mathcal{C}_1}(L_1 C, C_1) = \mathrm{Map}_{\mathcal{C}}(C, C_1) = 0$$

so by the Yoneda-Lemma already $L_1 C = 0$. We then calculate $L_1(C \to i_0 L_0 C) = (0 \to 0)$ and $L_0(C \to i_0 L_0 C) = \mathrm{id}_{L_0}$, so by joint conservativity the unit map $C \to i_0 L_0 C$ is an isomorphism and $C \in \mathcal{C}_0$. □

Generally, morphism spaces in localizations are difficult to calculate, unless we are dealing with a reflective localization which can be embedded into the original category. Since we will make use of it later, let us still develop an explicit description in the case of Verdier quotients (our arguments can however be adapted to general localizations, as indicated in the respective references).

**Definition 3.2.14** ([Lur09a, 5.3.5.4], [Lur09a, 5.1.6.8])**.** Let $\mathcal{C}$ be an arbitrary $\infty$-category. Its *Ind-completion* $\mathrm{Ind}(\mathcal{C})$ is defined as a the smallest fully subcategory of $\mathcal{PSh}(\mathcal{C})$ that contains the essential image of the Yoneda embedding $h : \mathcal{C} \hookrightarrow \mathcal{PSh}(\mathcal{C})$ (consisting of the representable presheaves) and is closed under filtered colimits. If $\mathcal{C}$ admits finite colimits, then this agrees with the full subcategory on those functors $\mathcal{C}^{op} \to \mathcal{S}$ that preserve finite limits (in $\mathcal{C}^{op}$).

Similarly, the *idempotent completion* or *Karoubi completion* of $\mathcal{C}$ is the full subcategory of $\mathcal{PSh}(\mathcal{C})$ spanned by retracts of representable presheaves. It agrees with the full subcategory on the *completely compact* presheaves, i.e. those $F \in \mathcal{PSh}(\mathcal{C})$ such that $\mathrm{Nat}(F, -)$ preserves (small) colimits in $\mathcal{PSh}(\mathcal{C})$.

*Remark.* The Ind-completion of a small $\infty$-category $\mathcal{C}$ is compactly generated (in particular presentable), as it is controlled by its small full subcategory $\mathcal{C}$. In fact, an $\infty$-category is compactly generated iff it admits colimits and is equivalent to the Ind-completion of a small $\infty$-category (its full subcategory on compact objects). More generally, an $\infty$-category $\mathcal{C}$ is presentable iff it admits colimits and there exists a regular cardinal $\kappa$ such that it is equivalent to the generalized Ind-completion $\mathrm{Ind}_\kappa(\mathcal{D})$ of some small $\infty$-category, compare [Lur09a, 5.5.1.1].

**Definition 3.2.15.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between arbitrary $\infty$-categories. Then, its *pseudo-left adjoint* is the functor $\mathcal{D} \to \mathcal{PSh}(\mathcal{C})$ informally given by sending $D \mapsto \mathrm{Map}_{\mathcal{D}}(F(-), D)$. Similarly, its *pseudo-right adjoint* is the functor $\mathcal{D} \to \mathrm{Fun}(\mathcal{C}, \mathcal{S})$ given by $D \mapsto \mathrm{Map}_{\mathcal{D}}(D, F(-))$.

If the pseudo-left adjoint of $F$ factors through $\mathrm{Ind}(\mathcal{C}) \subseteq \mathcal{PSh}(\mathcal{C})$, we call it the *pro-left adjoint* of $F$, and similarly for *pro-right adjoints* and the dual *Pro-completion* $\mathrm{Pro}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{S})$ of $\mathcal{C}$. Clearly, if it even factors through the essential image of the Yoneda embedding $\mathcal{C} \hookrightarrow \mathcal{PSh}(\mathcal{C})$, this factorization is an ordinary left adjoint of $F$.

**Proposition 3.2.16.** Every functor $F : \mathcal{C} \to \mathcal{D}$ between $\infty$-categories admits a pseudo-left and a pseudo-right adjoint. Similarly, every exact functor between stable $\infty$-categories admits a pro-left and a pro-right adjoint.

*Proof.* The statement for pseudo-adjoint follows by definition. Since stable $\infty$-categories admit finite (co)limits it remains to show that if $F$ is an exact functor between stable $\infty$-categories and $D \in \mathcal{D}$, the pseudo-left adjoint $\mathrm{Map}_{\mathcal{D}}(F(-), D)$ preserves finite limits in $\mathcal{D}^{op}$, which is clear since it is a composition of left exact functors. The statement for pseudo-right adjoints is dual. ☐

**Proposition 3.2.17** ([Lur09a, 5.3.5.10])**.** Let $\mathcal{C}$ be an $\infty$-category, and $h : \mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ be the Yoneda-embedding into its Ind-completion. Then, for any $\infty$-category $\mathcal{D}$ admitting filtered colimits, precomposing with $h$ induces an equivalence of categories

$$\mathrm{Fun}^{\mathrm{ind}}(\mathrm{Ind}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \tag{3.3}$$

between the full subcategory of functors $\mathrm{Ind}(\mathcal{C}) \to \mathcal{D}$ preserving filtered colimits and all functors $F : \mathcal{C} \to \mathcal{D}$, where the inverse is given by Yoneda extension $F \mapsto \mathrm{Lan}_h F =: \mathrm{Ind}(F)$.

*Proof.* This is an analogue of 1.1.6, and the proof is similar: By definition, every element of $\mathrm{Ind}(\mathcal{C})$ can be written as a filtered colimit of representable cosheaves, so a functor $\mathrm{Ind}(\mathcal{C}) \to \mathcal{D}$ preserving filtered colimits is determined by its value on the essential image of the Yoneda embedding. □

**Proposition 3.2.18** ([Lur09a, 5.5.1.1], [Lur17, 1.1.3.6])**.** If the $\infty$-category $\mathcal{C}$ has finite colimits, then $\mathrm{Ind}(\mathcal{C})$ is compactly generated, in particular admits all limits and colimits. If $\mathcal{C}$ is even a stable $\infty$-category, then $\mathrm{Ind}(\mathcal{C})$ is stable as well.

**Corollary 3.2.19** ([Lur09a, 5.3.5.13])**.** For $F : \mathcal{C} \to \mathcal{D}$ a functor between $\infty$-category, there is a canonical functor $F^* : \mathrm{Fun}(\mathcal{D}^{op}, \mathcal{S}) \to \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ between presheaf categories given by precomposition with $F$, which is right adjoint to the Yoneda extension of $F$. If $\mathcal{C}$ has finite colimits and those are preserved by $F$, this restricts to a functor $F^* : \mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{C})$ since the property of a presheaf preserving finite limits in $\mathcal{C}^{op}$ is preserved. By construction, it is again right adjoint to $\mathrm{Ind}(F)$; in particular $\mathrm{Ind}(F)$ preserves all, instead of just filtered, colimits. We obtain a restricted equivalence

$$\mathrm{Fun}^{\mathrm{colim}}(\mathrm{Ind}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{rex}}(\mathcal{C}, \mathcal{D}) \tag{3.4}$$

between the full subcategories of functors that preserve colimits or finite colimits, respectively.

**Lemma 3.2.20.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between $\infty$-categories. It induces two functors between presheaf categories $\mathcal{PSh}(\mathcal{C}) \to \mathcal{PSh}(\mathcal{D})$:

- The left Kan extension functor $\mathrm{Lan}_F$
- The Yoneda extension $\mathrm{Lan}_{h_\mathcal{C}}(h_\mathcal{D} \circ F)$

These functors agree.

*Proof.* The functor $\mathrm{Lan}_F$ is left adjoint to the precomposition functor $F^*$. Similarly, applying the nerve-realization paradigm 1.1.7, the Yoneda extension has a right adjoint realization functor sending a presheaf $S \in \mathrm{Fun}(\mathcal{D}^{op}, \mathcal{S})$ to $|S| \in \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ with

$$|S|(C) = \mathrm{Map}_{\mathcal{PSh}(\mathcal{D})}(h_\mathcal{D} \circ F(C), S) = \mathrm{Nat}(\mathrm{Map}_\mathcal{D}(-, F(C)), S) \cong S(F(C)) = (F^*S)(C)$$

by the Yoneda Lemma, so $|-| = F^*$ and our functors must agree by uniqueness of adjoints. □

**Construction 3.2.21** ([CDH+20b, A.3.11 and below], [NS18, I.3.5]). Let $\mathcal{C}_0 \xrightarrow{i_*} \mathcal{C} \xrightarrow{j^*} \mathcal{C}_1$ be a Verdier sequence of stable $\infty$-categories. Then, the exact composite functor $\mathcal{C} \to \mathcal{C}_1 \hookrightarrow \operatorname{Ind}(\mathcal{C}_1)$ can by the previous Corollary be extended to a colimit-preserving functor $\operatorname{Ind}(j^*) : \operatorname{Ind}(\mathcal{C}) \to \operatorname{Ind}(\mathcal{C}_1)$ with a right adjoint $\operatorname{Ind}(j)_*$ defined by precomposing presheaves with $j^*$, compare the above Lemma. Similarly for $i_*$, so the sequence

$$\operatorname{Ind}(\mathcal{C}_0) \xrightarrow{\operatorname{Ind}(i_*)} \operatorname{Ind}(\mathcal{C}) \xrightarrow{\operatorname{Ind}(j^*)} \operatorname{Ind}(\mathcal{C}_1) \tag{3.5}$$

is a right split Verdier sequence, because

- $\operatorname{Ind}(j^*)$ is a reflection, since its right adjoint $\operatorname{Ind}(j)_*$ is given by the precomposition $\operatorname{Fun}\left({}^{\mathcal{C}}\!/_{\mathcal{C}_0}, \mathcal{S}^{op}\right) \to \operatorname{Fun}(\mathcal{C}, \mathcal{S})$ with the localization $j^*$, which by definition is fully faithful. In particular,

  We find that ${}^{\mathcal{C}}\!/_{\mathcal{C}_0} \subseteq \operatorname{Ind}(\mathcal{C})$ via the embedding $\operatorname{Ind}(j)_* \circ h$.
- $\operatorname{Ind}(i_*)$ is fully faithful since it agrees with $\operatorname{Lan}_{i_*}$ on the presheaf category, and the left Kan extension along a fully faithful functor is so as well.
- It exhibits $\operatorname{Ind}(\mathcal{C}_0) \subseteq \operatorname{Ind}(\mathcal{C})$ as the kernel of $\operatorname{Ind}(j^*)$: Any object of $\operatorname{Ind}(\mathcal{C})$ can be written as a filtered colimit $I = \operatorname*{colim}_{k \in K} h_{p(k)}$ of representable presheaves for some diagram $p : K \to \mathcal{C}$, and we must show that $\operatorname{Ind}(j^*)(I) = \operatorname*{colim}_{k \in K} h_{j^* p(k)} = 0$ iff already $I \in \operatorname{Ind}(\mathcal{C}_0)$. The *if* direction is clear, for the *only if* we refer to the proof of the second reference, as it uses background we do not want to develop.

*Remark.* The references show that further right adjoints $\operatorname{Ind}(i_*) \dashv \operatorname{Ind}(i)^- \dashv \operatorname{Ind}(i)_-$ and $\operatorname{Ind}(j^*) \dashv \operatorname{Ind}(j)_* \dashv \operatorname{Ind}(j)^-$ exist, in particular the middle functors preserve colimits.

**Proposition 3.2.22** ([NS18, Proof of I.3.3]). As discussed above for any $\infty$-category $\mathcal{D}$ with all colimits, the functor $j^*$ induces a left Kan extension functor

$$\operatorname{Lan}_{j^*} : \operatorname{Fun}(\mathcal{C}, \mathcal{D}) \to \operatorname{Fun}\left({}^{\mathcal{C}}\!/_{\mathcal{C}_0}, \mathcal{D}\right) \tag{3.6}$$

that, since $j^*$ is exact, restricts to $\operatorname{Ind}(j^*)$ between the Ind-categories if $\mathcal{D} = \mathcal{S}$. We can give the explicit formula

$$\operatorname{Lan}_{j^*} F\,(C_1) = \operatorname*{colim}_{C_0 \in \mathcal{C}_{0\,/\hat{C}_1}} F\left(\operatorname{cofib}(C_0 \to \hat{C}_1)\right) \tag{3.7}$$

for any presheaf $F : \mathcal{C} \to \mathcal{D}$, where $\hat{C}_1$ is any object of $\mathcal{C}$ with $j^* \hat{C}_1 = C_1$.

*Remark.* Since $\mathcal{C}_0$ has finite limits, the involved colimit is filtered because the formation of limit lets us extend finite diagrams to cones.

*Proof.* Denote the above colimit expression by $L(F)$, we want to show that the functor $L$, which since a choice of lift is necessary is a priori defined on $\mathcal{C}$, factors through $\mathcal{C}/_{\mathcal{C}_0}$ where it is left adjoint to the fully faithful (by definition of a localization) precomposition functor $J = - \circ j^*$. There is a canonical natural transformation $(\eta_F : F \to L \circ J(F))_F$ induced by the compatible set of morphisms $F(C) \to F(\mathrm{cofib}(C_0 \to C))$ arising from the inclusion $C \to \mathrm{cofib}(C_0 \to C) = 0 \amalg_{C_0} C$. We are finished if we can verify the conditions in 3.2.6.

If $F$ lives in the full subcategory $\mathrm{Fun}\left(\mathcal{C}/_{\mathcal{C}_0}, \mathcal{D}\right) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$, then it must by definition send morphism with (co)fiber in $\mathcal{C}_0$ to isomorphisms. But $C \to \mathrm{cofib}(C_0 \to C)$ has fiber $C_0$, so $\eta_F$ is an isomorphism, which proves one of the conditions. On the other hand, $L(F)$ always lies in this subcategory (in particular it factors through the Verdier quotient, as claimed) since as a composition of exact functors it is exact, so it suffices to show that for $C_0' \in \mathcal{C}_0$, we have $L(F)(C_0') = 0$. Indeed we calculate

$$\operatorname*{colim}_{C_0 \in \mathcal{C}_{0\,/C_0'}} F\left(\mathrm{cofib}(C_0 \to C_0')\right) = F(\mathrm{cofib}(\mathrm{id}_{C_0'})) = 0$$

since the colimit diagram has a terminal object. □

*Remark.* The general formula for left Kan extensions also tells us that

$$\mathrm{Lan}_{j^*} F\,(C_1) = \operatorname*{colim}_{(C \in \mathcal{C},\, \alpha : j^*(C) \to C_1)} F(C)\ , \tag{3.8}$$

but it is more difficult to use since this requires knowledge about morphisms $\alpha$ in $\mathcal{C}_1$.

**Corollary 3.2.23** ([NS18, I.3.3])**.** The mapping space between objects $C_1, C_1'$ in $\mathcal{C}_1 = \mathcal{C}/_{\mathcal{C}_0}$ represented by $\hat{C}_1, \hat{C}_1'$ in $\mathcal{C}$ can be calculated as

$$\mathrm{Map}_{\mathcal{C}/_{\mathcal{C}_0}}(C_1, C_1') \simeq \operatorname*{colim}_{C_0 \in \mathcal{C}_{0\,/\hat{C}_1'}} \mathrm{Map}_{\mathcal{C}}(\hat{C}_1, \mathrm{cofib}(C_0 \to \hat{C}'_1))\ . \tag{3.9}$$

*Proof.* Applying the last Proposition 3.2.22, we need to show that

$$\mathrm{Map}_{\mathcal{C}/_{\mathcal{C}_0}}(C_1, -) \cong \mathrm{Lan}_{j^*}\, \mathrm{Map}_{\mathcal{C}}(\hat{C}_1, -) \tag{3.10}$$

as functors $\mathcal{C}/_{\mathcal{C}_0} \to \mathcal{S}$, in other words $\mathrm{Lan}_{j^*}$ intertwines the respective (dual) Yoneda embeddings. We apply the Yoneda Lemma in this functor category:

$$\mathrm{Nat}(\mathrm{Lan}_{j^*} \mathrm{Map}_{\mathcal{C}}(\hat{C}_1, -), F) \cong \mathrm{Nat}(\mathrm{Map}_{\mathcal{C}}(\hat{C}_1, -), F \circ j^*) \cong F(C_1) \tag{3.11}$$

for any $F : \mathcal{C}/_{\mathcal{C}_0} \to \mathcal{S}$, so since $\mathrm{Lan}_{j^*} \mathrm{Map}_{\mathcal{C}}(\hat{C}_1, -)$ is uniquely determined by this property, it can only depend on the class $C_1$ of $\hat{C}_1$. Our aim is to show that the factorization

$$C_1 \in \mathcal{C}/_{\mathcal{C}_0}^{\;op} \mapsto \mathrm{Lan}_{j^*} \mathrm{Map}_{\mathcal{C}}(\hat{C}_1, -) \in \mathrm{Fun}\left(\mathcal{C}/_{\mathcal{C}_0}, \mathcal{S}\right)$$

agrees with the Yoneda embedding, which is immediate from the universal property in Equation 3.11. □

## 3.3 Poincaré-Verdier Sequences

Let us translate this into the context of Poincaré $\infty$-categories, obtaining a long exact sequence of L-groups.

**Definition 3.3.1** ([CDH+20b, 1.1.5]). A *Poincaré-Verdier sequence* is a sequence of Poincaré $\infty$-categories and duality-preserving functors $(\mathcal{C}, \Upsilon) \xrightarrow{i_*} (\mathcal{D}, \Phi) \xrightarrow{j^*} (\mathcal{E}, \Psi)$ such that the underlying sequence of stable $\infty$-categories is a Verdier sequence, and additionally

- The canonical transformation $\Upsilon \Rightarrow i^*\Phi = \Phi \circ i^{*,op}$ that is part of the datum of a hermitian functor is an isomorphism, and
- The canonical transformation $\Phi \Rightarrow \Psi \circ j^{*,op}$ exhibits $\Psi$ as the left Kan extension $\mathrm{Lan}_{j^{*,op}} \Phi$.

It is called *split*, or *Poincaré recollement*, if the underlying Verdier sequence is split. In this case, the second condition is equivalent to the composition $\Phi \circ j_+ \to \Psi \circ j^* j_+ \to \Psi$ being an equivalence, because $j_+ \dashv j^*$ implies $\mathrm{Lan}_{j^{*,op}} = - \circ j_*^{op} \dashv - \circ j^{*,op}$.

*Remark.* In the non-split case, we can use the opposite version of 3.2.22 to rewrite

$$\Psi(C_1) \cong \mathrm{Lan}_{j^{*,op}} \Phi(C_1) \cong \operatorname*{colim}_{(C_0,\, \alpha: i_*C_0 \to \hat{C}_1) \in (\mathcal{C}_{0\,\hat{C}_1/})^{op}} \Phi(\mathrm{fib}(i_*C_0 \to \hat{C}_1)) \tag{3.12}$$

for $C_1 = j^*\hat{C}_1$. This expression should be handled with much care: The left Kan extension is explicitly along $p^{op}$, so we must work in the Pro-completion $\mathrm{Ind}(\mathcal{C})^{op} = \mathrm{Pro}(\mathcal{C})^{op}$ to calculate it (note $\mathcal{C}^{op}$ is still stable). The fiber in above expression is taken in $\mathcal{C}$, while the colimit is parametrized by arrows in $(\mathcal{C}_{0\,\hat{C}_1/})^{op} = (\mathcal{C}_0^{op})_{/\hat{C}_1}$.

*Warning.* Even though $\Phi \circ j_+ \simeq \Psi$ in the split case, the functor $j_+$ is usually not duality-preserving. If it were, this would mean

$$j_* \cong (j_+)^! = D_\Phi \circ j_+ \circ D_\Psi^{op} \cong D_\Phi^2 \circ j_+ \cong j_+ \tag{3.13}$$

so the adjunction $j_* \dashv j^* \dashv j_*$ becomes two-sided.

**Proposition 3.3.2** ([Lur11, Lecture 8, Proposition 6]). Let $(\mathcal{D}, \Phi)$ be a Poincaré $\infty$-category, and $\mathcal{C}$ a stable subcategory. If $\mathcal{C}$ is closed under duality $D_\Upsilon$, then

- The restriction $\Upsilon|_\mathcal{C}$ automatically makes $\mathcal{C}$ into a Poincaré $\infty$-category, and
- The left Kan extension $\mathrm{Lan}_{j^*} \Upsilon$ along the projection $j^* : \mathcal{D} \to \mathcal{D}/\mathcal{C}$ makes this Verdier quotient into a Poincaré $\infty$-category

so that $(\mathcal{C}, \Phi|_\mathcal{C}) \to (\mathcal{D}, \Phi) \to (\mathcal{C}/\mathcal{D}, \mathrm{Lan}_{j^*} \Phi)$ becomes a Poincaré-Verdier sequence.

*Remark.* In fact, quadratic functors can be restricted or left Kan extended along arbitrary functors, as shown in [CDH+20a, Section 1.4]. We do not give a proof, since it mostly consists of straightforward but not very enlightening calculations.

**Corollary 3.3.3.** If $(\mathcal{C}, \Upsilon) \xrightarrow{i_*} (\mathcal{D}, \Phi) \xrightarrow{j^*} (\mathcal{E}, \Psi)$ is a split Poincaré-Verier sequence, then the adjoint sequences

$$(\mathcal{E}, \Psi) \xrightarrow{j_+} (\mathcal{D}, \Phi) \xrightarrow{i^*} (\mathcal{C}, \mathrm{Lan}_{i^{*,op}} \Phi)$$
$$(\mathcal{E}, \Phi \circ j_*) \xrightarrow{j_*} (\mathcal{D}, \Phi) \xrightarrow{i^-} (\mathcal{C}, \Upsilon)$$

are right and left split Poincaré-Verdier sequences.

*Proof.* By 3.2.10 they are Verdier sequences, and as explained in the definition of Poincaré-Verdier sequences $\Psi \cong \Phi \circ j_+$. Similar to the reasoning there, $\mathrm{Lan}_{i^{-,op}} \cong (-\circ i_*^{op})$ since both functors are left adjoint to $(- \circ i^{-,op})$ and we are finished by applying the last Proposition. $\square$

**Definition 3.3.4.** For $F : (\mathcal{C}, \Upsilon) \to (\mathcal{D}, \Phi) : G$ any functor between stable $\infty$-categories with dualities, let $F^!$ be its dual functor

$$F^! := D_\Phi \circ F^{op} \circ D_\Upsilon^{op} : \mathcal{C} \to \mathcal{D} \; . \tag{3.14}$$

*Remark.* We will denote $(f_*)^! := f_!$ and $(f^*)^! := f^!$ in the rest of the text.

**Proposition 3.3.5.** Let $F : (\mathcal{C}, \Upsilon) \to (\mathcal{D}, \Phi) : G$ be an arbitrary functor between stable $\infty$-categories with dualities, and $G$ a right adjoint to $F$. Then, $G^!$ is left adjoint to $F^!$.

*Proof.* We use the fact that $D_\Upsilon$ and $D_\Phi$ are anti-autoequivalences and their own inverses, in particular $D_\Upsilon \dashv D_\Upsilon^{op}$ and $D_\Phi^{op} \dashv D_\Phi$. For $C \in \mathcal{C}$ and $D \in \mathcal{D}$,

$$\begin{aligned}\mathrm{Map}_{\mathcal{C}}(G^!(D), C) &= \mathrm{Map}_{\mathcal{C}}(D_\Upsilon G^{op} D_\Phi^{op}(D), C) \cong \mathrm{Map}_{\mathcal{C}^{op}}(G^{op} D_\Phi^{op}(D), D_\Upsilon^{op}(C)) \cong \\ &\cong \mathrm{Map}_{\mathcal{C}}(D_\Upsilon(C), G D_\Phi(D)) \cong \mathrm{Map}_{\mathcal{D}}(F D_\Upsilon(C), D_\Phi(D)) \cong \\ &\cong \mathrm{Map}_{\mathcal{D}}(D, D_\Phi F^{op} D_\Upsilon^{op}(C)) = \mathrm{Map}_{\mathcal{D}}(D, F^!(C))\end{aligned}$$

$\square$

*Remark.* Note that we have not actually used anything about $D_\Upsilon$ and $D_\Phi$ except for the two adjunctions. Hence, our statement holds more generally.

**Corollary 3.3.6.** Let $F : (\mathcal{C}, \Upsilon) \to (\mathcal{D}, \Phi)$ be a duality-preserving functor between stable $\infty$-categories with dualities. Then, $F$ admits a left adjoint iff it admits a right adjoint.

*Proof.* In 3.3.5, note that if $F^{op} \circ D_\Upsilon \cong D_\Phi \circ F$ since $F$ is duality-preserving, then $F^! \cong D_\Upsilon^{op} D_\Upsilon F \cong F$. $\square$

**Corollary 3.3.7.** A Poincaré-Verdier sequence $(\mathcal{C}, \Qoppa) \xrightarrow{i_*} (\mathcal{D}, \Phi) \xrightarrow{j^*} (\mathcal{E}, \Psi)$ is split iff one of the adjoints $j_*, j_+, i^*, i^-$ exists.

*Proof.* Since by assumption $i_*$ and $j^*$ are duality-preserving, combine the previous Corollary with 3.2.7 to construct the remaining adjoints. □

**Proposition 3.3.8.** (9-Lemma) Given a commutative diagram of stable (Poincaré) ∞-categories of the form

$$\begin{array}{ccccc}
\mathcal{C}_0 & \xrightarrow{i_*^0} & \mathcal{D}_0 & \xrightarrow{j_0^*} & \mathcal{E}_0 \\
\downarrow{\scriptstyle i_*^{\mathcal{C}}} & & \downarrow{\scriptstyle i_*^{\mathcal{D}}} & & \vdots\downarrow{\scriptstyle i_*^{\mathcal{E}}} \\
\mathcal{C} & \xrightarrow{i_*} & \mathcal{D} & \xrightarrow{j^*} & \mathcal{E} \\
\downarrow{\scriptstyle j_{\mathcal{C}}^*} & & \downarrow{\scriptstyle j_{\mathcal{D}}^*} & & \vdots\downarrow{\scriptstyle j_{\mathcal{E}}^*} \\
\mathcal{C}_1 & \xrightarrow{i_*^1} & \mathcal{D}_1 & \xrightarrow{j_1^*} & \mathcal{E}_1
\end{array}$$

where the rows and the first two columns are (split) (Poincaré-)Verdier sequences. Further, suppose that the any morphism $f : C \to D_0$ in $\mathcal{D}$, with $C \in \mathcal{C}$ and $D_0 \in \mathcal{D}_0$, can be factored as $C \to C_0 \to D_0$ for some $\mathcal{C}_0 \in \mathcal{C}_0$.Then, the dashed arrows determined by functoriality of the Verdier quotient make the right column into a (split) (Poincaré-)Verdier sequence. Similarly, given a commutative diagram

$$\begin{array}{ccccc}
\mathcal{C}_0 & \xrightarrow{i_*^0} & \mathcal{D}_0 & \xrightarrow{j_0^*} & \mathcal{E}_0 \\
\vdots\downarrow{\scriptstyle i_*^{\mathcal{C}}} & & \downarrow{\scriptstyle i_*^{\mathcal{D}}} & & \downarrow{\scriptstyle i_*^{\mathcal{E}}} \\
\mathcal{C} & \xrightarrow{i_*} & \mathcal{D} & \xrightarrow{j^*} & \mathcal{E} \\
\vdots\downarrow{\scriptstyle j_{\mathcal{C}}^*} & & \downarrow{\scriptstyle j_{\mathcal{D}}^*} & & \downarrow{\scriptstyle j_{\mathcal{E}}^*} \\
\mathcal{C}_1 & \xrightarrow{i_*^1} & \mathcal{D}_1 & \xrightarrow{j_1^*} & \mathcal{E}_1
\end{array}$$

where the rows and the last two columns are (split) (Poincaré-)Verdier sequences and assume that any morphism $g : D_0 \to C$ factors as $D_0 \to C_0 \to C$ for some $C_0 \in \mathcal{C}_0$. Then, the dashed arrows determined by functoriality of the kernel make the left column into a (split) (Poincaré-)Verdier sequence.

*Remark.* As will become clear in the proof, we can exchange the directions of both factorization conditions, i.e. in the first case it is sufficient if any morphism $f : D_0 \to C$ factors through a $C_0 \in \mathcal{C}_0$. Since they arise in different ways in the proof, we have already stated them in different directions. In fact, both factorization conditions follow immediately if suitable splittings (i.e. adjoints) exist making the upper left square Beck-Chevalley in either direction since we can then use those adjoints to factor morphisms; and the first condition also follows if the upper right square is Beck-Chevalley for simpler reasons.

*Remark.* The factorization conditions in this theorem seem weird when comparing with the analogous statement in abelian categories, but they are also present in isomorphism theorems for triangulated categories – see [Orl04, 1.3] or [KS13, 1.6.10]. In our setting the proof is however quite a bit more difficult, since it is harder to get a grip on the morphism spaces. The places where this condition enters in the first claim are the fully faithfulness of $i_*^{\mathcal{E}}$ and the condition on the quadratic functor on $\mathcal{E}_0$. The rest of the result actually still holds if we leave it out, and even more:

In the first diagram, choose a representant $D_0 \in \mathcal{D}_0$ of $E_0 \in \ker(i_*^{\mathcal{E}})$, i.e. $j_0^*(D_0) = E_0$. Then, $j^* \circ i_*^{\mathcal{D}}(D_0) = i_*^{\mathcal{E}} E_0 = 0$, so that $i_*^{\mathcal{D}}(D_0)$ must come from an element $C \in \mathcal{C}$ since it becomes 0 when localizing (see 3.1.2). But then $i_*^1 \circ j_{\mathcal{C}}^*(C) = j_{\mathcal{D}}^* \circ i_*(C) = j_{\mathcal{D}}^* i_*^{\mathcal{D}}(D_0) = 0$, and since $i_*^1$ is fully faithful, $j_{\mathcal{C}}^*(C) = 0$, so $C = i_*^{\mathcal{C}}(C_0)$ for a $C_0 \in \mathcal{C}_0$. It is now clear that $i_*^{\mathcal{D}} \circ i_*^0(C_0) = i_*^0(D_0)$ so since $i_*^{\mathcal{D}}$ is fully faithful, $D_0 = i_*^0(C_0)$. But this means $E_0 = j_0^* i_*^0(C_0) = 0$.

Hence, let $\alpha : E_0 \to E_0'$ be a morphism in $\mathcal{E}_0$, then $\alpha$ is an isomorphism iff $\mathrm{fib}(\alpha) \cong 0$ iff $i_*^{\mathcal{E}}(\mathrm{fib}(\alpha)) = \mathrm{fib}\, i_*^{\mathcal{E}}(\alpha) = 0$ (as $i_*^{\mathcal{E}}$ is exact) iff $i_*^{\mathcal{E}}(\alpha)$ is an isomorphism – this means that $i_*^{\mathcal{E}}$ is conservative. Without the factorization condition we are however not able to show that it is fully faithful.

In the second claim, a similar diagram chase shows that while $j_0^*$ might without the factorization not be a Verdier projection, it is still essentially surjective.

*Remark.* In the case of Verdier sequences, the first claim is a categorification of the third isomorphism theorem: Among other points, it entails that

$$\left(\mathcal{D}/\mathcal{D}_0\right) \Big/ \left(\mathcal{C}/\mathcal{C}_0\right) \simeq \left(\mathcal{D}/\mathcal{C}\right) \Big/ \left(\mathcal{D}_0/\mathcal{C}_0\right). \tag{3.15}$$

*Proof.* We only prove the first claim, the second is similar. Let us begin with the case of Verdier sequences. The dashed arrows exist and are exact functors since Verdier quotients are cofibers in $\mathrm{Cat}_\infty^{ex}$, and cofibers are functorial – alternatively, use the universal property of the localization as in [NS18, I.3.3].

To show that $i_*^{\mathcal{E}}$ is *fully faithful*, we need to check the following equivalence of mapping spaces, using the formula from 3.2.23:

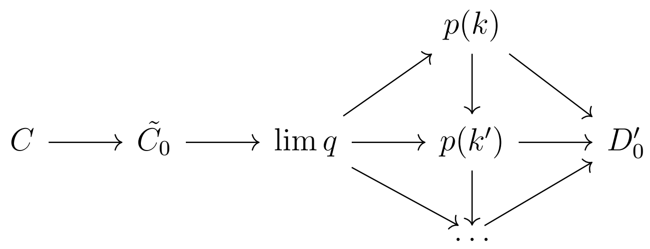
$$\begin{aligned}
\mathrm{Map}_{\mathcal{E}_0}(j_0^*(D_0), j_0^*(D_0')) &\simeq \operatorname*{colim}_{C_0 \in \mathcal{C}_{0\,/D_0'}} \mathrm{Map}_{\mathcal{D}_0}(D_0, \mathrm{cofib}(i_*^0 C_0 \to D_0')) \overset{!}{\simeq} \\
\simeq \mathrm{Map}_{\mathcal{E}}(i_*^{\mathcal{E}} j_0^* D_0, i_*^{\mathcal{E}} j_0^* D_0') &= \operatorname*{colim}_{C \in \mathcal{C}_{/i_*^{\mathcal{D}} D_0'}} \mathrm{Map}_{\mathcal{D}}(i_*^{\mathcal{D}} D_0, \mathrm{cofib}(i_* C \to i_*^{\mathcal{D}} D_0'))
\end{aligned}$$

Since $i_*^{\mathcal{D}}$ is fully faithful and exact, all we need to show is that in the second colimit, if suffices to only regard those $C$ that lie in the essential image of $i_*^{\mathcal{C}}$. In other words, we claim that the inclusion $\mathcal{C}_{0\,/D_0'} \to \mathcal{C}_{/i^0_* D_0'}$ is right cofinal. Applying Quillen's Theorem A 1.2.15, we need to show that for each $(C \to i^0_* D_0') \in \mathcal{C}_{/i^0_* D_0'}$, the simplicial set

$$\mathcal{C}_{0,\,C//D_0'} := \mathcal{C}_{0\,/D_0'} \times_{\mathcal{C}_{/i^0_* D_0'}} \left(\mathcal{C}_{/i^0_* D_0'}\right)_{(C \to i^0_* D_0')/}$$

is weakly contractible. We will show that it is cofiltered, in the sense that for any diagram $p : K \to \mathcal{C}_{0,\,C//D_0'}$ with $K$ a finite simplicial set, there exists an extension $\bar{p} : K^{\triangleleft} \to \mathcal{C}_{0,\,C//D_0'}$. If the reversed factorization condition holds, we can similarly show that it is filtered.

Since $\mathcal{D}_0$ is stable, it admits finite limits so the diagram $q : K^{\triangleright} \to \mathcal{D}_0$ induced by $p$ that sends the cone point to $D_0'$, and the rest of $K$ to the underlying objects in $\mathcal{C}_0$ of its image under $p$ admits a limit $\lim q \in \mathcal{D}_0$. Further, $p$ induces a natural transformation from the constant functor on $C$ to $q$ and thus a map $f : C \to \lim q$, which we can factor as $C \to \tilde{C}_0 \to \lim q$ using the factorization condition. Unwinding our construction, $\tilde{C}_0$ as a cone point allows us to lift $p$ to $\bar{p}$. We pictorially summarize our argument in the following diagram, where the right two columns describe the diagram $q$:



Next, let us show that $j^*_{\mathcal{E}}$ is a *Verdier projection*. Regard $\mathcal{C}$ and $\mathcal{D}_0$ as full subcategories of $\mathcal{D}$, and denote by $W_{\mathcal{D}_0}, W_{\mathcal{C}}, W_{\mathcal{C}_1}$ the classes of morphisms in $\mathcal{D}$ with fiber in $\mathcal{D}_0, \mathcal{C}$ or $\mathcal{C}_1$ respectively. We want to show that the composite maps $l_1 : \mathcal{D} \to \mathcal{D}_1 \to \mathcal{E}_1$ and $l_2 : \mathcal{D} \to \mathcal{E} \to \mathcal{E}/\mathcal{E}_0$ satisfy the same universal property, namely they both localize $\mathcal{D}$ at $W_{\mathcal{D}_0} \cup W_{\mathcal{C}}$.

Let $\mathcal{A}$ be another stable $\infty$-category. Precomposition with $\mathcal{D}_1 \to \mathcal{E}_1$ exhibits $\mathrm{Fun}(\mathcal{E}_1, \mathcal{A})$ as the full subcategory of $\mathrm{Fun}(\mathcal{D}_1, \mathcal{A})$ on the functors that send morphisms in $W_{\mathcal{C}_1}$ to isomorphisms. Therefore, precomposition with $l_1$ exhibits the former as the full subcategory of $\mathrm{Fun}(\mathcal{D}, \mathcal{A})$ on functors $F$ that send $W_{\mathcal{D}_0}$ to isomorphisms and, after being factored through $\mathcal{D}_1$, send $W_{\mathcal{C}_1}$ to isomorphisms.

If $F$ already sends $W_{\mathcal{C}}$ to isomorphisms, the latter is automatic by commutativity of the lower left square. Conversely, if the factorization of $F$ through $\mathcal{D}_1$ sends $W_{\mathcal{C}_1}$ to isomorphisms, note that any $f \in W_{\mathcal{C}}$ induces $j^*_{\mathcal{D}}(f)$ with $\mathrm{fib}(j^*_{\mathcal{D}} f) = j^*_{\mathcal{C}} \mathrm{fib}(f) \in \mathcal{C}_1$ since Verdier localizations are exact (see 3.2.2) and $j^*_{\mathcal{C}}$ is the (co-)restriction of $j^*_{\mathcal{D}}$ to the respective full subcategories; so $f$ is finally sent to an isomorphism. Using a similar argument for $l_2$, we are finished.

Next, let us show that if all involved sequences are split, the dotted sequence is so as well. This amounts to constructing left and right adjoints to the dashed arrows, which is straightforward by using commutativity of the right squares: For example, $i_*^{\mathcal{E}} \circ j_0^* \cong j^* \circ i_*^{\mathcal{D}}$ means that $i_*^{\mathcal{E}} \cong j^* \circ i_*^{\mathcal{D}} \circ j_+^0$ since $j_0^* \circ j_+^0 \cong \mathrm{id}_{\mathcal{E}_0}$ because $j_+^0$ is fully faithful, so a left adjoint is given by the composition $j_0^* \circ i_{\mathcal{D}}^- \circ j_*$.

Finally, we want to show that if all involved sequences are Poincaré, then the dotted sequence is as well. This amounts to showing that, for $\Qoppa$ the quadratic functor of $\mathcal{D}$,

$$\left(\mathrm{Lan}_{\mathcal{D}}^{\mathcal{E}} \Qoppa\right)|_{\mathcal{E}_0} \cong \mathrm{Lan}_{\mathcal{D}_0}^{\mathcal{E}_0} (\Qoppa|_{\mathcal{D}_0})$$
$$\mathrm{Lan}_{\mathcal{E}}^{\mathcal{E}} \mathrm{Lan}_{\mathcal{D}}^{\mathcal{E}} \Qoppa \cong \mathrm{Lan}_{\mathcal{D}_1}^{\mathcal{E}_1} \mathrm{Lan}_{\mathcal{D}}^{\mathcal{D}_1} \Qoppa$$

The latter follows from transitivity of Kan Extensions and commutativity of the lower right square; the former relies on equation 3.12 to write, for $E_0 \in \mathcal{E}_0$ with $E_0 = j^*(D_0)$ for some $D_0 \in \mathcal{D}$:

$$\mathrm{Lan}_{\mathcal{D}}^{\mathcal{E}} \Qoppa(E_0) \cong \operatorname*{colim}_{D_0 \to i_*(C)} \Qoppa(\mathrm{fib}(D_0 \to C))$$
$$\mathrm{Lan}_{\mathcal{D}_0}^{\mathcal{E}_0}(E_0)\,(\Qoppa|_{\mathcal{D}_0}) \cong \operatorname*{colim}_{D_0 \to i_*^0(C_0)} \Qoppa(\mathrm{fib}(D_0 \to C_0))$$

This follows by the same cofinality argument we had used to compare the mapping spaces in the beginning. □

Now, for the main reason we went through all these definitions:

**Theorem 3.3.9** ([CDH+20b, 4.4.6])**.** Given a Poincaré-Verdier sequence $(\mathcal{C}, \Qoppa) \to (\mathcal{D}, \Phi) \to (\mathcal{E}, \Psi)$, the by functoriality of $\mathbb{L}$ for duality-preserving functors associated sequence of L-spectra

$$\mathbb{L}(\mathcal{C}, \Qoppa) \longrightarrow \mathbb{L}(\mathcal{D}, \Phi) \longrightarrow \mathbb{L}(\mathcal{E}, \Psi) \tag{3.16}$$

is a fiber sequence of spectra. In particular, we obtain a long exact sequence of L-groups

$$\begin{array}{l} \dots \longrightarrow L_1(\mathcal{C}, \Qoppa) \longrightarrow L_1(\mathcal{D}, \Phi) \longrightarrow L_1(\mathcal{E}, \Psi) \\ \hookrightarrow L_0(\mathcal{C}, \Qoppa) \longrightarrow L_0(\mathcal{D}, \Phi) \longrightarrow L_0(\mathcal{E}, \Psi) \\ \hookrightarrow L_{-1}(\mathcal{C}, \Qoppa) \to L_{-1}(\mathcal{D}, \Phi) \to L_{-1}(\mathcal{E}, \Psi) \to \dots \end{array}$$

**Construction 3.3.10** ([CDH+20b], 4.4.7)**.** One can explicitly describe the value of the boundary operator in this long exact sequence on an $n$-dimensional Poincaré object $(E, q) \in L^n(\mathcal{E}, \Psi)$. For simplicity, choose $n = 0$ since other cases can be obtained by shifting the quadratic functor.

- First, since $j^*$ is essentially surjective, represent if as $E \cong j^* D'$ with $D' \in \mathcal{D}$.
- Apply equation 3.12 to write

$$\Psi(E) = \operatorname*{colim}_{(C_0, \alpha: i_* C_0 \to D') \in (\mathcal{C}_{0\ D'/})^{op}} \Phi(\mathrm{fib}(i_* C_0 \to D')) \ .$$

  Since $q$ is a point of this space, it must come from some $\hat{q} \in \Phi(\mathrm{fib}(i^* C_0 \to D')) = \Phi(D)$ with $D := \mathrm{fib}(i^* C \to D') \in \mathcal{D}$.

- If the considered Poincaré-Verdier sequence is split, we might simply choose $D = j_* E$ with $\hat{q}$ the image of $q$ under the natural isomorphism $\Phi \circ j_* \cong \Psi$.
- While $(D, \hat{q})$ is generally not a Poincaré object (even in the split case, $j_*$ is generally not duality-preserving), $\hat{q}$ induces a map $\hat{q}_\sharp : D \to D_\Phi D$. The cofiber of this map is self-dual up to a shift:

$$D_\Phi(\mathrm{cofib}\, \hat{q}_\sharp) \cong \mathrm{fib}\left( D_\Phi(D \xrightarrow{\hat{q}_\sharp} D_\Phi D) \right) = \mathrm{fib}(D_\Phi^2 D \to D_\Phi D) \cong \mathrm{cofib}(D \xrightarrow{\hat{q}_\sharp} D_\Phi D)[-1]$$

  after noticing $D\hat{q}_\sharp \cong p_\sharp$ composing with the biduality isomorphism, by its construction from $\hat{q}$. Using some gymnastics for quadratic functors (related to algebraic surgery/ the algebraic Thom-isomorphism as in [CDH+20b, Section 2.4]), one can show that this self-duality is induced by a quadratic form $q_c \in \Phi^{[1]}(\mathrm{cofib}\, \hat{q}_\sharp)$.

What the reference shows is that the boundary map $L^n(\mathcal{E}, \Psi) \to L^{n-1}(\mathcal{C}, \Qoppa)$ sends the class $[(E, q)]$ to the class represented by the Poincaré object $(\mathrm{cofib}\, \hat{q}_\sharp, q_c)$ we have just constructed.

## 3.4 Orthogonal Decompositions

**Proposition 3.4.1.** Let $\mathcal{C}$ be an $\infty$-category that admits finite limits and is the recollement of a closed subcategory $\mathcal{C}_0$ and an open subcategory $\mathcal{C}_1$ via the reflections $L_0$ and $L_1$. Suppose that the same functors also exhibit $\mathcal{C}$ as a recollement of $\mathcal{C}_1$ as closed, and $\mathcal{C}_0$ as open subcategory. Then, $L_0$ and $L_1$ induce an equivalence of categories $\mathcal{C} \simeq \mathcal{C}_0 \times \mathcal{C}_1$.

*Proof.* We claim that the functor $\mathcal{C} \to \mathcal{C}_0 \times \mathcal{C}_1$ mapping $C \mapsto (L_0 C, L_1 C)$ is an equivalence with inverse $(C_0, C_1) \mapsto C_0 \times C_1 \in \mathcal{C}$. The compositions of these functors map

$$(C_1, C_2) \mapsto (L_0(C_0 \times C_1), L_1(C_0 \times C_1)) \cong (C_0 \times *, * \times C_1) \cong (C_0, C_1) \tag{3.17}$$

and $C \mapsto L_0 C \times L_1 C$. The natural transformation $\alpha$ from $C$ to $L_0 C \times L_1 C$ given by the unit maps yields an isomorphism when applying $L_0$ or $L_1$, as $L_0 C \cong L_0 L_0 C \times L_0 L_1 C$. Since $L_0$ and $L_1$ are jointly conservative, $\alpha$ is a natural isomorphism as well. $\square$

*Remark.* Under $\mathcal{C} \simeq \mathcal{C}_0 \times \mathcal{C}_1$, the inclusions $i_0, i_1$ correspond to the functors $C_0 \mapsto (C_0, *)$ and $C_1 \mapsto (*, C_1)$.

**Corollary 3.4.2.** Let $\mathcal{C}_0 \xrightarrow{i_*} \mathcal{C} \xrightarrow{j^*} \mathcal{C}_1$ be a split Verdier sequence, so that the reversed sequence $\mathcal{C}_1 \xrightarrow{j_*} \mathcal{C} \xrightarrow{i^*} \mathcal{C}_0$ is also split Verdier. Then, $i^*$ and $j^*$ induce an equivalence $\mathcal{C} \simeq \mathcal{C}_0 \times \mathcal{C}_1$.

*Proof.* By 3.2.9, $\mathcal{C}$ is a stable recollement of $\mathcal{C}_0$ and $\mathcal{C}_1$, and similarly with the roles reversed. Therefore, we can apply 3.4.1 □

**Corollary 3.4.3.** If $\mathcal{C}_0 \subseteq \mathcal{C}$ is a reflective and coreflective stable subcategory of a stable $\infty$-category $\mathcal{C}$, and ${}^{\perp}\mathcal{C}_0$ and $\mathcal{C}_0^{\perp}$ coincide, then we can factor $\mathcal{C} \simeq \mathcal{C}_0 \times \mathcal{C}_0^{\perp}$.

*Proof.* By 3.2.12, we know $\mathcal{C}$ is a recollement of $\mathcal{C}_0$ and $\mathcal{C}_0^{\perp}$, in particular there are reflections $L_0 : \mathcal{C} \to \mathcal{C}_0$ and $L_1 : \mathcal{C} \to \mathcal{C}_0^{\perp}$ that are left exact and jointly conservative, such that $L_1\mathcal{C}_0 = 0$. We are finished if we can show $L_0(\mathcal{C}_0^{\perp}) = 0$, since then $\mathcal{C}$ is also a recollement of $\mathcal{C}_0^{\perp}$ and $\mathcal{C}$ with roles reversed, meaning we can apply 3.4.1. But this follows by the Yoneda lemma since for each $C_0 \in \mathcal{C}_0$ and $\mathcal{C}_1 \in \mathcal{C}_0^{\perp}$,

$$\mathrm{Map}_{\mathcal{C}_0}(L_0 C_1, C_0) = \mathrm{Map}_{\mathcal{C}}(C_1, C_0) = 0 \tag{3.18}$$

because $\mathcal{C}_0^{\perp} = {}^{\perp}\mathcal{C}_0$ by assumption. □

We develop an analogous statement for split Poincaré-Verdier sequences.

**Definition 3.4.4.** For $(\mathcal{C}, \Omega)$ and $(\mathcal{E}, \Psi)$ two Poincaré $\infty$-categories, their product $\mathcal{C} \times \mathcal{E}$ also admits the structure of a Poincaré $\infty$-category using the smashed quadratic functor $\Omega \wedge \Psi : \mathcal{C}^{op} \times \mathcal{E}^{op} \to \mathcal{S}p$ defined as the composition

$$\mathcal{C}^{op} \times \mathcal{E}^{op} \xrightarrow{\Omega \times \Psi} \mathcal{S}p \times \mathcal{S}p \xrightarrow{\oplus} \mathcal{S}p\ . \tag{3.19}$$

*Proof.* The associated polarization is

$$B_{\Omega \oplus \Psi}((C, E), (C', E')) = B_{\Omega}(C \oplus C') \oplus B_{\Psi}(E, E') \tag{3.20}$$

which is clearly still bilinear, and represented by the exact duality functor

$$D_{\Omega \oplus \Psi}(C, E) = (D_{\Omega}(C), D_{\Psi}(E)) \tag{3.21}$$

satisfying biduality, since the individual duality functors do. Finally,

$$\Lambda_{\Omega \oplus \Psi}(C, E) = \mathrm{fib}\left(\Omega(C) \oplus \Psi(E) \to (B_{\Omega}(C, C) \oplus B_{\Psi}(E, E)^{hS_2})\right) \cong \Lambda_{\Omega}(C) \oplus \Lambda_{\Psi}(E)$$

is still exact, so we have checked everything. □

*Remark.* This operation is both product and coproduct in $\mathcal{C}at_\infty^p$.

**Proposition 3.4.5.** If $(\mathcal{C}, \Omega) \xrightarrow{i_*} (\mathcal{D}, \Phi) \xrightarrow{j^*} (\mathcal{E}, \Psi)$ is a split Poincaré-Verdier sequence such that the reversed sequence $(\mathcal{E}, \Psi) \xrightarrow{j_*} (\mathcal{D}, \Phi) \xrightarrow{i^*} (\mathcal{C}, \Omega)$ is also split Poincaré-Verdier, then we can decompose

$$(\mathcal{D}, \Phi) \simeq (\mathcal{C} \times \mathcal{E}, \Omega \oplus \Psi) \tag{3.22}$$

such that $i_*, i^*, j_*, j^*$ are the canonical embeddings into and projections out of this product.

*Proof.* By 3.4.1, we know that on the underlying split Verdier sequences, $(i^*, j^*) : \mathcal{D} \to \mathcal{C} \times \mathcal{E}$ is an equivalence exhibiting $i_*, i^*, j_*, j^*$ as the canonical maps. In particular, any $D \in \mathcal{D}$ is isomorphic to $i_*i^*D \oplus j_*j^*D$. We only need to show that $(\Omega \oplus \Psi) \circ (i^*, j^*) \cong \Phi$, then we are finished. But

$$\begin{aligned}(\Omega \oplus \Psi) \circ (i^*, j^*)(D) = \Omega \circ i^* \oplus \Psi \circ j^*(i_*i^*D \oplus j_*j^*D) \cong \Omega(i^*D) \oplus \Psi(j^*D) \cong \\ \cong \Phi(i_*i^*D) \oplus \Phi(j_*j^*D) \oplus B_\Phi(i_*i^*D, j_*j^*D) \cong \Phi(i_*i^*D \oplus j_*j^*D) = \Phi(D)\end{aligned}$$

where we use that $j^*i_* = 0$ and $i^*j_*$ since both are the composites of Verdier sequences, and $B_\Phi(i_*i^*D, j_*j^*D) \cong \mathrm{Map}(i_*i^*D, D_\Phi j_*j^*D) \cong \mathrm{Map}(j^*i_*i^*D, D_\Psi j^*D) = 0$ since $j_*$ is duality preserving. $\square$

*Remark.* Using 3.3.5, we know that $i_! \dashv i^!$ and $j_! \dashv j^!$; but given our assumption in the last proposition that both directions are Poincaré-Verdier sequences, the functors $i_*, i^*, j_*, j^*$ must be duality-preserving. Hence, $i_! := D_\Phi \circ i_* \circ D_\Omega \cong D_\Phi^2 i_* \cong i_*$ and similarly for the others, so $i_* \dashv i^* \dashv i_*$ and $j_* \dashv j^* \dashv j_*$ are double-sided adjoints. This is not at all surprising, since by our proof they are just inclusions and projections into a biproduct, which always satisfy this property.

## 3.5 $P$-slicings and $P$-recollements

The above discussion has only involved splitting Poincaré $\infty$-categories into two components. Classically, semiorthogonal decompositions of triangulated categories can however consist of multiple subcategories. We translate this into our context since we could not find a full discussion in the literature (some results can be extracted from [FLM15]), and we will see a nice application ot this in 3.5.9.

Let us fix a parametrizing poset $P$, where $P = \{0 < 1\}$ corresponds to the case of two components.

**Definition 3.5.1.** A *slicing* of $P$ is a decomposition $P = P_- \sqcup P_+$ such that for every $p_- \in P_-$, $p_+ \in P_+$, we have $p_- < p_+$. The set $\mathcal{O}(P)$ of slicings of $P$ is partially ordered by setting $(P_-, P_+) \leq (P'_-, P'_+)$ iff $P_- \subseteq P'_-$ (and hence also $P_+ \supseteq P'_+$). Also, it has a minimal element $(\emptyset, P)$ and a maximal element $(P, \emptyset)$.

*Remark.* In this case, $P_-$ is downwards closed and $P_+$ is upwards closed.

**Lemma 3.5.2.** A slicing of $P$ is the same thing as an order-preserving map $P \to [1]$.

*Proof.* Given a slicing $(P_-, P_+)$ of $P$, we can define a map $f : P \to [1]$ sending all of $P_-$ to 0 and $P_+$ to 1. This is well-defined since $P_-, P_+$ are disjoint and cover all of $P$. Also, it is order preserving as we had assumed that $p_- < p_+$ for all $p_- \in P_-, p_+ \in P_+$.

Conversely, to any order-preserving map $f : P \to [1]$ we can associate a slicing $(f^{-1}(\{0\}), f^{-1}(\{1\}))$ of $P$ clearly satisfying $f^{-1}(\{0\}) \cap f^{-1}(\{1\}) = f^{-1}(\{0\} \cap \{1\}) = \emptyset$ and $f^{-1}(\{0\}) \sqcup f^{-1}(\{1\}) = f^{-1}([1]) = P$, as well as $p_- < p_+$ as above since $f$ is order-preserving. □

**Definition 3.5.3.** Let $\mathcal{C}$ be an $\infty$-category with finite limits. A *$P$-slicing* of $\mathcal{C}$ is a map that associates to every slicing $(P_-, P_+) \in \mathcal{O}(P)$ a pair of full subcategories $\mathcal{C}_{P_-}, \mathcal{C}_{P_+} \subseteq \mathcal{C}$ such that

- $\mathcal{C}$ is a recollement of $\mathcal{C}_{P_-}$ and $\mathcal{C}_{P_+}$,
- to $(\emptyset, P)$ and $(P, \emptyset)$ we associate the trivial recollements $\{*\}, \mathcal{C}$ and $\mathcal{C}, \{*\}$,
- and if $(P_-, P_+) \leq (P'_-, P'_+)$, then $\mathcal{C}_{P_-} \subseteq \mathcal{C}_{P'_-}$.

*Remark.* Again, of most interest is the case where $\mathcal{C}$ is stable. By 3.1.6, this implies that $\mathcal{C}_{P_+}, \mathcal{C}_{P_-}$ are stable subcategories and all involved functors are exact. In this setting, above notion was introduced in [FLM15, Chapter 6].

**Definition 3.5.4.** A *$P$-recollement* of an $\infty$-category $\mathcal{C}$ admitting finite limits consists of a collection of full subcategories $(\mathcal{C}_p)_{p \in P}$ such that:

- The inclusions $i_p : \mathcal{C}_p \hookrightarrow \mathcal{C}$ admit left exact left adjoints $L_p$, for all $p \in P$.
- For $C_p \in \mathcal{C}_p, C_q \in \mathcal{C}_q$ where $p, q \in P$ with $p < q$, the composition $L_q i_p = 0$ vanishes.
- The functors $(L_p)_{p \in P}$ are jointly conservative.

**Proposition 3.5.5.** If $P$ is finite, we may rephrase the last point by instead requiring that the smallest full subcategory of $\mathcal{C}$ closed under finite limits and containing every $\mathcal{C}_p$ is $\mathcal{C}$ itself.

*Proof.* For the *if* direction, if there was a morphism $\alpha$ in $\mathcal{C}$ such that all $L_p\alpha$ are isomorphisms, but $\alpha$ itself is not, then $L_p \operatorname{fib}(\alpha)$ for all $p$. Let $K$ denote the set of all objects in $\mathcal{C}$ with this property, then the true full subcategory of $\mathcal{C}$ spanned by objects that are not in $\mathcal{C}$ would violate above condition.

Conversely, if there were a true full subcategory $\mathcal{C}' \subseteq \mathcal{C}$ closed under finite limits and containing all of the $\mathcal{C}_p$, and $C \in \mathcal{C}$ were not in $\mathcal{C}$, then we map iteratively replace it by the finite limit $\mathrm{cofib}(C \to \lim_{p\in P} i_p L_p C)$ which is in the kernel of all $L_p$, violating joint conservativity. □

**Theorem 3.5.6.** For $P$ a finite poset, the following pieces of data are equivalent:

- An $\infty$-category $\mathcal{C}$ that admits all finite limits and is a $P$-recollement of $(\mathcal{C}_p)_{p\in P}$
- A functor $F : P^{op} \to \mathcal{C}at_\infty^{lex}$ into the non-full subcategory of $\mathcal{C}at_\infty$ spanned by $\infty$-categories that admit all finite limits, and left exact functors
- A $P$-slicing of an $\infty$-category $\mathcal{C}$ admitting finite limits

*Proof Sketch.* *(i)* ⇒ *(ii)*. If $\mathcal{C}$ is a $P$-recollement of $(\mathcal{C}_p)$ with functors denoted as above, and $p < q$, then we obtain a left exact functor $L_p \circ i_q : \mathcal{C}_q \to \mathcal{C}_p$. Also, $\mathcal{C}_p$ has all finite limits, as it is a reflective subcategory of a category that has them. We may thus assemble the required functor $F$ by $F(p) = \mathcal{C}_q$ and $F(q \geq p) = L_p \circ i_q$.

*(ii)* ⇒ *(i)*. Conversely, given such a functor $F : P^{op} \to \mathcal{C}at_\infty^{lex}$, let $\mathcal{C}$ be its lax limit. In other words, apply the Grothendieck construction to the underlying functor $F : P^{op} \to \mathcal{C}at_\infty$ to obtain a Cartesian fibration $C : \mathcal{M} \to P$, and define $\mathcal{C} := \mathrm{Fun}_P(P, \mathcal{M})$ as its space of sections. We now claim that $\mathcal{C}$ is a recollement of the categories $\mathcal{C}_p := F(p)$, with reflections $L_p$ given by evaluating a section $P \to \mathcal{M}$ at $p$, yielding an object of the fiber $\mathcal{M} \times_P \{p\} = \mathcal{C}_p$.

We only sketch this, the full proof is analogous to [Lur17, A.8.7]. The right adjoints $i_p : \mathcal{C}_p \to \mathcal{C}$ are given by constructing sections that consist only of terminal and $C$-Cartesian arrows, and the fact that they are fully faithful follows from the essential uniqueness of Cartesian lifts. From this construction, it is clear that $L_q i_p = *$ for $p < q$; and the fact that $L_p$ is left exact and $\mathcal{C}$ has finite limits follows from the fact that $F$ lands in $\mathrm{Cat}_\infty^{lex}$. Finally, the $L_p$ are jointly conservative since if $\alpha$ is a natural transformation between sections and $L_p(\alpha)$ are isomorphisms, then $\alpha$ is pointwise an isomorphism and therefore a natural isomorphism by [Lur18a, Tag 01DK].

*(ii)* ⇒ *(iii)*. Given any $P$-slicing $s : P \to [1]$, our goal is to base-change the functor $F : P^{op} \to \mathcal{C}at_\infty^{lex}$ along $s$. Informally, this should be done using a lax right Kan extension, so that

$$\mathcal{C} = \mathrm{laxlim}_{P^{op}}(F) \cong \mathrm{laxlim}_{[1]^{op}} \mathrm{laxRan}_{P^{op}}^{[1]^{op}}(F) \tag{3.23}$$

still decomposes $\mathcal{C}$. Just as a right Kan extension along a Cartesian fibration is calculated by taking the colimit over the fibers by [Lur18a, Tag 02ZM], we might expect the same formula to hold in the lax case, so we define $F' : [1]^{op} \to \mathcal{C}at_\infty^{lex}$ as the space of local sections $F'(0) := \mathrm{Fun}_P(P_-, \mathcal{M})$ on $P_- = s^{-1}(\{0\})$, and similarly for $P_+$. The required map $\mathrm{Fun}_P(P_+, \mathcal{M}) \to \mathrm{Fun}_P(P_-, \mathcal{M})$ is then a limit over the suitable contravariant transports. Concretely formalizing this proof is very heavy on combinatorics, we hope the reader is simplified with the simpler special case in 4.6.1.

*(iii)* ⇒ *(i)*. To any $p \in P$, we can associate two canonical $P$-slicings $(P_{\leq p}, P - P_{\leq p})$ and $(P_{<p}, P - P_{<p})$. By definition, they differ only by the side of the slicing that $p$ is on. If

we denote by $L_-^{\leq}$ the left exact reflection onto $\mathcal{C}_{P_{\leq p}}$ and similarly by $L_+^{<}$ the reflection onto $\mathcal{C}_{P-P_{<p}}$, define $\mathcal{C}_p := L_-^{\leq}\mathcal{C}_{P-P_{<p}} \subseteq \mathcal{C}$. This is a reflective subcategory with left exact reflection $L_p := L_-^{\leq}L_+^{<}$, and by definition for $q > p$ we have $L_q\mathcal{C}_p = 0$. It remains to show that together, the $L_p$ are jointly conservative, which is done by repeatedly slicing $P$ and reducing the statement to joint conservativity after localizing to $P_-$ and $P_+$, until by finiteness of $P$ we reach the case of [1] where the claim follows by assumption. □

**Definition 3.5.7.** A *$P$-decomposition* of a stable $\infty$-category $\mathcal{C}$ consists of a collection of stable subcategories $(\mathcal{C}_p)_{p\in P}$ such that:

- For $C_p \in \mathcal{C}_p, C_q \in \mathcal{C}_q$ where $p, q \in P$ with $p < q$, the mapping space $\mathrm{Map}_{\mathcal{C}}(C_p, C_q)$ is contractible.
- The smallest stable subcategory of $\mathcal{C}$ containing $\mathcal{C}_p$ for every $p \in P$ is $\mathcal{C}$ itself.

**Theorem 3.5.8.** For $P$ a finite poset, the following pieces of data are equivalent:

- A stable $\infty$-category $\mathcal{C}$ that is a $P$-decomposition of $(\mathcal{C}_p)_{p\in P}$
- A stable $\infty$-category $\mathcal{C}$ that is a $P$-recollement of $(\mathcal{C}_p)_{p\in P}$
- A functor $F : P^{op} \to \mathrm{Cat}_\infty^{ex}$ into the non-full subcategory of $\mathrm{Cat}_\infty$ spanned by stable $\infty$-categories and exact functors
- A $P$-slicing of a stable $\infty$-category $\mathcal{C}$

*Proof.* We had seen in Proposition 3.1.6 that a recollement of two subcategories is stable iff the two subcategories are stable. The first two data are hence equivalent by combining 3.5.5 with the observation that if $L_q i_p = 0$ for $p < q$, then $\mathrm{Map}(C_p, C_q) \simeq \mathrm{Map}(L_q i_p C_p) = 0$ automatically, and the converse following from the Yoneda lemma.

The remaining equivalences are also immediate by combining the arguments of the mentioned proposition with the previous Theorem. In particular, note that the recollements involved in the $P$-slicing in the last point are automatically stable. □

*Remark.* We expect partial results to still hold for $P$ a noetherian poset. For general $P$, the correct definition is that of a functor $P \to \mathcal{C}at_\infty^{lex}$ or into $\mathcal{C}at_\infty^{ex}$ respectively, whence we recover $\mathcal{C}$ as its lax limit. In particular, the notion of a $P$-recollement is generally stronger than the notion of a $P$-slicing, as indicated in the following proposition.

**Proposition 3.5.9** ([HPT20, 5.16])**.** Let $(X \to P)$ be a stratified space in the sense of 6, with $P$ potentially infinite. For $\mathcal{V}$ a compactly generated $\infty$-category, the hyperpullback functors $(\mathcal{S}h^{hyp}(X;\mathcal{V}) \to \mathcal{S}h^{hyp}(X_p;\mathcal{V}))_{p\in P}$ are jointly conservative. This implies that in our terms, the full subcategories $\mathcal{S}h^{hyp}(X_p;\mathcal{V})$ embedded into $\mathcal{S}h^{hyp}(X;\mathcal{V})$ by pushforwards form a $P$-recollement. However, they already form a $P$-slicing under the possibly weaker conditions of 3.1.5, e.g. if $\mathcal{V}$ is presentable and stable.

Now that we have discussed the case of $\infty$-categories with finite limits and stable $\infty$-categories, the latter generalizing the classical theory of semiorthogonal decomopositions by taking the homotopy category, we turn to the Poincaré case.

**Definition 3.5.10.** Let $(\mathcal{C}, \Omega)$ be a Poincaré $\infty$-category. A *Poincaré P-slicing* of $\mathcal{C}$ is a $P$-slicing of $\mathcal{C}$ such that for every slicing $(P_-, P_+)$ of $P$, the induced sequence $(\mathcal{C}_{P_-}, \Omega|_{\mathcal{C}_{P_-}}) \xrightarrow{i_*} (\mathcal{C}, \Omega) \xrightarrow{j^*} (\mathcal{C}_{P_+}, \Omega|_{\mathcal{C}_{P_+}})$ is a split Poincaré-Verdier sequence. In other words, $i_*$ and $j^*$ must be duality-preserving.

*Remark.* We are unsure if this can be reformulated as a functor $P^{op} \to \mathcal{C}at_\infty^p$; a first step would be to notice that the push-pull $i^* j_* E \cong i^- j_+ E[1] = D_\Omega i^* j_* D_\Omega E[1]$ is duality-preserving up to a shift by the argument of [Ban07, 8.2.6].

# 4 L-Groups of Simplicial Complexes and PL-spaces

While we have defined L-groups for arbitrary Poincaré $\infty$-categories, until now we mainly considered algebraic examples, like the perfect derived $\infty$-category or categories of module spectra. Our goal however is to apply these consideration to (stratified) topological spaces, a task that can get fairly difficult especially when the spaces involved are not well-behaved. As a first step, let us therefore take a look at one of the simplest classes of spaces, finite simplicial complexes, and then pass to compact PL spaces built from them.

## 4.1 Simplicial Sheaves

In this section, we define simplicial sheaves on a simplicial complex, which will turn out to be a special case of the constructible sheaves on stratified spaces we later consider. Further, we write down a Verdier duality functor on them allowing us to define L-groups. Let $\mathcal{V}$ be a stable $\infty$-category.

**Definition 4.1.1.** A *simplicial complex* $K$ consists of a set of *vertices* $K_0$, and a partially ordered set of *simplices* or faces denoted by $\mathcal{I}_K$ that is a collection of nonempty finite subsets of $K_0$, ordered by inclusion. We require that

- For each $v \in K_0$, we have $\{v\} \in \mathcal{I}_K$, and
- If $\sigma, \tau$ are nonempty finite subsets of $K_0$, such that $\sigma \subseteq \tau$ and $\tau \in \mathcal{I}_K$, then $\sigma \in \mathcal{I}_K$.

The *dimension* of a face $\sigma \in \mathcal{I}_K$ is defined as its cardinality minus 1. The dimension of $K$ is the maximum over the dimensions of all its faces. $K$ is called *finite* if the poset $\mathcal{I}_K$ is finite, which implies that $K_0$ is also finite.

**Definition 4.1.2.** A *map of simplicial complexes* $f : K \to L$ is a map of underlying sets $f : K_0 \to L_0$ such that the image $f(\sigma) \subseteq L_0$ of a simplex $\sigma \in \mathcal{I}_K$ is again a simplex in $\mathcal{I}_L$. One obtains a category of simplicial complexes.

A simplicial complex $K$ should be regarded as special case of a simplicial set, where

- no ordering is fixed on the faces of an $n$-simplex,

- the gluing of simplices is *regular*, i.e. all $n$-simplices of $K$ are isomorphic to the standard simplex $\Delta^n$, and
- the intersection of two simplices is again a *single* simplex.

In particular, we can associate a simplicial set to any simplicial complex $K$ (which is unique after fixing an order on the vertices), with non-degenerate simplices precisely the simplices of $K$. We use it to define e.g. the geometric realization $|K|$ or the (barycentric) subdivision $\mathrm{sd}(K)$, which agree with the usual definitions from topology.

*Warning.* We also associate a different simplicial set to $K$, namely the $\infty$-category obtained as the nerve of the partially ordered set of simplices $\mathcal{I}_K$. This simplicial set has one vertex for every simplex of $K$ and one edge for every inclusion relation among them; so it is the subdivision of the construction above. In particular, their geometric realization are homotopy equivalent (even homeomorphic).

**Definition 4.1.3.** For $K$ a simplicial complex and $\mathcal{V}$ an $\infty$-category, a *simplicial sheaf* on $K$ is a functor $\mathcal{I}_K \to \mathcal{V}$, where we regard the poset $\mathcal{I}_K$ as a (thin) $\infty$-category using the nerve construction. We write $\mathcal{S}h^{simp}(K;\mathcal{V}) := \mathrm{Fun}(\mathcal{I}_K, \mathcal{V})$, and if $\mathcal{V}$ has a terminal object $*$ we denote by $\mathcal{S}h_c^{simp}(K;\mathcal{V})$ its full subcategory on *compactly supported* simplicial sheaves $F$, meaning that $F(\sigma) \cong * \in \mathcal{V}$ for all but finitely many $\sigma \in \mathcal{I}_K$.

*Remark.* If $\mathcal{V}$ is compactly generated, or presentable stable, then this is equivalent to the $\infty$-category of constructible sheaves on the stratified space $|K| \to \mathcal{I}_K$, as we show in 6.2.13. More generally, if $\mathcal{V}$ itself is not presentable but a full subcategory of an $\infty$-category $\mathcal{W}$ satisfying these properties, we will see in 6.3.5 that we may identify $\mathcal{S}h^{simp}(K;\mathcal{V})$ with the full subcategory of $\mathcal{S}h^{cbl}(|K|;\mathcal{W})$ on those sheaves with stalks in $\mathcal{V}$. This often happens for Poincaré $\infty$-categories, for example $D^{\mathrm{perf}}(R) \subseteq D(R)$.

**Observation 4.1.4.** If $\mathcal{V}$ possesses all (co)limits, the $\infty$-category $\mathcal{S}h^{simp}(K;\mathcal{V})$ does so as well. Also, every functor $F : \mathcal{I}_K \to \mathcal{V}$ agrees with the filtered colimit

$$F \cong \operatorname*{colim}_{K' \subseteq K \text{ finite}} i_{K',*} i_{K'}^* F \; , \tag{4.1}$$

where $i_{K',*} i_{K'}^* F$ is defined to agree with $F$ on all simplices in the sub-poset $\mathcal{I}_{K'} \subseteq \mathcal{I}_K$, and is zero otherwise. Thus, $\mathcal{S}h_c^{simp}(K;\mathcal{V})$ generates $\mathcal{S}h^{simp}(K;\mathcal{V})$ under colimits.

**Example 4.1.5.** For $\tau \in \mathcal{I}_K$ and $C \in \mathcal{V}$, denote by $F_{\tau,V} : \mathcal{I}_K \to \mathcal{V}$ the sheaf that sends each face $\sigma \subseteq \tau$ to $V$, and all other simplices of $K$ to 0. This is compactly supported on the simplex $\tau$ and its faces.

**Example 4.1.6.** Conversely, denote by $F^{\tau,V} : \mathcal{I}_K \to \mathcal{V}$ the sheaf that sends every $\sigma \supseteq \tau$ to $V$ and all other simplices of $K$ to 0. As in the last example, the transition maps are either identities or zero maps.

**Example 4.1.7.** Both examples above still make sense if we replace $\tau$ by any subcomplex $L \subseteq K$, yielding $F_{L,V}$ and $F^{L,V}$.

**Definition 4.1.8.** A simplicial sheaf $F : \mathcal{I}_K \to \mathcal{V}$ on a simplicial complex $K$ is called *locally constant* if for each $\sigma, \tau \in \mathcal{I}_K$ with $\sigma \subseteq \tau$, the image $F(\sigma \subseteq \tau) : F(\sigma) \to F(\tau)$ is an isomorphism. For every $V \in \mathcal{V}$, we can define a locally constant sheaf $\underline{V} : K \to \mathcal{V}$ as the constant functor with value $V$; let us call simplicial sheaves of this form *constant*.

**Definition 4.1.9.** Given a simplicial sheaf $F : \mathcal{I}_K \to \mathcal{V}$ on a simplicial complex $K$, its *global sections* or *simplicial cochain complex* are defined as

$$\Gamma(F) = C^*(F) := \lim_{\sigma \in \mathcal{I}_K} F(\sigma)\ . \tag{4.2}$$

Dually, we define its *simplicial chain complex* $C_*(F) := \operatorname{colim}_{\sigma \in \mathcal{I}_K} F(\sigma)$.

**Proposition 4.1.10.** It $\mathcal{V}$ admits limits and colimits, the functors $C_* \dashv \underline{(-)} \dashv C^*$ form an adjoint triple, by definition of limit and colimit:

$$\mathcal{S}h^{simp}(K;\mathcal{V}) = \operatorname{Fun}(\mathcal{I}_K, \mathcal{V}) \underset{\xrightarrow{\qquad C^* \qquad}}{\overset{\xrightarrow{\qquad C_* \qquad}}{\xleftarrow{\qquad \underline{(-)} \qquad}}} \mathcal{V}$$

Hence, the compositions $C_* \circ \underline{(-)} \dashv C^* \circ \underline{(-)}$ are adjoint functors $\mathcal{V} \to \mathcal{V}$, also denoted by $C_*(K; -)$ and $C^*(K; -)$.

*Remark.* For $\mathcal{V} = D(R)$ the derived category of a ring or the category of chain complexes $\mathcal{C}h(R)$, the groups $C^*(K;R)$ and $C_*(K;R)$ agree with the usual simplicial (co)chain complexes since homotopy (co)limits in the derived category of a Grothendieck abelian category can be calculated using the *bar construction* as explained in 1.5.18. More generally, $C_*$ and $C^*$ calculate the simplicial (co)chain complexes with values in a local system. We will in 5.3.11 and 6.3.8 apply the same construction for the (co)homology of a topological space with values in a local system or constructible sheaf; one can also obtain the simplicial (co)chain complexes of a (regular, locally finite) CW complex in this manner using 6.2.12.

*Technical Remark.* If we regard $\mathcal{S}h^{simp}(K;\mathcal{S})$ as a (presheaf) $\infty$-topos, then $\underline{(-)} \dashv C^*$ agrees with the global sections geometric morphism, and their composition is a left-exact functor $\mathcal{S} \to \mathcal{S}$ called the *shape* of this topos. The last remark shows how this encodes topological invariants of the simplicial complex $K$. We will observe something similar for the topological setting in 5.1.10.

**Definition 4.1.11.** If $K$ is a simplicial complex and $\sigma \in \mathcal{I}_K$ a simplex, the *open star* of $\sigma$ is the set of simplices in $K$ that have a non-empty intersection with $\sigma$, regarded as an open subset of $|K|$ by taking the union of their interiors. The *star* of $\sigma$ is the closure of the open star under the operation of taking faces, so that it becomes a simplicial complex itself (or a closed subset of $|K|$). The *link* of $\sigma$ consists of those simplices in the star that do not intersect $\sigma$.

**Example 4.1.12.** Usually, we are interested in the case where $\sigma = \{v\}$, so the star of $\{v\}$ is the closure of all simplices containing $v$ as a vertex. If we for example let $v = 0 \in \Delta^n$, then its star is all of $\Delta^n$; its link consists of those simplices in $\Delta^n$ that do not contain $v$, making up $\Delta^{1<\cdots<n} \cong \Delta^{n-1}$; and its open star is $\Delta^n - \Delta^{1<\cdots<n}$.

**Example 4.1.13** ([Lur11, Lecture 18, Example 7])**.** Let us calculate the global sections for the exemplary sheaves above:

$$C^*(F_{L,V}) = \lim_{\sigma \in \mathcal{I}_K} F_{L,V} = \lim_{\sigma \in \mathcal{I}_L} V = V^{\mathcal{I}_L} \tag{4.3}$$

where $V^{\mathcal{I}_L}$ is the *mapping object* defined by

$$\mathrm{Map}_{\mathcal{V}}(V', V^{\mathcal{I}_L}) = \lim_{\sigma \in \mathcal{I}_L} \mathrm{Map}_{\mathcal{V}}(V', V) = \mathrm{Map}_{\mathcal{S}}(\mathcal{I}_L, \mathrm{Map}_{\mathcal{V}}(V', V)) \tag{4.4}$$

for all $V' \in \mathcal{V}$. To calculate $C^*(F^{\tau,V})$, let $L$ be the subcomplex of $K$ defined as the complement of the open star of $\tau$. The sequence

$$F^{\tau,V} \to F_{K,V} = \underline{V} \to F_{L,V} \tag{4.5}$$

of functors is a fiber and cofiber sequence, since this can be checked simplex-wise. Hence,

$$C^*(F^{\tau,V}) = \mathrm{fib}(C^*(F_{K,V} \to F_{L,V})) = \mathrm{fib}(V^{\mathcal{I}_K} \to V^{\mathcal{I}_L}) =: V^{(\mathcal{I}_K, \mathcal{I}_L)} \tag{4.6}$$

and since the mapping objects are defined by mapping $K, L$ into the Kan complex $\mathrm{Map}_{\mathcal{V}}(V', V)$, they are invariant under weak homotopy by definition 1.2.16 so we can contract the open star of $\tau$ to a point $x$ in the interior of $\tau$, obtaining $C^*(F^{\tau,V}) = V^{(|K|,|K|-\{x\})}$ where we identify the topological spaces with their (singular) Kan complexes.

From now on, let $(\mathcal{V}, \mathfrak{Q})$ be a Poincaré $\infty$-category. This allows us to equip $\mathcal{S}h^{simp}(K; \mathcal{V})$ with the tensor hermitian structure, making it into a Poincaré $\infty$-category as well by 2.3.8. We can even twist the quadratic functor by a local system:

**Definition 4.1.14.** A spectrum $X \in \mathcal{S}p$ is called *invertible* if there is another spectrum $Y \in \mathcal{S}p$ such that $X \wedge Y \cong \mathbb{S}$. This already implies that $X \cong \mathbb{S}^n$ for some $n \in \mathbb{Z}$. Denote by $\mathcal{S}p^{inv} \subseteq \mathcal{S}p$ the full subcategory on invertible spectra.

**Definition 4.1.15.** Given a simplicial complex $K$, a *spherical fibration* on $K$ is a locally constant sheaf $\chi : \mathcal{I}_K \to \mathcal{S}p^{inv}$.

From now on, we always assume that $K$ is finite.

**Proposition 4.1.16** ([Lur11, Lecture 21])**.** For $(\mathcal{V}, \Omega)$ a hermitian $\infty$-category, $K$ a simplicial complex and $\zeta : \mathcal{I}_K \to \mathcal{S}p^{inv}$ a spherical fibration, the category of simplicial sheaves $\mathcal{S}h^{simp}(K; \mathcal{V})$ equipped with the *twisted Verdier duality* functor

$$\Omega_{K,\zeta}(F) := \operatorname*{colim}_{\sigma \in \mathcal{I}_K} \zeta(\sigma) \wedge \Omega(F(\sigma)) \tag{4.7}$$

for any $F : \mathcal{I}_K \to \mathcal{V}$ is a hermitian $\infty$-category. Its associated bilinear functor is

$$B_{K,\zeta}(F, G) = \operatorname*{colim}_{\sigma \in \mathcal{I}_K} \zeta(\sigma) \wedge B_{\Omega}(F(\sigma), G(\sigma)) \tag{4.8}$$

and if $\Omega$ admits a duality functor $D_{\Omega}$, we obtain an associated duality functor

$$D_{K,\zeta}(F)(\sigma) = \zeta(\sigma) \wedge D_K(\sigma) = \zeta(\sigma) \wedge \operatorname*{colim}_{\tau \in \mathcal{I}_K} \begin{cases} D_{\Omega}(F(\tau)) & \text{for } \tau \supseteq \sigma \\ 0 & \text{otherwise} \end{cases} \tag{4.9}$$

where we denote the non-twisted tensor duality functor by $D_{K,\underline{\mathbb{S}}} =: D_K$.

*Remark.* The $\zeta(\sigma) \wedge -$ in this formula is the tensoring $\mathcal{S}p^{\text{fin}} \otimes \mathcal{V} \to \mathcal{V}$ from 1.6.22, determined by the universal property

$$\mathrm{Map}_{\mathcal{V}}(E \wedge V, V') \simeq \mathrm{Map}_{\mathcal{S}p}(E, \mathrm{map}_{\mathcal{V}}(V, V'))\ . \tag{4.10}$$

Since $\zeta(\sigma) = \Sigma^n \mathbb{S}$ for some $n \in \mathbb{Z}$, this tensoring on objects simply acts as $\zeta(\sigma) \wedge V = \Sigma^n V$, but a priori it is not clear that this is functorial in $\mathcal{S}p^{inv}$.

**Lemma 4.1.17.** For $E \in \mathcal{S}p^{\text{fin}}$, and $\mathcal{V}$ a stable $\infty$-category with $V, V' \in \mathcal{V}$,

$$E \wedge \mathrm{map}_{\mathcal{V}}(V, V') \cong \mathrm{map}_{\mathcal{V}}(V, E \wedge V') \tag{4.11}$$

*Proof.* Every finite spectrum can we written as a finite (co)limit over the sphere spectrum $\mathbb{S} = \Sigma^\infty S^0$ (essentially by definition, compare [CDH+20a, 4.1.2]), and regarded as functors in $E$, both sides of the above equality preserve finite limits. We can therefore reduce to $E = \mathbb{S}$, which is a unit for the smash product so the result is automatic. □

*Proof of 4.1.16.* Note that all involved colimits are finite, so they exist in a stable $\infty$-category. In the case where $\zeta = \underline{\mathbb{S}} : \mathcal{I}_K \to \mathcal{S}p^{inv}$ is the constant functor on $\mathbb{S}$, this Proposition is just a special case of 2.3.7. Generally, $\Omega_{K,\zeta}(0) = 0$ is reduced,

$$\begin{aligned}
\Omega_{K,\zeta}(F \oplus G) &= \operatorname*{colim}_{\sigma \in K} \zeta(\sigma) \wedge (\Omega(F(\sigma)) \oplus \Omega(G(\sigma)) \oplus B_{\Omega}(F(\sigma), G(\sigma))) \cong \\
&\cong \operatorname*{colim}_{\sigma \in \mathcal{I}_K} \zeta(\sigma) \wedge \Omega(F(\sigma)) \oplus \operatorname*{colim}_{\sigma \in \mathcal{I}_K} \zeta(\sigma) \wedge \Omega(G(\sigma)) \oplus \operatorname*{colim}_{\sigma \in \mathcal{I}_K} \zeta(\sigma) \wedge B_{\Omega}(F(\sigma), G(\sigma)) = \\
&= \Omega_{K,\zeta}(F) \oplus \Omega_{K,\zeta}(G) \oplus B_{K,\zeta}(F, G)
\end{aligned}$$

exhibits $B_{K,\zeta}(F,G)$ as the correct polarization, and

$$B_{K,\zeta}(F,G) = \operatorname*{colim}_{\sigma\in\mathcal{I}_K} \zeta(\sigma)\wedge \mathrm{map}_{\mathcal{V}}\left(F(\sigma), D_{\mathfrak{Q}}G(\sigma)\right) \cong$$
$$\cong \operatorname*{colim}_{\sigma\in\mathcal{I}_K} \mathrm{map}_{\mathcal{V}}(F(\sigma), \zeta(\sigma)\wedge D_{\mathfrak{Q}}G(\sigma))$$

using the Lemma above. From there, we can follow the proof of 2.3.7. Further, $D_{K,\zeta}$ is exact since the smash product preserves colimits in both variables and we already know $D_K$ is exact, so $B_{\mathfrak{Q}}$ is bilinear and automatically symmetric as it arises as a polarization. We calculate

$$\Lambda_{K,\zeta}(F) := \mathrm{fib}\left(\mathfrak{Q}_{K,\zeta}(F)\to B_{K,\zeta}(F,F)^{hS_2}\right) = \operatorname*{colim}_{\sigma\in\mathcal{I}_K}\zeta(\sigma)\wedge\Lambda_{\mathfrak{Q}}(F(\sigma))$$

since $\wedge$ preserves colimits, making it exact as well, so $\mathfrak{Q}_{K,\zeta}$ is a non-degenerate quadratic functor. □

**Example 4.1.18** ([Lur11, Lecture 19, Example 2])**.** Let $\underline{V}:\mathcal{I}_K\to\mathcal{V}$ be the constant sheaf on $V\in\mathcal{V}$. Its Verdier dual can be calculated as

$$D_K\underline{V}(\sigma) = \operatorname*{colim}_{\tau\in\mathcal{I}_K}\begin{cases} D_{\mathfrak{Q}}(V) & \text{for } \tau\supseteq\sigma\\ 0 & \text{otherwise}\end{cases} = D_{\mathfrak{Q}}\lim_{\tau\in\mathcal{I}_K}\begin{cases} V & \text{for } \tau\supseteq\sigma\\ 0 & \text{otherwise}\end{cases} =$$
$$= D_{\mathfrak{Q}}C^*F^{\sigma,V} = D_{\mathfrak{Q}}V^{(|K|,|K|-\{x\})}$$

for $x$ in the interior of $\sigma$, using example 4.1.13.

**Theorem 4.1.19** ([Lur11, Lecture 19, Proposition 3])**.** *If $\mathcal{V}$ is a Poincaré $\infty$-category and $\zeta$ a spherical fibration on a finite simplicial complex $K$, then the hermitian $\infty$-category $(\mathcal{S}h^{simp}(K,\mathcal{V}),\mathfrak{Q}_{K,\zeta})$ is Poincaré as well.*

We will need further preparations to prove the biduality statement included in this theorem. Let us first examine the functoriality of our constructions.

**Definition 4.1.20.** For $f:\mathcal{I}_K\to\mathcal{I}_L$ a map of simplicial sets as defined in 4.1.2, we define the *pullback*

$$f^*:\mathcal{S}h^{simp}(L;\mathcal{V})\to\mathcal{S}h^{simp}(K;\mathcal{V}) \tag{4.12}$$

by precomposing $G:\mathcal{I}_L\to\mathcal{V}$ with $f$, in the sense that $f^*G(\tau):=G(f(\tau))$. In fact, this even makes sense if $f$ is just a map of posets.

**Definition 4.1.21.** As a precomposition functor $f^*$ has adjoints $f_+\dashv f^*\dashv f_*$ given by left and right Kan extension along $f$, assuming that $\mathcal{V}$ has the required colimits or limits (e.g. $K$ is finite and $\mathcal{V}$ stable). We call $f_*=\mathrm{Ran}_f:\mathcal{S}h^{simp}(K;\mathcal{V})\to\mathcal{S}h^{simp}(L;\mathcal{V})$ the *pushforward* along $f$, it is explicitly given by

$$(f_*F)(\tau) := \lim_{f(\sigma)\supseteq\tau} F(\sigma)\ . \tag{4.13}$$

**Observation 4.1.22.** If $f : K \to L$ is even a map of simplicial complexes and $\tau \in \mathcal{I}_L$, then there exists a $\sigma \in \mathcal{I}_K$ with $f(\sigma) \supseteq \tau$ iff there exists a $\sigma' \in \mathcal{I}_K$ with $f(\sigma') = \tau$, as we can choose $\sigma' \subseteq \sigma$ as a subset making the restricted map on vertices bijective. Therefore, the set of $\sigma'$ with $f(\sigma') = \tau$ is a left cofinal subposet of the indexing poset of the above limit, as either both are empty or each $\sigma$ there exists an appropriate face $\sigma'$. We may then write

$$(f_* F)(\tau) := \lim_{f(\sigma)=\tau} F(\sigma) \,. \tag{4.14}$$

In particular, this implies that $f^* F$ exists if $\mathcal{V}$ admits finite limits (e.g. it is stable) and the preimages $f^{-1}(\tau)$ are finite. We might call such maps of simplicial complexes *proper*.

**Lemma 4.1.23.** For $f : \mathcal{I}_K \to \mathcal{I}_L$, $g : \mathcal{I}_L \to \mathcal{I}_M$ maps of posets, we have:

$$(g \circ f)_+ = g_+ \circ f_+, \quad (g \circ f)^* = f^* \circ g^*, \quad (g \circ f)_* = g_* \circ f_* \tag{4.15}$$

Also, if $t : L \to \Delta^0$ is the terminal map, then $t_* = C^*$ and $t_+ = C_*$ under the identification $\mathcal{S}h^{simp}(\Delta^0; \mathcal{V}) = \mathrm{Fun}(\Delta^0, \mathcal{V}) \simeq \mathcal{V}$. In particular, $C^* \circ f_* = C^*$ and $C_* \circ f_+ = C_*$.

*Proof.* This statement is clearly true for the precompositions. For the pushforwards, this it is due to transitivity of Kan extensions. □

*Technical Remark.* In fact, the expression $\mathrm{Fun}(-, \mathcal{V}) : \mathrm{PoSet}^{\mathrm{fin},op} \to \mathcal{C}at_\infty$ sending maps to the respective pullbacks is a functor on all finite posets, and classifies the coCartesian fibration $\mathcal{M} \to \mathrm{PoSet}^{\mathrm{fin},op}$ which is also Cartesian, and the Cartesian fibration $\mathcal{M}^\vee \to \mathrm{PoSet}^{\mathrm{fin}}$ which is also coCartesian. This yields the adjunction $f_+ \dashv f^* \dashv f_*$ when restricting to the non-full subcategory on $(\mathcal{I}_K, \mathcal{I}_L, f)$ in $\mathrm{PoSet}^{\mathrm{fin}}$, and the previous Lemma when restricting to the span of $(\mathcal{I}_K, \mathcal{I}_L, \mathcal{I}_M, f, g)$ and composing the (co)Cartesian lifts.

**Proposition 4.1.24** ([Lur11, Lecture 19])**.** For $f : K \to L$ a map of finite simplicial complexes, $\zeta : \mathcal{I}_L \to \mathcal{S}p^{inv}$ a spherical fibration and $\mathcal{V}$ a Poincaré $\infty$-category, the induced pushforward functor

$$f_* : \big(\mathcal{S}h^{simp}(K; \mathcal{V}), \Omega_{K,\zeta\circ f}\big) \to \big(\mathcal{S}h^{simp}(L; \mathcal{V}), \Omega_{K,\zeta}\big) \tag{4.16}$$

is duality-preserving in the sense that $f_* \circ D_{K,\zeta\circ f} \cong D_{L,\zeta} \circ f_*$.

*Proof.* By the Yoneda-Lemma, it suffices to show that for $F : \mathcal{I}_K \to \mathcal{V}$ and $G : \mathcal{I}_L \to \mathcal{V}$,

$$\mathrm{map}(G, D_{L,\zeta} f_* F) \cong \mathrm{map}(f^* G, D_{K,\zeta\circ f} F)$$

naturally in $F$ and $G$. But these are precisely the expressions for the associated bilinear functors; we calculate

$$\begin{aligned} B_{L,\zeta}(G, f_* F) = \operatorname*{colim}_{\tau \in \mathcal{I}_L} \zeta(\tau) \wedge B_\Omega \left( G(\tau), \lim_{f(\sigma)=\tau} F(\sigma) \right) &\cong \operatorname*{colim}_{\tau, f(\sigma)=\tau} \zeta(\tau) \wedge B_\Omega \left( G(\tau), F(\sigma) \right) \cong \\ \cong \operatorname*{colim}_{\sigma \in \mathcal{I}_K} \zeta(f(\sigma)) \wedge B_\Omega \left( G(f(\sigma)), F(\tau) \right) &= B_{K,\zeta\circ f}(f^* G, F) \,. \end{aligned}$$

□

*Remark.* A different proof can be found in [CDH+20a, 6.5.13, 6.6.1] relying on the observation that the map $(\mathcal{I}_K)_{\sigma/} \to (\mathcal{I}_L)_{f(\sigma)/}$ induced by $f$ is always right cofinal.

**Lemma 4.1.25.** The sheaves $(F^{\tau,V})_{\tau\in K, V\in\mathcal{V}}$ generate $\mathcal{S}h_c^{simp}(K;\mathcal{V})$ under cofibers, and $\mathcal{S}h^{simp}(K;\mathcal{V})$ under all colimits. Actually, for any $F \in \mathcal{S}h^{simp}(K;\mathcal{V})$ we have

$$\mathrm{Map}_{\mathcal{S}h^{simp}(K;\mathcal{V})}(F^{\tau,V}, F) \cong \mathrm{Map}_{\mathcal{C}}(V, F(\tau))\ . \tag{4.17}$$

If we regard the simplex $\tau \in \mathcal{I}_K$ as a simplicial complex and let $i_\tau : \tau \to K$ be the canonical inclusion and $\underline{V} : \tau \to \mathcal{V}$ the constant simplicial sheaf on $\tau$ with value $V$, we can identify $i_{\tau,*}\underline{V} = F^{\tau,V}$.

*Proof.* Given a compactly supported simplicial sheaf $F$ on $K$, its support $\mathrm{supp}(F)$ is a finite downwards-closed subset of $\mathcal{I}_K$. If it is empty, then $F \cong F^{\tau,0}$ and we are finished, otherwise we proceed by induction on its cardinality and choose any maximal $\tau_0 \in \mathrm{supp}(F)$. The fiber $F' := \mathrm{fib}(F \to F^{\tau_0,F(\tau_0)})$ has a smaller support $\mathrm{supp}(F) - \{\tau_0\}$, and $F = \mathrm{cofib}(F^{\tau_0,F(\tau_0)} \to F')$ so we are finished. We have already observed in 4.1.4 that the compactly supported sheaves generate all sheaves under filtered colimits, showing the second claim.

The third claim follows either from the Yoneda-Lemma, as $F^{\tau,V}$ is just the Yoneda-embedding of $\tau$ tensored with $V$, or by an explicit calculation. Then, the last claim follows by applying the Yoneda-Lemma to the calculation

$$\begin{aligned}\mathrm{Map}_{\mathcal{S}h^{simp}(K;\mathcal{V})}(i_{*,\tau}\underline{V}, F) &\cong \mathrm{Map}_{\mathcal{S}h^{simp}(\tau;\mathcal{V})}(\underline{V}, i_\tau^* F) \cong \mathrm{Map}_{\mathcal{V}}(V, \lim_{\mathcal{I}_\tau} F|_\tau) \cong \\ &\cong \mathrm{Map}_{\mathcal{V}}(V, F(\tau)) \cong \mathrm{Map}_{\mathcal{S}h^{simp}(K;\mathcal{V})}(F^{\tau,V}, F)\end{aligned}$$

since the limit diagram has an initial object. □

*Proof of 4.1.19.* By the previous proposition 4.1.16, we only need to show that $\mathrm{Id} \cong D^2_{K,\zeta}$ is perfect. By its definition, $D_{K,\zeta}$ commutes with colimits, so it suffices to show this on a class of objects generating $\mathcal{S}h^{simp}(K;\mathcal{V})$ under colimits. We use the sheaves $F^{\tau,V} = i_{\tau,*}\underline{V}$ for this purpose, and since pushforwards commute with $D_{K,\zeta}$ by 4.1.24 we can reduce to the constant sheaf $\underline{V}$ on an $n$-simplex $\tau$.

In 4.1.18 we have calculated $D_K\underline{V}(\sigma) = D_{\mathcal{Q}} V^{(\Delta^n, \Delta^n - \{x\})}$ for $x$ a point in the interior of $\sigma$. If $\sigma \subseteq \tau$ is a proper face, this homotopy cofiber is trivial; for $\sigma = \tau$ it is homotopy equivalent to $D_{\mathcal{Q}} V^{(D^n, S^{n-1})} \simeq D_{\mathcal{Q}} V^{S^n} = D_{\mathcal{Q}}\Sigma^{-n} V = \Sigma^n D_{\mathcal{Q}} V$. Finally, we calculate

$$\begin{aligned}D^2_{K,\zeta}\underline{V}(\sigma) &= \zeta(\sigma) \wedge \operatorname*{colim}_{\sigma'\in\mathcal{I}_\tau} \begin{cases} D_{\mathcal{Q}}\left(\zeta(\sigma') \wedge D_K\underline{V}(\sigma')\right) & \text{for } \sigma' \supseteq \sigma \\ 0 & \text{otherwise}\end{cases} = \\ &= \zeta(\sigma) \wedge \operatorname*{colim}_{\sigma'\in\mathcal{I}_\tau} \begin{cases} D_{\mathcal{Q}}\left(\zeta(\sigma') \wedge \Sigma^n D_{\mathcal{Q}} V\right) & \text{for } \tau = \sigma' \supseteq \sigma \\ 0 & \text{otherwise}\end{cases} = \\ &= \zeta(\sigma) \wedge \zeta^{-1}(\sigma) \wedge \Sigma^{-n} V \wedge \operatorname*{colim}_{\sigma'\in\mathcal{I}_\tau} \begin{cases} D_{\mathcal{Q}}\Sigma^\infty S^0 & \text{for } \tau = \sigma' \\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

using that $D_{\mathcal{Q}}$ sends $\Sigma^n$ to $\Sigma^{-n}$, and thus an invertible spectrum to its inverse. The last colimit can be identified with $\Sigma^\infty \operatorname{fib}(|\partial\tau| \to |\tau|) = \Sigma^\infty S^n$, so we are left with $\Sigma^{-n}V \wedge \Sigma^n\mathbb{S} = V$, as claimed. □

## 4.2 Combinatorial Sheaves on PL spaces

**Definition 4.2.1.** For $K, L$ simplicial complexes, a homeomorphism $r : |K| \to |L|$ exhibits $K$ as a *refinement* of $L$ if

- It embeds the realization of a simplex $\sigma$ in $K$ into the realization of some simplex in $L$. This induces a map of partially ordered sets $r : \mathcal{I}_K \to \mathcal{I}_L$ sending $\sigma$ to the smallest such $\tau \in \mathcal{I}_L$.
- The realization of a simplex in $L$ is the union of such embedded simplices from $K$.

We will often only specify the map of posets $r : \mathcal{I}_K \to \mathcal{I}_L$, be aware that this is never a map of simplicial complexes unless it is an isomorphism.

**Definition 4.2.2.** A *piecewise linear space*, or *PL space* in short, is a topological space $X$ together with a set $\mathcal{T}$ of locally finite triangulations such that

- if $T \in \mathcal{T}$, then the barycentric subdivision $\operatorname{sd}(T)$ is also in $\mathcal{T}$, and
- any $T, T' \in \mathcal{T}$ possess a common refinement $T'' \in \mathcal{T}$.

The set $\mathcal{T}$ is partially ordered by refinement of triangulations. We refer to the literature (e.g. [Lur11, Lecture 17]) for more information, and different characterizations. In particular, a *PL map* between PL spaces is a map of simplicial complexes on some triangulations of them, making PL spaces into a category.

**Proposition 4.2.3** ([Lur11, Lecture 17, Remark 7])**.** For $X$ a PL space, the following are equivalent:

- $X$ is compact as a topological space
- $X$ admits a finite triangulation
- Every triangulation of $X$ is finite

**Definition 4.2.4.** An *n-dimensional PL manifold* is a PL space $X$ that is locally isomorphic to $\mathbb{R}^n$ with its canonical PL structure, in the sense that any point $x \in X$ has an open neighborhood that, together with its restricted PL structure, possesses mutually inverse PL maps to and from $\mathbb{R}^n$. Similarly, we define *n-dimensional PL manifolds with boundary.*

**Proposition 4.2.5.** A PL space $X$ is an $n$-dimensional PL manifold iff it possesses a triangulation $K$ such that for every vertex $v \in K_0$, (the geometric realization of) its link is homeomorphic to $S^{n-1}$. This then automatically holds for every triangulation of $X$, and even for the link of any simplex (not just vertices). In particular, (Whitehead) triangulations of smooth manifolds yield PL manifolds, but triangulations of topological manifolds may have non-spherical links.

Similarly, a PL space $X$ is an $n$-dimensional PL manifold with boundary iff for some (or for every) triangulation $K$ the link of every vertex (or simplex) $v \in K_0$ is either homeomorphic to $S^{n-1}$ or $D^{n-1}$. The subcomplex spanned by all vertices with link $D^{n-1}$ makes up the *boundary* of $X$.

**Definition 4.2.6.** For $X$ a PL space with family of triangulations $\mathcal{T}$ and $\mathcal{V}$ an $\infty$-category, we define the $\infty$-category of *combinatorial sheaves* on $X$ with respect to $\mathcal{T}$

$$\mathcal{S}h^{comb}(X;\mathcal{V}) := \lim_{T\in\mathcal{T}} \mathcal{S}h^{simp}(T;\mathcal{V}) = \lim_{T\in\mathcal{T}} \mathrm{Fun}(T,\mathcal{V}) \ , \tag{4.18}$$

where the transition maps in this limit diagram are given by pushforward along refinements. Similarly, we define the $\infty$-category of *compactly supported combinatorial sheaves* as a colimit along pullbacks of refinements

$$\mathcal{S}h_c^{comb}(X;\mathcal{V}) := \operatorname*{colim}_{T\in\mathcal{T}} \mathcal{S}h_c^{simp}(T;\mathcal{V}) \ . \tag{4.19}$$

To be a bit more precise, denote by $\mathcal{I} : \mathcal{T} \to \mathcal{C}at_\infty$ the functor from the poset of triangulations of $X$ into $\mathcal{C}at_\infty$ that sends a triangulation $T$ to the nerve of its poset $\mathcal{I}_T$ of simplices, and a refinement $r$ of triangulations to the underlying map on posets. Postcomposing with $\mathrm{Fun}(-,\mathcal{V})$ this induces a functor $\mathrm{Fun}(\mathcal{I}^*(-),\mathcal{V}) : \mathcal{T}^{op} \to \mathcal{C}at_\infty$ sending $T \mapsto \mathrm{Fun}(\mathcal{I}_T,\mathcal{V}) = \mathcal{S}h^{simp}(T;\mathcal{V})$ and $r$ to the pullback $r^*$. Sending $r$ to $r_*$ instead, we obtain a functor $\mathrm{Fun}(\mathcal{I}_*(-),\mathcal{V}) : T \to \mathcal{C}at_\infty$.

Note that $r^*$ preserves the property of being compactly supported, since every simplex consists of finitely many refined simplices by 4.2.3, so we can restrict to a functor $\mathrm{Fun}_c(\mathcal{I}^*(-),\mathcal{V})$ sending $T$ to the full subcategory on functors $\mathcal{I}_T \to \mathcal{V}$ that send all but finitely many simplices to 0. Then, we define

$$\mathcal{S}h^{comb}(X;\mathcal{V}) := \lim_{\mathcal{T}} \mathrm{Fun}(\mathcal{I}_*(-),\mathcal{V}), \quad \mathcal{S}h_c^{comb}(X;\mathcal{V}) := \operatorname*{colim}_{\mathcal{T}} \mathrm{Fun}_c(\mathcal{I}^*(-),\mathcal{V}) \ . \tag{4.20}$$

*Technical Remark.* To be even more precise, the coCartesian fibration $\mathcal{M} \to \mathcal{T}^{op}$ classifying $\mathrm{Fun}(\mathcal{I}^*(-),\mathcal{V})$ is also Cartesian, classifying a functor $\mathrm{Fun}(\mathcal{I}_*(-),\mathcal{V})$ whose application to each refinement is right adjoint to the application of the first functor.

*Remark.* Compare this with the definition 1.5.24 of finite spectra as the sequential colimit of finite pointed spaces with transitions maps given by the suspension functor $\Sigma$, while general spectra were a sequential limit over pointed spaces with right adjoint transition map $\Omega$. A similar argument to the discussion there shows that $\mathcal{S}h_c^{simp}(K;\mathcal{V}) \subseteq \mathcal{S}h^{simp}(K;\mathcal{V})$ is canonically embedded as a full subcategory. If $X$ is compact, both categories are equivalent since every triangulation of $X$ is finite by 4.2.3. We will still mostly work with the limit definition, however as in the case of spectra the construction of $\mathcal{S}h_c^{simp}(K;\mathcal{V})$ seems more intuitive.

To define a Poincaré structure on the category of combinatorial sheaves on a compact PL space with values in a Poincaré $\infty$-category $(\mathcal{V}, \Omega)$, we need some structure theorems about refinement maps between simplicial complexes.

**Proposition 4.2.7** ([Lur11, Lecture 19, Construction 1])**.** Let $r : \mathcal{I}_K \to \mathcal{I}_L$ be a refinement of simplicial complexes and $\zeta : \mathcal{I}_L \to \mathcal{S}p^{inv}$ a spherical fibration. Then, the quadratic functor $\Omega_{L,\zeta}$ agrees with the composition $\Omega_{K,\zeta\circ r} \circ r^*$.

*Proof.* We have to see that the following colimits agree, for $F : L \to \mathcal{V}$:

$$\Omega_{L,\zeta} = \operatorname*{colim}_{\sigma' \in L} \zeta(\sigma') \wedge \Omega(F(\sigma')) \overset{!}{\cong} \Omega_{K,\zeta\circ r} \circ r^*(F) = \operatorname*{colim}_{\sigma \in K} \zeta(r(\sigma)) \wedge \Omega(F(r(\sigma)))$$

It suffices to show that the $r : K \to L$ is a right cofinal map of $\infty$-categories, which applying Quillen's Theorem A 1.2.15 is equivalent to the partially ordered set $\{\sigma \in K \mid i(\sigma) \subseteq \tau\}$ for each $\tau \in L$ being weakly contractible. But the geometric realization of $\tau$ is by assumption the union the union of the geometric realizations of the simplices $\sigma$ in this set, i.e. the geometric realization of (the nerve of) this simplicial set. Since $|\tau| = |\Delta^k|$, this is always a contractible space. □

**Corollary 4.2.8.** This implies $B_{\Omega_{L,\zeta}}(F, F') \cong B_{\Omega_{K,\zeta\circ r}}(r^*F, r^*F')$ by construction of the polarization, so

$$\operatorname{map}(F, D_{L,\zeta}F') \cong \operatorname{map}(r^*F, D_{K,\zeta\circ r} r^*F') \tag{4.21}$$

and we have $D_{L,\zeta} \cong r_* D_{K,\zeta\circ r} r^*$ by the Yoneda Lemma.

**Proposition 4.2.9** ([Lur11, Lecture 19])**.** For $r : \mathcal{I}_K \to \mathcal{I}_L$ a refinement of simplicial complexes and $\zeta : \mathcal{I}_L \to \mathcal{S}p^{inv}$ a spherical fibration, the pullback $r^*$ is duality-preserving in the sense that $D_{K,\zeta\circ r} r^* \cong r^* D_{L,\zeta}$.

*Proof.* Long but not very illuminating, see the reference. □

**Proposition 4.2.10** ([Lur11, Lecture 18, Proposition 8])**.** If $r : \mathcal{I}_K \to \mathcal{I}_L$ is a refinement, then the pullback functor $r^*$ is fully faithful.

*Proof.* We show that the unit map $\mathrm{Id} \Rightarrow i_* i^*$ is an isomorphism. Explicitly for $G : \mathcal{I}_L \to \mathcal{V}$, it can be expressed by

$$G(\tau) \cong \lim_{i(\sigma) \supseteq \tau} G(\tau) \to i_* i^* G(\tau) = \lim_{i(\sigma) \supseteq \tau} G(i(\sigma)) \tag{4.22}$$

where the left isomorphism follows because we take the limit of a constant functor over a weakly contractible diagram, as the geometric realization of the poset of $\sigma$ with $\tau \subseteq i(\sigma)$ by definition of a refinement makes up $|\tau|$, which is contractible. If $r$ was a map of simplicial sets, we could apply the same cofinality argument as in 4.1.22 to reduce to the subdiagrams with $i(\sigma) = \tau$, where this transformation clearly is an isomorphism. For refinements, the following Lemma applied to $\tau$ regarded as a PL manifold with boundary supplies a similar result, finishing the proof. □

**Lemma 4.2.11** ([Lur11, Lecture 18, Lemma 9])**.** Let $\mathcal{T}$ be a simplicial complex where the link of any vertex either looks like $S^{n-1}$ or $D^{n-1}$, i.e. a triangulation of a PL $n$-manifold with boundary. Also, let $\mathcal{I}_T^\circ$ be the sub-poset of $\mathcal{I}_T$ on those simplices $\tau$ that are not contained in the boundary, i.e. not every vertex of $\tau$ has link $D^{n-1}$. Then, the inclusion $\mathcal{I}_T^\circ \subseteq \mathcal{I}_T$ is left cofinal.

*Proof.* We work by induction on $n$, where the case $n = 0$ is trivial. For $\sigma \in \mathcal{I}_T$, we need to show that $P := \{\tau \in \mathcal{I}_T^\circ \,|\, \sigma \subseteq \tau\}$ weakly contractible. For $\sigma \in \mathcal{I}_T^\circ$ this is trivial as it makes up an initial element of this set, so assume $\sigma$ is in the boundary. Then, $P$ consists of those simplices in the open star of $\sigma$ that are not contained in the boundary. But simplices in the open star, excluding $\sigma$ itself, are in bijection with simplices in the link, via the map $\tau \mapsto \tau - \sigma$ if we regard $\tau, \sigma \subseteq T_0$. By 4.2.5, this link is a triangulation of $D^{n-1}$, so we can identify $P$ with the subset of those simplices in it that are not contained in the boundary $\partial D^{n-1}$. This is left cofinal by the inductive step, so it is weakly contractible since $D^{n-1}$ is. □

**Corollary 4.2.12.** If $r : \mathcal{I}_K \to \mathcal{I}_L$ is a refinement of simplicial complexes, the functors $C^* \circ r^*$ and $C^* : \mathcal{S}h^{simp}(L; \mathcal{V}) \to \mathcal{V}$ are naturally isomorphic, assuming that the respective limits exist.

*Proof.* This is a special case of the last proposition 4.2.10: Let $t_K : \mathcal{I}_K \to \Delta^0$ and $t_L : \mathcal{I}_L \to \Delta^0$ be terminal maps of simplicial sets, then

$$C^* \circ r^* = t_{K,*} \circ r^* = t_{L,*} \circ r_* \circ r^* = t_{L,*} = C^* \,.$$ □

*Remark.* We can not simply use a cofinality argument to show $\lim_{\tau \in L} F(\tau) \cong \lim_{\sigma \in K} F(r(\sigma))$, as it is not clear that $r$ is left cofinal (while it was easy to verify that it is right cofinal).

Similarly to this result, we know that $C^* \circ r_* = C^*$ by 4.1.23, and even $C^* r_+ \cong C^*$ by [Lur11, Lecture 18, Proposition 11].

**Definition 4.2.13.** Since we have seen in 4.1.23 that the global sections functor $C^*$ is compatible with pushing forward along maps of posets, its value on any two triangulations must agree as we can compare them on a common refinement, so it defines a functor

$$C^* : \mathcal{S}h^{comb}(X; \mathcal{V}) \to \mathcal{V} . \tag{4.23}$$

Formally, $C^*$ defines a natural transformation $\operatorname{Fun}(\mathcal{I}_*, \mathcal{V}) \Rightarrow \underline{\mathcal{V}}$ of functors $\mathcal{T} \to \mathcal{C}at_\infty$, and taking the limit on both sides yields a functor $\mathcal{S}h^{comb}(X; \mathcal{V}) \to \lim_{\mathcal{T}} \underline{\mathcal{V}} = \mathcal{V}$ as $\mathcal{T}$ is filtered an therefore weakly contractible.

**Definition 4.2.14.** A *spherical fibration* $\zeta$ on a PL space $(X, \mathcal{T})$ consists of

- A right cofinal subset $\mathcal{T}_\zeta \subseteq \mathcal{T}$, in the sense that every admissible triangulation of $X$ possesses a refinement in $\mathcal{T}_\zeta$
- A natural transformation from $\mathcal{I}^* : \mathcal{T}_\zeta^{op} \to \mathcal{C}at_\infty$ to the constant functor on $\mathcal{S}p^{inv}$.

In other words, we need to choose spherical fibrations compatibly on a cofinal subset of all triangulations of $X$.

**Construction 4.2.15.** For $r : \mathcal{I}_K \to \mathcal{I}_L$ a refinement and $\zeta : \mathcal{I}_L \to \mathcal{S}p^{inv}$ a spherical fibration, we have seen in 4.2.7 that $\text{Ϙ}_{L,\zeta} \cong \text{Ϙ}_{K,\zeta\circ r} \circ r^* : L \to \mathcal{V}$ are naturally isomorphic. Hence, given a spherical fibration $\zeta$ on a cofinal set of triangulations $\mathcal{T}_\zeta$ for a PL space $X$, we can glue these quadratic functor together to obtain a map $\operatorname{Fun}_c(\mathcal{I}_*(-), \mathcal{V})^{op} \Rightarrow \underline{\mathcal{S}p}$ between functors $\mathcal{T}_\zeta \to \mathcal{C}at_\infty$, which by definition is equivalent to a functor

$$\text{Ϙ}_{X,\zeta} : \mathcal{S}h_c^{comb}(X; \mathcal{V})^{op} \to \mathcal{S}p \tag{4.24}$$

out of the colimit. Similarly, by 4.2.9 we know that the duality functors $D_{L,\zeta}$ are compatible with pushforwards along refinements, so they glue to a functor

$$\mathbb{D}_{X,\zeta} : \mathcal{S}h_c^{comb}(X; \mathcal{V})^{op} \to \mathcal{S}h_c^{comb}(X; \mathcal{V}) \tag{4.25}$$

satisfying $\mathbb{D}^2_{X,\zeta} = \operatorname{Id}$ since this holds on all components by 4.1.19. In fact, we can check on components of the colimit all conditions that are necessary to exhibit $\text{Ϙ}_{X,\zeta}$ as a quadratic functor and $\mathcal{S}h_c^{comb}(X; \mathcal{V})$ as a Poincaré $\infty$-category. See below for a more abstract argument.

**Construction 4.2.16** ([Lur11, Lecture 20]). The isomorphism $\text{Ϙ}_{K,\zeta\circ r} \circ r^* \cong \text{Ϙ}_{L,\zeta}$ induces[1] an adjoint morphism $\text{Ϙ}_{K,\zeta\circ r} \to \text{Ϙ}_{L,\zeta} \circ r^*$ that generally is not an isomorphism. Explicitly, we obtain it as the composition

$$\text{Ϙ}_{K,\zeta\circ r} F \cong \operatorname*{colim}_{\tau \in L, r(\sigma)=\tau} \zeta(\tau) \wedge \text{Ϙ}(F(\sigma)) \to \operatorname*{colim}_{\tau\in L} \zeta(\tau) \wedge \text{Ϙ}\left(\lim_{r(\sigma)=\tau} F(\sigma)\right) = \text{Ϙ}_{L,\zeta}(r_* F) \tag{4.26}$$

[1]Note that we actually precompose $\text{Ϙ}_{K,\zeta\circ r}$ with $(r^*)^{op}$, and $(- \circ (r^*)^{op}) \dashv (- \circ (r_*)^{op})$ with (co)units induced by precomposition with the original (co)units.

which even works for arbitrary maps of posets instead of $r$. This means that $r_*$ becomes a hermitian functor, which is even duality-preserving by 4.1.24. For any PL space $(X, \mathcal{T})$ with spherical fibration $(\zeta, \mathcal{T}_\zeta)$ with $\mathcal{T}_\zeta = \mathcal{T}$ we thus obtain a (filtered) diagram $\mathcal{T} \to \mathcal{C}at_\infty^p$ of Poincaré $\infty$-categories, which by [CDH+20a, Proposition 6.1.4] admits a limit. Explicitly by [CDH+20a, Remark 6.1.3], we need to form the limit $\mathcal{S}h^{comb}(X; \mathcal{V})$ of the underlying $\infty$-categories along the pushforwards $r_*$, and equip it with the limit of the quadratic functors pulled back to this limit cone: If we denote by $\pi_T : \mathcal{S}h^{comb}(X; \mathcal{V}) \to \mathcal{S}h^{simp}(T; \mathcal{V})$ the canonical projections, then

$$Ϙ_{X,\zeta}(F) := \lim_{T \in \mathcal{T}} Ϙ_{T,\zeta_T} \circ \pi_T(F) \ . \tag{4.27}$$

*Remark.* The restriction to $\mathcal{T}_\zeta = \mathcal{T}$ is necessary since otherwise, the limit over the diagram $\mathcal{T} \to \mathcal{C}at_\infty^p$ need not yield combinatorial sheaf, as $\mathcal{T}_\zeta$ is right and not necessarily left cofinal. We do not know how to fix this, unless of course $X$ is compact.

**Proposition 4.2.17.** For $(X, \mathcal{T})$ a compact PL space with spherical fibration $(\zeta, T_\zeta)$, both functors $Ϙ_{X,\zeta}$ we have just constructed on $\mathcal{S}h^{comb}(X; \mathcal{V}) \cong \mathcal{S}h_c^{comb}(X; \mathcal{V})$ agree and equip it with the structure of a Poincaré $\infty$-category.

*Proof.* Since for $r$ a refinement, both $r_*$ and $r^*$ are duality-preserving, the duality functors agree by construction. The case of the quadratic functors follows from our construction of the inclusion $\mathcal{S}h_c^{comb}(X; \mathcal{V}) \subseteq \mathcal{S}h^{comb}(X; \mathcal{V})$, where the left was defined as a colimit over pullbacks and the right as a limit over the right adjoint pushforwards. Since the comparison maps $Ϙ_{K,\zeta\circ r} \circ r^* \cong Ϙ_{L,\zeta}$ and $Ϙ_{K,\zeta\circ r} \to Ϙ_{L,\zeta} \circ r^*$ correspond to each other under this adjunction $r^* \dashv r_*$, they must induce the same functor on the colimit/ full subcategory of the limit. ☐

By this theorem and 4.1.19, we can for any Poincaré $\infty$-category $(\mathcal{V}, Ϙ)$ define L-spectra

- $\mathbb{L}(\mathcal{S}h^{simp}(K; \mathcal{V}), Ϙ_{K,\zeta})$ for any finite simplicial complex $K$ and spherical fibration $\zeta : \mathcal{I}_K \to \mathcal{S}p^{inv}$,
- $\mathbb{L}(\mathcal{S}h^{comb}(X; \mathcal{V}), Ϙ_{X,\zeta})$ for any compact PL space $X$ and spherical fibration $\zeta$ on $X$.

We will study them, and related L-spectra, in the next sections.

## 4.3 Locally Constant Sheaves and their L-spectrum

Recall that we have defined a simplicial sheaf $F : \mathcal{I}_K \to \mathcal{V}$ to be locally constant if for any $\sigma \subseteq \tau$ in $\mathcal{I}_K$, the image $F(\sigma \subseteq \tau)$ is an isomorphism.

**Proposition 4.3.1.** If $K, L$ are simplicial complexes and $f : \mathcal{I}_K \to \mathcal{I}_L$ is a map of posets, then the pullback of simplicial sheaves $f^*$ preserves locally constant sheaves.

*Proof.* If $F$ is a locally constant sheaf on $L$ and $\sigma \subseteq \tau$ in $\mathcal{I}_K$, then $f^*F(\sigma \subseteq \tau) = F(f(\sigma) \subseteq f(\tau))$ must also be an equivalence, so $f^*F$ is indeed locally constant. $\square$

**Proposition 4.3.2.** For $r : \mathcal{I}_T \to \mathcal{I}_{T'}$ a refinement, the pushforward $r_* : \mathcal{S}h^{simp}(T; \mathcal{V}) \to \mathcal{S}h^{simp}(T'; \mathcal{V})$ restricts to the full subcategories of locally constant sheaves $\mathcal{S}h^{lc}(T; \mathcal{V}) \to \mathcal{S}h^{lc}(T', \mathcal{V})$ where it induces an equivalence of categories. Similarly for the precomposition $r^*$ which induces the inverse to this equivalence, and for $r_+$.

*Proof.* As $r_+ \dashv r^* \dashv r_*$, it suffices to prove the claim for $r^*$ since on any full subcategory that one adjoint is an equivalence on, the other is an equivalence as well by [Lur18a, Tag 02EX]. By the previous proposition, $r^*$ preserves locally constant sheaves since we can factor $r$ through the localization

$$\bar{r} : \mathcal{I}_T[W_T^{-1}] \longrightarrow \mathcal{I}_{T'}[W_{T'}^{-1}]$$

where $W_T, W_{T'}$ denote the classes of all morphisms in $\mathcal{I}_T, \mathcal{I}_{T'}$ respectively. Precomposition with this map of simplicial sets agrees with $r^*$ on locally constant sheaves. It suffices to show that $\bar{r}$ is an equivalence of categories, which as both sides are Kan complexes is equivalent to $\bar{r}$ being a homotopy equivalence. Localizing at all morphisms is a form of Quillen fibrant replacement, just as $\mathrm{Sing}\,|-|$, so we can identify $\bar{r}$ with

$$\mathrm{Sing}\,|r| : \mathrm{Sing}\,|\mathcal{I}_T| \to \mathrm{Sing}\,|\mathcal{I}_{T'}|\ .$$

Since we assume $r$ to be a refinement, this map on partially ordered sets is induced by a homeomorphism $r : |K| \to |L|$, and one can check that, since we know how everything is glued together and simplices are contractible, $|r|$ must be homotopic to this homeomorphisms and hence is a homotopy equivalence itself (actually, it is even a homeomorphism). Since Sing sends those to homotopy equivalences of Kan complexes, we are finished. $\square$

**Definition 4.3.3.** For $(X, \mathcal{T})$ a PL space and $T$ any triangulation of it, we define the $\infty$-category of *locally constant* sheaves on $X$ by $\mathcal{S}h^{lc}(X; \mathcal{V}) = \mathcal{S}h^{lc}(T; \mathcal{V})$. Since the partially ordered set $\mathcal{T}$ is filtered, this is by the last Proposition independent of $T$ up to isomorphism since any two triangulations can be compared on a common refinement. Taking a limit over all triangulations, we obtain a fully faithful subcategory

$$\mathcal{S}h^{lc}(X; \mathcal{V}) \subseteq \mathcal{S}h^{comb}(X; \mathcal{V}) \tag{4.28}$$

since fully faithful functors are closed under limits.

*Remark.* Alternatively, we could define $\mathcal{S}h^{lc}(X; \mathcal{V}) := \lim_{T \in \mathcal{T}} \mathcal{S}h^{lc}(T; \mathcal{V})$ since we know this limit diagram is essentially constant, and $\mathcal{T}$ is filtered and hence weakly contractible. We could also take a colimit.

**Proposition 4.3.4.** If $\mathcal{V}$ has limits and colimits, the inclusion of locally constant into all simplicial sheaves $\mathcal{S}h^{lc}(K;\mathcal{V}) \subseteq \mathcal{S}h^{simp}(K;\mathcal{V})$ has a left adjoint $L_{lc}$ and a right adjoint $R_{lc}$, and similarly for $\mathcal{S}h^{lc}(X;\mathcal{V}) \subseteq \mathcal{S}h^{comb}(X;\mathcal{V})$.

*Proof.* As discussed above, the first inclusion agrees with the map $\mathrm{Fun}(\mathcal{I}_K[W^{-1}],\mathcal{V}) \to \mathrm{Fun}(\mathcal{I}_K,\mathcal{V})$ induced by precomposing with the localization functor $L : \mathcal{I}_K \to \mathcal{I}_K[W^{-1}]$ at all morphisms in $\mathcal{I}_K$. Since $\mathcal{V}$ has limits and colimits, this precomposition has a left adjoint $L_{lc} = \mathrm{Lan}_L$ and right adjoint $R_{lc} = \mathrm{Ran}_L$.

In the PL case, using $\mathcal{S}h^{lc}(X;\mathcal{V}) = \lim_{T\in\mathcal{T}} \mathcal{S}h^{lc}(T;\mathcal{V})$ we can form the limit

$$L_{lc} := \lim_{T\in\mathcal{T}} L_{lc} : \mathcal{S}h^{comb}(X;\mathcal{V}) \to \mathcal{S}h^{lc}(X;\mathcal{V})$$

which is still left adjoint to the inclusion, since adjunctions are preserved under (co)limits of $\infty$-categories. To see this, take the (co)limit of the respective units and counits and notice that by functoriality, the triangle identities are still fulfilled. Similarly for $R_{lc}$. □

**Corollary 4.3.5.** In particular, locally constant sheaves are closed under fibers, direct sums and contain the zero sheaf, so they form a stable subcategory of $\mathcal{V}$.

**Proposition 4.3.6.** On any simplicial complex $K$, there is an equivalence of categories

$$\mathcal{S}h^{lc}(K;\mathcal{V}) \simeq \mathrm{Fun}(\mathrm{Sing}\,|K|,\mathcal{V}) \ . \tag{4.29}$$

In particular, any locally constant sheaf on a simplex $K = \Delta^n$ is constant, as $\mathrm{Sing}\,|\Delta^n|$ is contractible.

*Proof.* As in the last proof, $\mathcal{S}h^{lc}(K;\mathcal{V}) = \mathrm{Fun}(\mathcal{I}_K[W^{-1}],\mathcal{V})$. But $\mathcal{I}_K[W^{-1}]$ and $\mathrm{Sing}\,|K|$ are both Quillen-replacements of $\mathcal{I}_K$, since $\mathcal{I}_K$ is the subdivision of $K$ regarded as a simplicial set so they are weakly equivalent, meaning that $|K| \simeq |\mathcal{I}_K|$. Thus, $\mathcal{I}_K[W^{-1}] \simeq \mathrm{Sing}\,|K|$ are homotopy equivalent Kan complex and therefore in particular categorically equivalent. □

**Definition 4.3.7.** A simplicial sheaf $F : \mathcal{I}_K \to \mathcal{V}$ is called *balanced* if for every locally constant sheaf $S$, the mapping space $\mathrm{Map}(F,S) \simeq \Delta^0$ is contractible. Let us denote their full subcategory by $\mathcal{S}h^{\perp lc}(K;\mathcal{V}) \subseteq \mathcal{S}h^{simp}(K;\mathcal{V})$. If $\mathcal{V}$ admits colimits, this is equivalent to $L_{lc}F = 0$ since we can identify above mapping space with $\mathrm{Map}(L_{lc}F,S)$ and apply the Yoneda Lemma.

**Definition 4.3.8.** If $r : \mathcal{I}_K \to \mathcal{I}_L$ is a refinement and $G$ a balanced sheaf on $L$, then $\mathrm{Map}(r^*F,S) \cong \mathrm{Map}(F,r_*S) = 0$ since $r_*$ preserves locally constant sheaves. We thus obtain a full subcategory of balanced sheaves $\mathcal{S}h^{\perp lc}(X;\mathcal{V}) \subseteq \mathcal{S}h^{comb}(X;\mathcal{V})$ on any compact PL space $(X,\mathcal{T})$ by taking a colimit over pullbacks along refinements, consisting of the kernel of $L_{lc}$ if $\mathcal{V}$ admits colimits.

*Remark.* A similar argument shows that for any map of posets $f : K \to L$, the functor $f_+$ preserves balanced sheaves.

**Example 4.3.9.** If $K = \Delta^n$ is a simplex, then $F \in \mathcal{S}h^{simp}(K; \mathcal{V})$ is balanced iff its simplicial chain complex $C_*F$ is trivial. This is because we have seen in 4.3.6 that every locally constant sheaf on $\mathcal{V}$ is constant, so $F$ is balanced iff for any $V \in \mathcal{V}$

$$\mathrm{Map}(F, \underline{V}) \simeq \mathrm{Map}(\operatorname*{colim}_{\mathcal{I}_K} F, V) \simeq \Delta^0$$

which by the Yoneda-Lemma implies that the colimit $C_*F = 0$. A dual argument shows that $F \in \mathcal{S}h^{lc}(K; \mathcal{V})^\perp$ iff the global sections $C^*F$ vanish.

**Example 4.3.10.** Let $\tau \subseteq \tau'$ be two arbitrary simplices in $K$, $V \in \mathcal{V}$ and $F^{\tau,V}, F^{\tau',V}$ the sheaves we defined in 4.1.6. There is a canonical map $F^{\tau',X} \to F^{\tau,X}$ acting on simplices containing $\tau'$ as the identity and as zero otherwise, whose (co)fiber is a balanced sheaf. To see this, let $S$ be an arbitrary locally constant sheaf on $K$. Then,

$$\mathrm{Map}(\mathrm{cofib}(F^{\tau',V} \to F^{\tau,V}), S) \simeq \mathrm{fib}\left(\mathrm{Map}(F^{\tau,V}, S) \to \mathrm{Map}(F^{\tau',V}, S)\right)$$

which by 4.1.25 agrees with $\mathrm{fib}\left(\mathrm{Map}(V, S(\tau) \to \mathrm{Map}(V, S(\tau')))\right) = 0$ since this map is an isomorphism by the assumption that $S$ is locally constant. In fact, using that the $F^{\tau,V}$ generate $\mathcal{S}h^{simp}(K; \mathcal{V})$ under colimits, we see that such cofibers generate all balanced sheaves.

**Proposition 4.3.11.** Let $\mathcal{V}$ be stable and bicomplete, and $K$ be a simplicial set. The sequence of stable $\infty$-categories

$$\mathcal{S}h^{\perp lc}(K; \mathcal{V}) \hookrightarrow \mathcal{S}h^{simp}(K; \mathcal{V}) \overset{L_{lc}}{\to} \mathcal{S}h^{lc}(K; \mathcal{V}) \tag{4.30}$$

is a right split Verdier sequence, and similarly if we replace $K$ by a PL space $X$. Dually, the sequence

$$\mathcal{S}h^{lc}(K; \mathcal{V}) \hookrightarrow \mathcal{S}h^{simp}(K; \mathcal{V}) \to \mathcal{S}h^{lc}(K; \mathcal{V})^\perp \tag{4.31}$$

using the right orthogonal subcategory is a split Verdier sequence.

*Proof.* This is immediate from 3.2.12 and its proof, since we know about the existence of adjoints by 4.3.4. □

Our goal is to refine this to a Poincaré-Verdier sequence, but the issue in doing this is that Poincaré $\infty$-categories are usually not bicomplete. In many cases, we can however embed them into a stable bicomplete $\infty$-category, denote this as $i : \mathcal{V} \hookrightarrow \mathcal{W}$. A natural candidate is the Ind-completion $\mathrm{Ind}(\mathcal{V})$ which satisfies these properties by 3.2.18. In fact, this is a universal candidate since by 3.2.19, the functor $i$ factors uniquely through a colimit-preserving functor $\mathrm{Ind}(\mathcal{V}) \to \mathcal{W}$ that is fully faithful if the essential image of $i$ consists of compact objects by [Lur09a, 5.3.5.11].

**Construction 4.3.12.** Let $\mathcal{V}$ be a Poincaré $\infty$-category that is embedded inside the stable bicomplete $\infty$-category $\mathcal{W} = \mathrm{Ind}(\mathcal{V})$. For $K$ a finite simplicial complex and $\zeta : \mathcal{I}_K \to \mathcal{S}p^{inv}$ a spherical fibration, define the $\infty$-category of *$\mathcal{V}$-generated locally constant sheaves* as the Verdier quotient

$$\mathcal{S}h^{lc}(K;\mathcal{W})^{(\mathcal{V})} := \mathcal{S}h^{simp}(K;\mathcal{V})\big/\mathcal{S}h^{\perp lc}(K;\mathcal{V}) \subseteq \mathcal{S}h^{simp}(K;\mathcal{W}) \tag{4.32}$$

where the last inclusion follows from 3.2.21. Note that

$$\mathcal{S}h^{\perp lc}(K;\mathcal{V}) = \mathcal{S}h^{\perp lc}(K;\mathcal{W}) \cap \mathcal{S}h^{simp}(K;\mathcal{V}) \tag{4.33}$$

since $\mathcal{S}h^{lc}(K;\mathcal{W}) = \mathrm{Fun}(\mathcal{I}_K[W^{-1}], \mathrm{Ind}\,\mathcal{V}) = \mathrm{Ind}\,\mathcal{S}h^{lc}(K;\mathcal{V})$ so every locally constant sheaf in $\mathcal{W}$ is a filtered colimit of locally constant sheaves in $\mathcal{V}$, meaning that a simplicial sheaf $F$ that is orthogonal to the latter and also a compact object, is orthogonal to the former class of sheaves.

According to [Lur11, Lecture 21, p.2], we can further identify the full subcategory $\mathcal{S}h^{lc}(K;\mathcal{W})^{(\mathcal{V})}$ in $\mathcal{S}h^{simp}(K;\mathcal{W})$ as the essential image of $\mathcal{S}h^{simp}(K;\mathcal{V})$ under the functor $L_{lc}$. [We do however not understand his argument; while it is possible to show this using our 9-Lemma 3.3.8, this seems highly non-trivial and we are not able to verify our factorization condition.]

**Theorem 4.3.13** ([Lur11, Lecture 22, Lemma 3])**.** In the situation described above, the sequence

$$\mathcal{S}h^{\perp lc}(K;\mathcal{V}) \longrightarrow \mathcal{S}h^{simp}(K;\mathcal{V}) \xrightarrow{L_{lc}} \mathcal{S}h^{lc}(K;\mathcal{W})^{(\mathcal{V})} \tag{4.34}$$

is a Poincaré-Verdier sequence, where we equip the middle with the quadratic functor $\Omega_{K,\zeta}$, the left with its restriction and the right with its restriction from $\mathcal{S}h^{simp}(K;\mathcal{W})$.

*Proof.* We know that the middle entry is a Poincaré $\infty$-category by 4.1.19, and the sequence is Verdier by definition. By 3.3.2, it suffices to show that $\mathcal{S}h^{\perp lc}(K;\mathcal{V})$ is closed under duality, since the left Kan extension of $\Omega_{K,\zeta}$ agrees with the restriction from $\mathcal{S}h^{simp}(K;\mathcal{W})$ by our derivation of equation 3.12 from 3.2.21.

Given $F^0 \in \mathcal{S}h^{\perp lc}(K;\mathcal{V})$, we have to show that for any $S \in \mathcal{S}h^{lc}(K;\mathcal{V})$,

$$\mathrm{Map}(D_{K,\zeta}F^0, S) \simeq \mathrm{Map}(D_K F, \zeta^{-1} \wedge S) \simeq \Delta^0$$

so since $\zeta$ is locally constant, we may reduce to $\zeta = \underline{\mathbb{S}}$. Further, we know by 4.3.10 that the category of balanced sheaves is generated under colimits by the objects $\mathrm{cofib}(F^{\tau',V} \to F^{\tau,V}) =: F^{\tau'/\tau,V}$ for $\tau \subseteq \tau'$ in $\mathcal{I}_K$ and $V \in \mathcal{V}$. In fact, for $\tau \subseteq \tau' \subseteq \tau''$, applying $\mathrm{Map}(D_K(-), S)$ to the fiber sequence

$$F^{\tau''/\tau',V} \longrightarrow F^{\tau''/\tau,V} \longrightarrow F^{\tau'/\tau,V}$$

tells us that is suffices to prove our statement for its outer entries. Inductively, we restrict to the case where $\tau'$ is an $n$-simplex and $\tau$ a codimension-1-face.

Denote by $\Lambda^{\tau'}_\tau$ the horn $\partial\tau' - \tau$. We then calculate

$$D_K F_{\tau'/\tau,V}(\sigma) = \operatorname*{colim}_{\sigma'\in K} \begin{cases} D_\Qoppa V & \text{for } \tau' \nsubseteq \sigma' \supseteq \sigma, \tau \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \Sigma^n D_\Qoppa V, & \text{for } \sigma \subseteq \tau', \sigma \nsubseteq \Lambda^{\tau'}_\tau \\ 0 & \text{otherwise} \end{cases}$$

similarly to the proof of 4.1.19. We can pull this back from a coarser triangulation where both $\tau'$ and $\Lambda^{\tau'}_\tau$ are simplices, where this sheaf is described by $F_{\tau'/\Lambda^{\tau'}_\tau, \Sigma^n D_\Qoppa V}$. This is balanced, and $r^*$ preserves balanced sheaves as we have seen, so we are finished. □

**Theorem 4.3.14.** Let $\mathcal{V}, \mathcal{W}$ be as above and $(X, \mathcal{T})$ be a compact PL space with spherical fibration $(\zeta, \mathcal{T}_\zeta)$. Define

$$\mathcal{S}h^{lc}(X;\mathcal{W})^{(\mathcal{V})} := \mathcal{S}h^{comb}(X;\mathcal{V}) / \mathcal{S}h^{\perp lc}(X;\mathcal{V}) \subseteq \mathcal{S}h^{lc}(X;\mathcal{W}) \tag{4.35}$$

which, as in the simplicial case, consists of the essential image of $L_{lc}$. Then, the sequence

$$\mathcal{S}h^{\perp lc}(X;\mathcal{V}) \longrightarrow \mathcal{S}h^{comb}(X;\mathcal{V}) \xrightarrow{L_{lc}} \mathcal{S}h^{lc}(X;\mathcal{W})^{(\mathcal{V})} \tag{4.36}$$

is a Poincaré-Verdier sequence, where we equip the middle with the quadratic functor $\Qoppa_{X,\zeta}$, the left with its restriction and the right with its restriction from $\mathcal{S}h^{comb}(X;\mathcal{W})$.

*Proof.* It is a Verdier sequence by definition, and since the quadratic functors are glued together from compatible quadratic functors on components the rest can be checked on triangulations and thus follows from the last proof. □

These sequences are not very useful unless we have some control over $\mathcal{S}h^{lc}(K;\mathcal{W})^{(\mathcal{V})}$. If we work with $\mathcal{V} = \mathrm{LMod}^{\mathrm{fp}}_R$ and $\mathcal{W} = \mathrm{Ind}(\mathcal{V}) = \mathrm{LMod}_R$ over a ring spectrum $R$ that is an algebra over a commutative ring spectrum $k$ (potentially $k = \mathbb{S}$), this can be achieved. The proofs of the following statements are similar to the topological case, so we postpone them until 5.4.

**Proposition 4.3.15** ([Lur11, Lecture 21])**.** For $K$ a connected simplicial complex, there is an equivalence of categories

$$\mathcal{S}h^{lc}(K;\mathrm{LMod}_R) \simeq \mathrm{LMod}_{\Sigma^\infty\Omega|K|\wedge R} . \tag{4.37}$$

Similarly for $X$ a PL space, $\mathcal{S}h^{lc}(X;\mathrm{LMod}_R) \simeq \mathrm{LMod}_{\Sigma^\infty\Omega X\wedge R}$.

**Proposition 4.3.16** ([Lur11, Lecture 21, Theorem 2])**.** For $K$ a connected simplicial complex, there is an equivalence of categories

$$\mathcal{S}h^{lc}(K;\mathrm{LMod}_R)^{(\mathrm{fp})} \simeq \mathrm{LMod}^{\mathrm{fp}}_{\Sigma^\infty\Omega|K|\wedge R} \tag{4.38}$$

where the $(fp)$ in the exponents denotes $\mathrm{LMod}^{\mathrm{fp}}_R$-generated sheaves. Similarly for $X$ a PL space, $\mathcal{S}h^{lc}(X;\mathrm{LMod}_R)^{(\mathrm{fp})} \simeq \mathrm{LMod}^{\mathrm{fp}}_{\Sigma^\infty\Omega X\wedge R}$.

If $M$ is an invertible module over $R$, we can use the duality functors $\Qoppa^q_M, \Qoppa^s_M$ to induce $\Qoppa^q_{M,K,\zeta}, \Qoppa^s_{M,K,\zeta}$ on $K$. Their left Kan extension to locally constant sheaves is described by equipping left modules over $\Sigma^\infty\Omega|K| \wedge R$ with an involution combined from

- the involution in $M$,
- the loop-reversing involution on $\Omega X$,
- the non-triviality of $\zeta$, e.g. the obstruction along a loop to orienting it.

See the mentioned later discussion for a more precise statement and proof, the only difference in the latter is that we use $F_{\tau,V}$ as a generating set.

**Corollary 4.3.17** ([Lur11, Lecture 22, Proposition 6])**.** If we consider the case $(\mathcal{V}, Ϙ) = (\mathrm{LMod}_R^{fp}, Ϙ_R^s)$ for connected $X$, we can rewrite

$$\mathbb{L}^{vs}(X; R) := \mathbb{L}\left(\mathcal{S}h^{lc}(X; \mathrm{LMod}_R^{fp}), Ϙ_{K,\zeta}\right) \simeq \mathbb{L}(\Sigma^\infty \Omega X \wedge R, Ϙ_{M,K,\zeta}^s) \tag{4.39}$$

which we call the *visible symmetric L-groups* of $X$ with coefficients in $R$. Similarly, we can define the *visible quadratic L-groups* $\mathbb{L}^{vq}(X; R)$ of $X$. If $R$ is connective, we may apply 2.5.10 to rewrite

$$\mathbb{L}^{vq}(X; R) \cong \mathbb{L}^q(\pi_0(\Sigma^\infty \Omega X \wedge R)) = \mathbb{L}^q(\pi_0 R[\pi_1 X]) \ . \tag{4.40}$$

Though we suppress it, in both cases the involution must be kept in mind.

*Remark.* If we even assume that $X$ is simply connected, we have $\mathbb{L}^{vq}(X; R) = \mathbb{L}^q(\pi_0 R)$. This is commonly exploited in algebraic topology for $R = H\mathbb{Z}$ or $R = H\mathbb{Q}$, where we obtain the signature and Arf invariant of the space $X$ comparing with the respective L-groups 2.4.11.

## 4.4 Assembly

Until now, we have defined several different Poincaré $\infty$-categories of sheaves associated to simplicial complexes and PL spaces, allowing us to calculate L-spectra fitting into a fiber sequence

$$\mathbb{L}(\mathcal{S}h^{\perp lc}(X; \mathcal{V})) \longrightarrow \mathbb{L}(\mathcal{S}h^{simp}(X; \mathcal{V})) \longrightarrow \mathbb{L}(\mathcal{S}h^{lc}(X; \mathcal{W})^{(\mathcal{V})}) \tag{4.41}$$

obtained by applying 3.3.9 to 4.3.14. We have also seen that the last L-spectrum agrees with $\mathbb{L}(\pi_0 R[\pi_1 X])$, i.e. it can be calculated from the group ring $(\pi_0 R)[\pi_1 X]$ so it only depends on the fundamental group of $X$. Our goal in this section will be to learn more about the L-spectrum in the middle.

**Construction 4.4.1.** From 4.1.24 we know that given a map of simplicial complexes $f : K \to L$, the pushforward functor $f_*$ is duality preserving. Therefore, it induces a map between L-spectra

$$f_* : \mathbb{L}(\mathcal{S}h^{simp}(K; \mathcal{V}), Ϙ_{K,\zeta\circ f}) \longrightarrow \mathbb{L}(\mathcal{S}h^{simp}(L; \mathcal{V}), Ϙ_{L,\zeta}) \tag{4.42}$$

and we obtain a functor $\mathbb{L}(\mathcal{S}h^{simp}(-;\mathcal{V}), \Qoppa_{-})$ from the ordinary category of pairs of finite simplicial complexes and spherical fibrations into $\mathcal{S}p$. If we set $\zeta = \mathbb{S}$ to be trivial, we denote this functor by $\mathbb{L}(K, \mathcal{V}, \Qoppa)$.

Now, let us assume that our finite simplicial complexes are pointed. Noting $\mathbb{L}(\Delta^0, \mathcal{V}, \Qoppa) = \mathbb{L}(\mathcal{V}, \Qoppa)$ we form the reduced L-groups

$$\mathbb{L}^{\mathrm{red}}(K, \mathcal{V}, \Qoppa) := \mathrm{cofib}\left(\mathbb{L}(\mathcal{V}, \Qoppa) \to \mathbb{L}(K, \mathcal{V}, \Qoppa)\right) \tag{4.43}$$

where the map is induced by the pointing of $K$.

**Construction 4.4.2.** Just as in the last construction, we can define a functor $\mathbb{L}(-, \mathcal{V}, \Qoppa)$ from the ordinary category Poly of compact PL spaces and PL maps to spectra sending $X$ to the L-spectrum of $\mathcal{S}h^{simp}(X;\mathcal{V})$ with respect to $\Qoppa_X$. On pointed PL spaces, we also define a reduced version of this functor.

*Remark.* Let $W$ denote the class of morphisms in Poly that are PL homotopy equivalences. Then, the $\infty$-categorical localization Poly$[W^{-1}]$ agrees with the $\infty$-category $\mathcal{S}^{\mathrm{fin}}$ of finite spaces.

**Theorem 4.4.3** ([Lur11, Lecture 20])**.** The functor $\mathbb{L}(-, \mathcal{V}, \Qoppa) : \mathrm{Poly} \to \mathcal{S}p$ as well as its reduced variant are invariant under PL homotopies, in particular they send the morphism in $W$ to isomorphisms. We obtain a functor

$$\Omega^\infty \mathbb{L}^{\mathrm{red}}(-, \mathcal{V}, \Qoppa) : \mathcal{S}^{\mathrm{fin}} \to \mathcal{S} \tag{4.44}$$

that is reduced and excisive, so that it defines a spectrum, which agrees with $\mathbb{L}(\mathcal{V}, \Qoppa)$.

*Proof Sketch.* Let $f : X \to Y$ be a PL map between compact PL spaces, which we rewrite as $f : T \to S$ on triangulations. Potentially after further refinement, let $h : T \times \Delta^1 \to S$ be a PL homotopy from $f$ to $g$. It suffices to show that for $F \in \mathcal{S}h^{simp}(K;\mathcal{V})$ a Poincaré object, the pushforwards $f_*F$ and $g_*F$ are bordant, since the same argument can then also be applied to Poincaré objects in the $\rho$-construction. For this purpose, denote by $i_0, i_1 : K \to K \times \Delta^1$ the inclusions at 0 and 1, and by $p : K \times \Delta^1 \to X$ the projection. Since $h_*$ is duality-preserving, it in particular preserves Poincaré objects and bordisms, so our goal is to show that $i_{0,*}F$ and $i_{1,*}F$ are bordant. But $i_{0,*}F$ by definition agrees with $F$ on $K \times \{0\}$ and vanishes over 1, similarly for $i_{1,*}F$. Clearly $p^*F$ supplies the wanted bordism, as it agrees with $F$ over 0 and 1 so dividing out one of these pushforwards yields the other.

The functor $\Omega^\infty \mathbb{L}^{\mathrm{red}}(-, \mathcal{V}, \Qoppa)$ is reduced by definition, so we only need to show that it sends pushout squares to pullback squares. If $X = X' \amalg_{Y'} Y$ is a (homotopy) pushout of finite pointed spaces, it suffices to show that

$$\frac{\mathbb{L}^{\mathrm{red}}(X, \mathcal{V}, \Qoppa)}{\mathbb{L}^{\mathrm{red}}(X', \mathcal{V}, \Qoppa)} \cong \frac{\mathbb{L}^{\mathrm{red}}(Y, \mathcal{V}, \Qoppa)}{\mathbb{L}^{\mathrm{red}}(Y', \mathcal{V}, \Qoppa)}$$

where by construction both sides can be rewritten as

$$\mathbb{L}^{\mathrm{red}}(X/X', \mathcal{V}, \mathfrak{Q}) \cong \mathbb{L}^{\mathrm{red}}(Y/Y', \mathcal{V}, \mathfrak{Q})$$

which follows by homotopy invariance and the assumption $X/X' \simeq Y/Y'$.

By this analysis, the functor $\mathbb{L}^{\mathrm{red}}(-, \mathcal{V}, \mathfrak{Q}) : \mathcal{S}^{\mathrm{fin}} \to \mathcal{S}p$ is itself reduced and excisive, so it defines an object in the spectrification $\mathcal{S}p(\mathcal{S}p) = \mathcal{S}p \otimes \mathcal{S}p \cong \mathcal{S}p$ as it is the unit of this tensor product. Unwinding the definitions, under this equivalence $\mathbb{L}(-, \mathcal{V}, \mathfrak{Q})$ corresponds to its infinite loop space

$$\Omega^\infty_{\mathcal{S}p}(\mathbb{L}(-, \mathcal{V}, \mathfrak{Q}) = \mathbb{L}^{\mathrm{red}}(S^0, \mathcal{V}, \mathfrak{Q}) = \mathbb{L}(\Delta^0, \mathcal{V}, \mathfrak{Q}) = \mathbb{L}(\mathcal{V}, \mathfrak{Q})$$

proving the last claim. One could have also shown that both functors agree on spheres by 1.5.21, but this would be more complicated. □

**Corollary 4.4.4.** The spectra $\mathbb{L}^{\mathrm{red}}(X, \mathcal{V}, \mathfrak{Q}) \simeq \Sigma^\infty X \wedge \mathbb{L}(\mathcal{V}, \mathfrak{Q})$ are equivalent for any pointed PL space $X$. In particular,

$$\mathbb{L}(X, \mathcal{V}, \mathfrak{Q}) \simeq \Sigma^\infty X_+ \wedge \mathbb{L}(\mathcal{V}, \mathfrak{Q}) \ . \tag{4.45}$$

**Theorem 4.4.5** ([WW93, Theorem 1.1])**.** Given any functor $F : \mathcal{S}^{\mathrm{fin}} \to \mathcal{S}p$, there exists a unique reduced functor $F^{\%} : \mathcal{S}^{\mathrm{fin}} \to \mathcal{S}p$ preserving pushout squares, equipped with a natural transformation

$$A : F^{\%} \longrightarrow F \tag{4.46}$$

such that $A_{\Delta^0} : F^{\%}(\Delta^0) \to F(\Delta^0)$ is a homotopy invariance. In fact, this functor is given by $F^{\%}(X) = \Sigma^\infty X_+ \wedge F(\Delta^0)$. Transformations $A$ that arise in this way, and the morphisms they consist of, are called *assembly maps*. The association $F \mapsto F^{\%}$ is functorial, and even a reflection of $\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}, \mathcal{S}p)$ on the full subcategory on functors preserving finite colimits.

In the same way, any functor $F : \mathcal{S} \to \mathcal{S}p$ can be uniquely approximated by a functor $F^{\%} : \mathcal{S} \to \mathcal{S}p$ preserve pullback squares and arbitrary wedge products (called *strongly excisive* in the reference).

*Proof.* We translate the proof in the reference to our $\infty$-categorical language. Since $\mathcal{S}^{\mathrm{fin}}$ is generated by the point $\Delta^0$ under finite colimits, and $\mathcal{S} = \mathrm{Ind}(\mathcal{S}^{\mathrm{fin}}) = \mathcal{PS}h(\Delta^0)$ is generated by the point under colimits, we have (combining 1.1.6 and 3.2.17)

$$\mathrm{Fun}^{\mathrm{colim}}(\mathcal{S}, \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{rex}}(\mathcal{S}^{\mathrm{fin}}, \mathcal{D}) \simeq \mathrm{Fun}(\Delta^0, \mathcal{D}) \simeq \mathcal{D} \tag{4.47}$$

for any $\infty$-category $\mathcal{D}$ admitting colimits, in particular $\mathcal{S}p$. If we let $j : \{\Delta^0\} \hookrightarrow \mathcal{S}$ be the canonical inclusion, then under this identification, the approximation $F^{\%}$ is constructed as the counit $F^{\%} := j_+ j^* F \to F$ where $j_+$ denotes the left Kan extension. In other words, we evaluate $F$ on the point and Yoneda-extend the result using $\mathcal{S}p \simeq \mathrm{Fun}^{\mathrm{colim}}(\mathcal{S}, \mathcal{S}p)$. This agrees with $X \mapsto \Sigma^\infty X_+ \wedge F(\Delta^0)$, since both preserve colimits (by the abstract definition of the smash product) and agree on the point. □

*Remark.* This is again related to Goodwillie-Weiss-Calculus, in the sense of 2.1.3: Any functor $F : \mathcal{S}_*^{\text{fin}} \to \mathcal{S}p^{op}$ can be approximated, either from the left or from the right, by a unique $n$-excisive functor (see 2.1.2), just as any smooth function $\mathbb{R}^n \to \mathbb{R}^m$ can be approximated by its Taylor series. In fact, this holds for functors between much more general $\infty$-categories. In the case $n = 1$, the approximation from the left agrees with $F^{\%}$ as defined above.

**Theorem 4.4.6.** The map $\mathbb{L}(\mathcal{S}h^{simp}(X;\mathcal{V}), \Omega_X) \to \mathbb{L}(\mathcal{S}h^{simp}(X;\mathcal{V}), \Omega_X)$ induced by $L_{lc}$ in 4.3.14 induces an *assembly map*

$$A : \Sigma^\infty X_+ \wedge \mathbb{L}(\mathcal{V}, \Omega) \longrightarrow \mathbb{L}(\mathcal{S}h^{lc}(X;\mathcal{V}), \Omega_X) \tag{4.48}$$

in the spirit of the last theorem. In particular, if $(\mathcal{V}, \Omega) = (\mathrm{LMod}_R^{\mathrm{fp}}, \Omega_R)$ for a ring spectrum $R$, we obtain

$$A : \Sigma^\infty X_+ \wedge \mathbb{L}^s(R) \longrightarrow \mathbb{L}^s(\Sigma^\infty \Omega X \wedge R)$$
$$A : \Sigma^\infty X_+ \wedge \mathbb{L}^q(\pi_0 R) \longrightarrow \mathbb{L}^q(\pi_0 R[\pi_1 X])$$

where the second row agrees with Ranicki's Assembly map from [Ran92].

*Proof.* This is immediate since the assembly of $F$ was always constructed as the wedge product $\Sigma^\infty X_+ \wedge F(\Delta^0)$, and is unique. □

**Proposition 4.4.7** ([Lur11, Lecture 22, Proposition 7])**.** In the above situation, the commutative square

$$\begin{array}{ccc} X \wedge \mathbb{L}^q(\pi_0 R) & \longrightarrow & X \wedge \mathbb{L}^s(R) \\ \downarrow & & \downarrow \\ \mathbb{L}(\pi_0 R[\pi_1 X]) & \longrightarrow & \mathbb{L}^s(\mathcal{S}h^{lc}(X; \mathrm{LMod}_R)) \end{array}$$

where the horizontal maps are induced by the norm map is a pullback square in $\mathcal{S}p$.

## 4.5 Stratifications and Constructible Sheaves

For this section, let us fix a poset $P$ and a stable $\infty$-category $\mathcal{V}$.

**Definition 4.5.1.** A *stratification* on a simplicial complex $K$ with stratification poset $P$ is an order-preserving map $f : \mathcal{I}_K \to P$. Denote by $K_p$ the preimage $f^{-1}(\{p\})$, by $K_{\leq p}$ the preimage $f^{-1}(\{q \in P \mid q \leq p\})$, and similarly $K_{\geq p}$.

**Proposition 4.5.2.** For each $p \in P$, the poset $K_{\leq p}$ determines the simplices of a simplicial subcomplex of $K$.

*Proof.* Define the set of vertices $(K_{\leq p})_0 := \{v \in K_0 | f(\{v\}) \leq p\}$, which can be used to describe every simplex in $K_{\leq p}$ since $\{v\} \subseteq \sigma \in K_{\leq p}$ implies $\{v\} \in K_{\leq p}$ as $f$ is order-preserving. For the same reason, $\tau \in K_{\leq p}$ and $\sigma \subseteq \tau$ implies $\sigma \in K_{\leq p}$, so $K_{\leq p}$ is indeed a simplicial complex. □

*Warning.* The subsets $K_p$ and $K_{\geq p}$ of $\mathcal{I}_K$ in general do not define simplicial subcomplexes. This situation improves a bit if we instead define a stratification as a map $K_0 \to P$ labeling each vertex by a stratum, instead of all simplices, since we can then consider the subcomplex spanned by the vertices in a fixed stratum. We can always reduce from our definition to this more rigid situation by taking the barycentric subdivision.

**Example 4.5.3.**

- Every simplicial complex $K$ admits the trivial stratification $\mathcal{I}_K \to [0]$ sending everything to 0, and the identity stratification $\mathcal{I}_K \to \mathcal{I}_K$ putting each simplex into a different stratum. The former is the *coarsest*, the latter the *finest* stratification on $K$, where we call $\mathcal{I}_K \to P$ *finer* than $\mathcal{I}_K \to Q$ if we can factor the latter map as $\mathcal{I}_K \to P \to Q$ for some order-preserving map $P \to Q$.
- Every simplicial complex $K$ admits the *skeletal stratification* (usually called skeletal filtration) $\mathcal{I}_K \to \mathbb{N}_0$ sending every simplex to its dimension.
- Given a subcomplex $K' \subseteq K$, we may stratify $\mathcal{I}_K \to [1]$ sending simplices in $K'$ to 0, and everything else to 1. In fact, any [1]-stratification arises in this manner.
- Similarly, an $\mathbb{N}_0$-stratification of a complex $K$ can be thought of as a filtration by a system of sub-complexes

$$K_0 \subseteq K_{\leq 1} \subseteq K_{\leq 2} \subseteq K_{\leq 3} \subseteq \cdots \subseteq K\ .$$

- As a special case of the last examples, we can stratify a triangulation of a manifold with boundary by sending the interior to 1, and simplices contained in the boundary to 0. This is also an instance of the intrinsic stratification of a PL space we introduce below.

**Definition 4.5.4.** A simplicial sheaf $F : K \to \mathcal{V}$ on $K$ is called *constructible* with respect to the stratification $f : K \to P$ if for each $p \in P$, the restriction $F|_{K_p} : K_p \to \mathcal{V}$ is locally constant in the sense that for $\sigma \subseteq \tau$ any simplices of $K_p$, the map $F(\sigma \subseteq \tau) : F(\sigma) \to F(\tau)$ is an isomorphism. Denote the full subcategory on constructible sheaves by $\mathcal{S}h^{cbl}(K; \mathcal{V}) \subseteq \mathcal{S}h^{simp}(K; \mathcal{V})$.

**Definition 4.5.5.** Let $(X, \mathcal{T})$ be a PL space. A *stratification* on $X$ over a poset $P$ consists of a right cofinal sub-poset $\mathcal{T}_P \subseteq \mathcal{T}$ and, for any $T \in \mathcal{T}_P$, a map of posets $T \to P$ compatible with refinements in $\mathcal{T}$. In other words, we need to choose a natural transformation $\mathcal{I}|_{\mathcal{T}_P} \Rightarrow \underline{P}$ between functors $\mathcal{T}_P \to \mathcal{C}at_\infty$, which is by right cofinality equivalent to a map out of the colimit

$$\operatorname*{colim}_{\mathcal{T}} \mathcal{I} \cong \operatorname*{colim}_{\mathcal{T}_P} \mathcal{I} \longrightarrow P \ .$$

**Example 4.5.6.**

- The *trivial stratification* on $X$ over $[0]$ is induced by the terminal map $T \to [0]$ on every triangulation $T$.
- There is an *intrinsic stratification* on $X$ that, given any triangulation $T$, sends points with PL homeomorphic (i.e. isomorphic after further refinement) links to the same stratum. In particular, any PL manifold with boundary is canonically stratified over $[1]$.
- Any stratification of $X$ as a PL space determines a stratification of the underlying topological space, in the sense of 6.1.2.
- A *PL subspace* $X' \subseteq X$ is determined by a stratification of $X$ by $[1]$. If $T \in \mathcal{T}_{[1]}$ is any triangulation in the defining sub-poset of this stratification, then the preimage of $0 \in [1]$ under $T \to [1]$ is a triangulation of $X'$, and together these determine its PL structure.

**Proposition 4.5.7.** For $(X, \mathcal{T})$ a $P$-stratified PL space as above, any refinement $r : \mathcal{I}_T \to \mathcal{I}_{T'}$ in $\mathcal{T}_P$ induces an equivalence of categories

$$\mathcal{S}h^{cbl}(T; \mathcal{V}) \simeq \mathcal{S}h^{cbl}(T'; \mathcal{V}) \tag{4.49}$$

so that these categories agree or all $T \in \mathcal{T}_P$ since for any two, we can choose a common refinement. We thus define

$$\mathcal{S}h^{cbl}(X; \mathcal{V}) := \mathcal{S}h^{cbl}(T; \mathcal{V}) = \operatorname*{colim}_{T \in T_P} \mathcal{S}h^{cbl}(T; \mathcal{V}) = \lim_{T \in T_P} \mathcal{S}h^{cbl}(T; \mathcal{V}) \tag{4.50}$$

since $\mathcal{T}_P \subseteq \mathcal{T}_P$ is cofinal and both are filtered, hence weakly contractible so the (co)limit of a constant diagram agrees with its constant value.

*Proof.* Both $\mathcal{I}_T, \mathcal{I}_{T'}$ are by definition equipped with maps to $P$ that factor through the localizations $\mathcal{I}_T[W_P^{-1}], \mathcal{I}_{T'}[W_P^{-1}]$, where $W_P$ respectively denotes the class of morphisms determined by containment relations between simplices in the same stratum. As in 4.3.2, it suffices to show that the functor $\bar{r}$ induced by $r$ in the diagram

$$\begin{array}{ccc} \mathcal{I}_T[W_P^{-1}] & \xrightarrow{\quad \bar{r} \quad} & \mathcal{I}_{T'}[W_P^{-1}] \\ & \searrow \quad \swarrow & \\ & P & \end{array}$$

is an equivalence of $\infty$-categories. The proof there shows that this map is a weak equivalence, but since the above localizations are not Kan complexes, we are not finished. One way to proceed is to use the stratified homotopy theory we introduce in 6.2: On stratified realizations, $r : |\mathcal{I}_T[W_P^{-1}]| \simeq |\mathcal{I}_T|_P \to |\mathcal{I}_{T'}|_P \simeq |\mathcal{I}_{T'}[W_P^{-1}]|_P$ is (as it is a refinement, and compatible with the stratifications) a stratified homeomorphism so by [DW21, Corollary 4.22], the original map in $\bar{r}$ is a weak equivalence in $\mathcal{S}_P$, i.e. a categorical equivalence. □

**Proposition 4.5.8.** If $\mathcal{V}$ admits all limits and colimits, the inclusion $\mathcal{S}h^{cbl}(K;\mathcal{V}) \subseteq \mathcal{S}h^{simp}(K;\mathcal{V})$ has a left adjoint $L_{cbl}$ and a right adjoint $R_{cbl}$. Similarly for a stratified PL space $X$, the inclusion $\mathcal{S}h^{cbl}(X;\mathcal{V}) \subseteq \mathcal{S}h^{simp}(X;\mathcal{V})$ has a left and a right adjoint.

*Proof.* This is completely analogous to 4.3.4. Denote by $W_P$ the class of morphisms $\sigma \subseteq \tau$ in $K$ such that $f(\sigma) = f(\tau)$. Then, we can identify the above inclusion with the embedding

$$\mathrm{Fun}(K[W_P^{-1}], \mathcal{V}) \subseteq \mathrm{Fun}(K, \mathcal{V})$$

induced by the universal property of a localization.This is explicitly given by precomposing with the localization functor $L : K \to K[W_P^{-1}]$, so it possesses left and right adjoints $\mathrm{Lan}_L$ and $\mathrm{Ran}_L$ since $\mathcal{V}$ is bicomplete. The PL case again follows from the fact that adjunctions are preserved under (co)limits. □

**Definition 4.5.9.** A simplicial sheaf $F : K \to \mathcal{V}$ is called *constructibly balanced* if for any constructible sheaf $S$, the mapping space $\mathrm{Map}(F, S) \cong \Delta^0$ is contractible. If $\mathcal{V}$ admits colimits, this is again equivalent $L_{cbl}(F) = 0$ by the Yoneda Lemma. Denote the full subcategory on constructibly balanced sheaves by $\mathcal{S}h^{\perp cbl}(K, \mathcal{V}) \subseteq \mathcal{S}h^{simp}(K, \mathcal{V})$.

**Proposition 4.5.10.** For $K$ a $P$-stratified simplicial complex and $\mathcal{V}$ a bicomplete stable $\infty$-category, the sequence

$$\mathcal{S}h^{\perp cbl}(K;\mathcal{V}) \hookrightarrow \mathcal{S}h^{simp}(K;\mathcal{V}) \xrightarrow{L_{cbl}} \mathcal{S}h^{cbl}(K;\mathcal{V}) \tag{4.51}$$

is a right split Verdier sequence and the sequence

$$\mathcal{S}h^{cbl}(K;\mathcal{V}) \hookrightarrow \mathcal{S}h^{simp}(K;\mathcal{V}) \longrightarrow \mathcal{S}h^{cbl}(K;\mathcal{V})^{\perp} \tag{4.52}$$

is split Verdier. Similarly if we replace $K$ by a $P$-stratified PL space.

This follows precisely as in the locally constant case. We obtain a commutative diagram of stable $\infty$-categories, where the the rows are right split Verdier sequences:

$$\begin{array}{ccccc}
\mathcal{S}h^{\perp cbl}(K;\mathcal{V}) & \hookrightarrow & \mathcal{S}h^{simp}(K;\mathcal{V}) & \longrightarrow & \mathcal{S}h^{cbl}(K;\mathcal{V}) \\
\uparrow & & \| & & \downarrow \\
\mathcal{S}h^{\perp lc}(K;\mathcal{V}) & \hookrightarrow & \mathcal{S}h^{simp}(K;\mathcal{V}) & \longrightarrow & \mathcal{S}h^{lc}(K;\mathcal{V})
\end{array}$$

**Definition 4.5.11.** For $K$ a $P$-stratified simplicial complex, $\mathcal{V}$ a stable $\infty$-category and $\mathcal{W} := \mathrm{Ind}(\mathcal{V})$, we define the $\infty$-category of $\mathcal{V}$-generated constructible sheaves on $K$ with values in $\mathcal{W}$ as the Verdier quotient

$$\mathcal{S}h^{cbl}(K;\mathcal{W})^{(\mathcal{V})} := \mathcal{S}h^{simp}(K;\mathcal{V}) \big/ \mathcal{S}h^{\perp cbl}(K;\mathcal{V}) \subseteq \mathcal{S}h^{cbl}(K;\mathcal{W}) \tag{4.53}$$

where the last inclusion follows analogously to the locally constant case. Its essential image again consists of precisely those constructible sheaves of the form $L_{cbl}F$ for some $F \in \mathcal{S}h^{simp}(K,\mathcal{W})$.

**Theorem 4.5.12.** If $K$ is a finite simplicial complex, $(\mathcal{V},\Omega)$ a Poincaré $\infty$-category with $\mathcal{W} = \mathrm{Ind}(\mathcal{V})$ and we equip $\mathcal{S}h^{\perp cbl}(K;\mathcal{V})$ and $\mathcal{S}h^{cbl}(K;\mathcal{W})^{(\mathcal{V})}$ with the respective restrictions of the quadratic functor $\Omega_K$, the following is a Poincaré-Verdier sequence:

$$\mathcal{S}h^{\perp cbl}(K;\mathcal{V}) \hookrightarrow \mathcal{S}h^{simp}(K;\mathcal{V}) \xrightarrow{L_{cbl}} \mathcal{S}h^{cbl}(K;\mathcal{W})^{(\mathcal{V})} \tag{4.54}$$

Similarly, passing to a (co)limit over all triangulations on a compact PL space, the sequence

$$\mathcal{S}h^{\perp cbl}(X;\mathcal{V}) \hookrightarrow \mathcal{S}h^{comb}(X;\mathcal{V}) \xrightarrow{L_{cbl}} \mathcal{S}h^{cbl}(X;\mathcal{W})^{(\mathcal{V})} \tag{4.55}$$

is Poincaré-Verdier as well.

*Proof.* Analogous to the proof of 4.3.13. The only difference is that we now must show that $\mathcal{S}h^{\perp cbl}(K;\mathcal{V})$ is closed under duality, so we have to find a suitable generating set for it. With little effort if becomes evident that the sheaves $F^{\tau'/\tau,V}$ still work, given that we restrict to $\tau \subseteq \tau'$ lying in the same stratum. □

*Remark.* While we have not introduced a spherical fibration for this theorem, the result is still true if we add one. If we want to apply the L-groups constructed in this manner to stratified surgery, that is however not the right way to go: Spherical fibration should be spherical, because the link of a point in a PL manifold is a sphere. This is no longer true for PL pseudomanifolds, and the right approach is to use the dualizing complex $\omega_X$ instead, which is indeed constructible by 6.4.1. We will elaborate on this in the topological case.

To summarize the last sections, given a compact PL space $X$ or a finite simplicial complex $K$ equipped with a spherical fibration $\zeta$ we have constructed the following commutative diagram of L-groups, where the labels $A$ denote assembly maps, the rows are fiber sequences and all quadratic functors are induced by $\Omega^q_{M,X,\zeta}$.

$$
\begin{array}{ccccc}
\mathbb{L}^q\left(\mathcal{S}h_{\mathrm{fp}}^{\perp cbl}(X;R)\right) & \longrightarrow & \mathbb{L}^q\left(\mathcal{S}h_{\mathrm{fp}}^{comb}(X;R)\right) & \longrightarrow & \mathbb{L}^q\left(\mathcal{S}h^{cbl}(X;R)^{(\mathrm{fp})}\right) \\
\uparrow & & \| & & \downarrow \\
\mathbb{L}^q\left(\mathcal{S}h_{\mathrm{fp}}^{\perp lc}(X;R)\right) & \longrightarrow & \mathbb{L}^q\left(\mathcal{S}h_{\mathrm{fp}}^{comb}(X;R)\right) & \xrightarrow{A} & \mathbb{L}^q\left(\mathcal{S}h^{lc}(X;R)^{(\mathrm{fp})}\right) \\
 & & \downarrow\cong & & \downarrow\cong \\
 & & \Sigma^\infty X_+ \wedge \mathbb{L}^q(\pi_0 R) & \xrightarrow{A} & \mathbb{L}^q(\pi_0 R[\pi_1 X])
\end{array}
$$

We use the shorthands $\mathcal{S}h(X;R) := \mathcal{S}h(X;\mathrm{LMod}_R)$ and $\mathcal{S}h_{\mathrm{fp}}(X;R) := \mathcal{S}h(X, \mathrm{LMod}_R^{\mathrm{fp}})$ denoting that our sheaves have finitely presented stalks (regarded as sheaves on a topological space via the exodromy correspondence 6.3.4). The second row can be regarded as a special case of the first row for the coarsest stratification on $X$, in fact any finer or coarser stratification of $X$ can be added as a row into this commutative diagram. A similar diagram can be drawn for symmetric L-groups or in fact for any Poincaré $\infty$-category; only the last row generally does not work. An immediate question arising during inspection of above diagram is whether the constructible analogues $\mathbb{L}^q\left(\mathcal{S}h^{cbl}(X;R)\right)$ of the quadratic L-groups of locally constant sheaves can be expressed in a similar, calculable way. We turn to this in the next section.

## 4.6 Decomposition into Strata

Let $s : P \to [1]$ a slicing of a partially ordered set $P$, and fix a $P$-stratified simplicial complex $K$ and a stable $\infty$-category $\mathcal{V}$. The composition $K \to P \to [1]$ divides the set of simplices in $K$ into two disjoint classes. Let us denote by $K_+ \subseteq K$ to sub-poset of simplices in the preimage of 1, and $K_- := (s \circ f)^{-1}(\{0\})$ similarly. We have seen that $K_- \subseteq K$ forms a subcomplex.

**Proposition 4.6.1.** The precomposition functors $i^*, j^*$ with the inclusions $i : K_- \hookrightarrow K, j : K_+ \hookrightarrow K$ possess right adjoints $i_*, j_*$ exhibiting $\mathcal{S}h^{simp}(K;\mathcal{V})$ as a recollement of $\mathrm{Fun}(K_-, \mathcal{V})$ and $\mathrm{Fun}(K_+, \mathcal{V})$. In other words, the sequence

$$\mathrm{Fun}(K_-, \mathcal{V}) \overset{i_*}{\hookrightarrow} \mathcal{S}h^{simp}(K;\mathcal{V}) \overset{j^*}{\to} \mathrm{Fun}(K_+, \mathcal{V}) \tag{4.56}$$

is split Verdier sequence.

*Proof.* It suffices to show that $\mathcal{S}h^{simp}(K;\mathcal{V})$ is a recollement of these full subcategories. Since $K_-$ is downward closed, the left Kan extension functor applied to $F : K_+ \to \mathcal{V}$

$$j_+ F(k_-) = \operatorname*{colim}_{k_+ \in K_+, k_+ \leq k_-} F(k_+) = 0 \tag{4.57}$$

vanishes on simplices of $K_-$, and similarly the right Kan extension $i_* G$ of $G \in \mathrm{Fun}(K_-, \mathcal{V})$ vanishes on simplices of $K_+$. In particular, they both exist, and are fully faithful since $i, j$ are fully faithful. Further, it is immediate to check $\text{Fun}(K_+, \mathcal{V}) = \text{Fun}(K_-, \mathcal{V})^{\perp}$ since the $\subseteq$ direction is clear by definition of a zero object, and the $\supseteq$ direction follows by restricting a functor to $K_+$ and applying the Yoneda Lemma. We can apply 3.2.12 since the remaining adjunctions are induced as in 3.2.7. □

**Corollary 4.6.2.** The recollements of the form $(\text{Fun}(K_-, \mathcal{V}), \text{Fun}(K_+, \mathcal{V}))$ for any slicing of $P$ form a $P$-slicing of $\mathcal{S}h^{simp}(K; \mathcal{V})$.

This is a special case of an even stronger result:

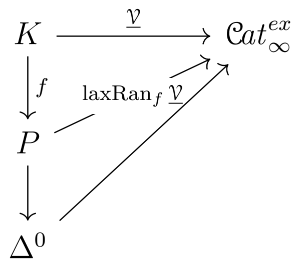
**Proposition 4.6.3.** Precomposing with the inclusions $i_p : K_p = f^{-1}(\{p\}) \hookrightarrow K$ for all $p \in P$ yields restriction functors

$$i_p^* : \mathcal{S}h^{simp}(K; \mathcal{V}) \to \text{Fun}(K_p, \mathcal{V}) \tag{4.58}$$

possessing fully faithful right adjoints $i_{p,*} = \text{Ran}_{i_p}$. These functors exhibit $\mathcal{S}h^{simp}(K; \mathcal{V})$ as a $P$-decomposition of $\text{Fun}(K_p, \mathcal{V})$ for all $p \in P$.

The previous result follow from this by postcomposing the respective stratification with the slicing $P \to [1]$ to obtain a [1]-stratified simplicial set.

*Proof Sketch.* We could do essentially the same calculation as in the last proof, but we rather use it as an opportunity to informally show off lax right Kan extensions. By their transitivity, the diagram




commutes, where $\mathcal{C}at_\infty^{ex}$ is the $\infty$-category of stable $\infty$-categories and exact functors, $\underline{\mathcal{V}} : K \to \mathcal{C}at_\infty^{ex}$ is the constant functor with value $\mathcal{V}$, and the lowest arrow classifies $\text{Fun}(K, \mathcal{V}) = \text{Fun}(\text{laxcolim}_K \underline{\Delta}^0, \mathcal{V}) = \text{laxlim}_K \underline{V}$, as the lax right Kan extension along the terminal functor obtains the lax limit. It then remains to calculate that the functor $P \to \mathcal{C}at_\infty^{ex}$ sends $p$ to $\text{Fun}(K_p, \mathcal{V})$. □

A similar result holds for constructible sheaves:

**Proposition 4.6.4.** The above sequence reduces to a split Verdier sequence

$$\mathcal{S}h^{cbl}(K_-; \mathcal{V}) \longrightarrow \mathcal{S}h^{cbl}(K; \mathcal{V}) \longrightarrow \mathcal{S}h^{cbl}(K_+; \mathcal{V}) \tag{4.59}$$

where $\mathcal{S}h^{cbl}(K_+, \mathcal{V})$ denotes the functors $K_+ \to \mathcal{V}$ that send morphisms in $K_+$ that are constant over $P$ to isomorphisms, and similarly for $K_-$.

*Proof.* Since there are still no morphisms from $K_-[W_P^{-1}]$ to $K_+[W_P^{-1}]$, this follows from a proof analogous to 4.6.1. $\square$

**Theorem 4.6.5.** Fix a $P$-stratified simplicial complex $K$, a bicomplete stable $\infty$-category $\mathcal{W}$ and a slicing $(P_-, P_+)$ of $P$ with inverse images $K_-, K_+ \subseteq K$. Then, there is a square diagram

$$\begin{array}{ccccc}
\mathcal{S}h^{\perp cbl}(K_-;\mathcal{W}) & \longrightarrow & \mathcal{S}h(K_-;\mathcal{W}) & \longrightarrow & \mathcal{S}h^{cbl}(K_-;\mathcal{W}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{S}h^{\perp cbl}(K;\mathcal{W}) & \longrightarrow & \mathcal{S}h(K;\mathcal{W}) & \longrightarrow & \mathcal{S}h^{cbl}(K;\mathcal{W}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{S}h^{\perp cbl}(K_+;\mathcal{W}) & \longrightarrow & \mathcal{S}h(K_+;\mathcal{W}) & \longrightarrow & \mathcal{S}h^{cbl}(K_+;\mathcal{W})
\end{array}$$

where all rows are right split and columns are split Verdier sequences. In particular, the vertical sequences induce $P$-slicings of $\mathcal{S}h^{cbl}(K;\mathcal{W})$, $\mathcal{S}h(K;\mathcal{W})$ and $\mathcal{S}h^{\perp cbl}(K;\mathcal{W})$.

*Proof.* We have just checked that the middle and right vertical sequences are split Verdier, and the horizontal sequences are right split Verdier by 4.5.10. We are finished if we can apply the 9-Lemma 3.3.8 to deduce that the first sequence is split Verdier; the fact that the horizontal sequences are only right split is no problem as can be checked by going through its proof or applying [Lur18a, Tag 02EX].

In other words, we need to show the factorization condition: Any morphism of sheaves $F^0 \to G$ with $F$ constructibly balanced and $G$ supported on $K_-$ factors through a $G^0$ that is both. Clearly, defining $G^0 := i_*i^*F \in \mathrm{im}(i_*)$ as the restriction of $F$ to $K_-$, sending everything else to 0, works since $i^* \dashv i_*$ implies that composing $i_*i^*F \to G^0$ with the unit map $F \to i_*i^*F$ yields the adjoint map $i^*F \to i^*G^0$ meaning that this is actually a factorization. Further, $i^*F$ is constructibly balanced on $K_-$ because for $S \in \mathcal{S}h^{cbl}(K_-;\mathcal{W})$,

$$\mathrm{Map}(i^*F, S) = \mathrm{Map}(F, i_*S) \simeq \Delta^0$$

is contractible as $i_*S$ is clearly still constructible. $\square$

**Theorem 4.6.6.** For a finite $P$-stratified simplicial complex $K$ with spherical fibration $\zeta$, a Poincaré $\infty$-category $\mathcal{V}$ with $\mathcal{W} = \mathrm{Ind}(\mathcal{V})$ and a slicing $(P_-, P_+)$ of $P$ with inverse images $K_-, K_+ \subseteq K$, the above diagram restricts to

$$\begin{array}{ccccc}
\mathcal{S}h^{\perp cbl}(K_-;\mathcal{V}) & \longrightarrow & \mathcal{S}h^{simp}(K_-;\mathcal{V}) & \longrightarrow & \mathcal{S}h^{cbl}(K_+;\mathcal{W})^{(\mathcal{V})} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{S}h^{\perp cbl}(K;\mathcal{V}) & \longrightarrow & \mathcal{S}h^{simp}(K;\mathcal{V}) & \longrightarrow & \mathcal{S}h^{cbl}(K;\mathcal{W})^{(\mathcal{V})} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{S}h^{\perp cbl}(K_+;\mathcal{V}) & \longrightarrow & \mathcal{S}h^{simp}(K_+;\mathcal{V}) & \longrightarrow & \mathcal{S}h^{cbl}(K_-;\mathcal{W})^{(\mathcal{V})}
\end{array}$$

where all rows and columns are Poincaré-Verdier sequences with respect to Verdier duality $\mathfrak{P}_K$ suitably restricted from $\mathcal{S}h^{simp}(K,\mathcal{W})$.

*Proof.* We know that the horizontal sequences are right split Poincaré-Verdier by 4.3.13. The left and middle vertical sequences are split Verdier since they are restricted from the respective sequences in the last proof of 4.6.5, and all adjoint functors preserve the full subcategory of sheaves in $\mathcal{V}$ since they only involve finite limits. We have also seen there that the factorization condition of the 9-Lemma 3.3.8 is fulfilled, so we are finished if we can show that the left and middle vertical sequences are Poincaré-Verdier.

The functors $i_*$ and $j^*$ restrict to constructibly balanced sheaves by 4.6.5, so by the definition of the respective quadratic functors as the correct restrictions all we need to show is that $i_*$ and $j^*$ are duality-preserving. For $i_*$ this follows from 4.1.24, while for $j^*$ it is a tedious computation we omit. □

*Remark.* We even suspect that the columns are all split, but were not able to show this for the last column.

The right vertical sequence can be used to calculate $\mathbb{L}^q(\mathcal{S}h^{cbl}(K;\mathcal{V})^{(fp)})$ in many special cases; we return to this in 6.5. Let us mention that this theorem still holds if we introduce a spherical fibration, or switch to symmetric L-theory. Also, everything works out similarly in the PL setting:

**Theorem 4.6.7.** All results of this section still hold if we replace the (finite) $P$-stratified simplicial complex $K$ with a $P$-stratified compact PL space.

*Proof.* As in the locally constant case, everything can be checked on components of the respective colimit over the cofinal subset $\mathcal{T}_P$ of triangulations. □

# 5 L-Groups of Manifolds

After this extensive discussion of the piecewise linear case, let us generalize some of our results to good topological spaces, in particular topological manifolds and CW complexes. We begin with a discussion of Verdier duality and the six-functor formalism for $\infty$-sheaves, following [Vol21] and [Lur17]. An important technical tool is also the monodromy correspondence, which we use to construct some of the functors involved in the split Verdier sequences of interest. We further discuss the characterization of locally constant sheaves of $R$-modules as modules over $\Sigma^\infty \Omega X \wedge R$ that was teased in the last section.

## 5.1 Verdier Duality

**Definition 5.1.1.** A topological space $X$ is called *locally compact* if for every $x \in X$ and every open subset $x \in U \subseteq X$, there exists a compact neighborhood $x \in K \subseteq U$. Note that as a neighborhood, $K$ must contain an open neighborhood of $x$.

**Proposition 5.1.2.** Given a topological space $X$, define another topological space $X^+$ called its *one-point compactification* with underlying set $X \cup \{\infty\}$, and $U \subseteq X^+$ open iff either

- $\infty \notin U$ and $U \subseteq X$ open, or
- $\infty \in U$ and $X^+ - U \subseteq X$ compact.

If $X$ is locally compact Hausdorff, then $X^+$ is also locally compact Hausdorff.

**Definition 5.1.3.** Let $\mathcal{V}$ be a pointed $\infty$-category that admits all limits and colimits, $X$ be a Hausdorff space, and $F$ be a $\mathcal{V}$-valued sheaf on $X$. Then, given a closed subset $A \subseteq X$, denote by $\Gamma_A(X, F) := F(X) \times_{F(X-A)} 0$ the space of sections of $F$ supported in $A$. Using this, we define the space of compactly supported sections on an open $U \subseteq X$ as

$$\Gamma_c(U, F) := \operatorname*{colim}_{K \subseteq U \text{ cpt}} \Gamma_K(X, F) . \tag{5.1}$$

This induces a covariant functor $\Gamma_c(-, F) : \mathrm{Open}(X) \to \mathcal{V}$.

**Theorem 5.1.4** ([Lur17, 5.5.5.1])**.** Given a stable $\infty$-category $\mathcal{V}$ with all limits and colimits, a locally compact Hausdorff space $X$ and a sheaf $F \in \mathcal{S}h(X;\mathcal{V})$, the functor $F_c := \Gamma_c(-, F)$ is a $\mathcal{V}$-valued cosheaf on $X$. In fact, $F \mapsto F_c$ induces a contravariant equivalence of categories

$$\mathcal{S}h(X;\mathcal{V})^{op} \simeq \mathcal{S}h(X;\mathcal{V}^{op}) \;, \tag{5.2}$$

where the inverse is again given by taking the cosheaf of compactly supported sections, regarding a cosheaf as a $\mathcal{V}^{op}$-valued sheaf.

*Proof Sketch.* Since $X$ is locally compact Hausdorff, a sheaf $F$ on $X$ is determined by its values on all compact subsets $K$. To be more precise, let $K(X)$ be the partially ordered set of compact subsets in $X$ regarded as an $\infty$-category, then a functor $F : K(X)^{op} \to \mathcal{V}$ is called a *K-sheaf* if

- $F(\emptyset) = 0$ is final,
- For any $K, K' \in K(X)$, we can write $F(K \cup K') \cong F(K) \cap_{F(K\cap K')} F(K')$ via the maps induced by functoriality of $F$,
- We have $F(K) = \underset{K'\supseteq K}{\operatorname{colim}} F(K')$ where the colimit ranges over compact $K'$ containing an open neighborhood of $K$.

Let us denote the full subcategory of $\operatorname{Fun}(K(X)^{op}, \mathcal{V})$ on K-sheaves by $\mathcal{S}h_K(X;\mathcal{V})$, then [Lur17, 5.5.5.3] shows that the canonical map $\mathcal{S}h(X;\mathcal{V}) \to \mathcal{S}h_{\mathcal{K}}(X;\mathcal{V})$ sending $F \mapsto (K \mapsto \Gamma_K(X, F))$ is an equivalence.

The main idea of the proof is now to exploit a sort of duality between compact subsets and complements of compact subsets (note the role that the latter play in the one-point compactification). To this purpose, we define a set $M$ as a subset of $\{0 < 1 < 2\} \times \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set, on those pairs $(i, S)$ where

- For $i = 0$, the subset $S \subseteq X$ is compact
- For $i = 2$, the subset $X - S \subseteq X$ is compact

and order $M$ by defining $(i, S) \leq (j, T)$ if either $i \leq j$ and $S \subseteq T$, or $i = 0$ and $j = 2$. In particular, $M_0 := M \times_{[2]} \{0\} = K(X)$ and $M_2 \cong K(X)^{op}$. Now, the main technical work done in [Lur17, 5.5.5.7] lies in showing that the following are equivalent, for $F : M \to \mathcal{C}$ a functor:

- The restriction $F|_{M_0}$ is a K-cosheaf (a K-sheaf with values in $\mathcal{V}^{op}$), $F|_{M_1} = 0$ and $F$ agrees with the left Kan extension of $F|_{M_0 \cup M_1}$ where $M_0 \cup M_1 := M \times_{[2]} \{0 < 1\}$
- The restriction $F|_{M_2}$ is a K-sheaf, $F|_{M_1} = 0$ and $F$ agrees with the right Kan extension of $F|_{M_1 \cup M_2}$

This proves the theorem, since the first class of functors is determined by the value on $F_{M_0}$ which is an arbitrary K-cosheaf, while the second class is determined by the value on $F_{M_2}$ which is an arbitrary K-sheaf. The right (and similarly, left) Kan extension translating between them can be written out as

$$F(0,K) = \mathrm{Ran}^{M}_{M_1 \cup M_2} F(0,K) = \lim_{(i,L)\in (M_1\cup M_2)_{(0,K)/}} \begin{cases} S(X-L), \text{for } i=2 \\ 0, \text{for } i=1 \end{cases} \tag{5.3}$$

for $S : M_2 \cong K(X)^{op} \to \mathcal{V}$ an arbitrary K-sheaf, which contains the left cofinal subdiagram

$$\begin{array}{ccc} F(0,K) & \dashrightarrow & F(1,K)=0 \\ \downarrow & & \downarrow \\ F(2,\emptyset) = S(X) & \longrightarrow & F(2,K) = S(X-K) \end{array}$$

by Quillen's Theorem A (each of the appearing slice categories admits an initial object, so they are weakly contractible). This agrees with $S_c(K) = \Gamma_K(X,S)$ as claimed. □

Now, let $\mathcal{V}$ be a bicomplete stable $\infty$-category and $D : \mathcal{V}^{op} \to \mathcal{V}$ be functor such that $D^{op} \dashv D$, i.e. there is an isomorphism in $X, Y \in \mathcal{V}$

$$\mathrm{Map}_{\mathcal{V}}(X, DY) \simeq \mathrm{Map}_{\mathcal{V}}(Y, DX) \tag{5.4}$$

natural in $X, Y$. In particular, $D$ sends colimits in $\mathcal{V}$ to limits in $\mathcal{V}$. For example, these conditions are satisfied if

- $D$ is a duality functor, for example in a Poincaré $\infty$-category;
- we additionally require $\mathcal{V}$ to be symmetric monoidal closed with unit $1_{\mathcal{V}}$, and set

$$DX := \underline{\mathrm{Hom}}(X, 1_{\mathcal{V}}) \; ; \tag{5.5}$$

- or if $\mathcal{V} = \mathrm{LMod}_R$ where $R$ is a ring spectrum, $M$ an invertible module over $R$, and $DX := \underline{\mathrm{Hom}}_R(X, M)$. Note that this is not included in the first case, since we do not restrict to perfect modules.

**Definition 5.1.5.** Using the fact that $D : \mathcal{V}^{op} \to \mathcal{V}$ preserves limits, we see that post-composing above equivalence of categories with $D$ preserves the sheaf condition, yielding a functor

$$\mathbb{D} : \mathcal{S}h(X;\mathcal{V})^{op} \to \mathcal{S}h(X;\mathcal{V}) \,, \; F \mapsto (U \mapsto D(\Gamma_c(U,F))) \tag{5.6}$$

which we will call the *Verdier duality functor* associated to $D$. If $D$ is an equivalence of categories, then $\mathbb{D}$ is as well.

**Lemma 5.1.6.** Equivalently, $\mathbb{D} = D \circ (-)_c \cong (-)_c \circ D^{op}$.

*Proof.* Given $F \in \mathcal{S}h(X; \mathcal{V})$ and $U \in \mathrm{Open}(X)$, we can write

$$\begin{aligned} DF_c(U) =& D\left(\operatorname*{colim}_{K \subseteq U \text{ cpt}} \Gamma_K(U, F)\right) \cong \operatorname*{colim}_{K \subseteq U \text{ cpt}} D^{op}\left(F(U) \times_{F(U-K)} 0\right) \cong \\ &\cong \operatorname*{colim}_{K \subseteq U \text{ cpt}} \left(D^{op}F(U) \times_{D^{op}F(U-K)} 0\right) = (D^{op}F)_c(U) \ . \end{aligned}$$

Note that $D^{op}F(X)$ for example is regarded as an object of $\mathcal{V}^{op}$, so the colimit is actually a limit in $\mathcal{V}$. □

**Lemma 5.1.7.** $\mathbb{D}^{op} \dashv \mathbb{D}$ is true as well. In particular if $D$ is an equivalence with inverse $D^{op}$, the same holds for $\mathbb{D}$.

*Proof.* As we have seen $(-)_c$ is an equivalence with inverse also given by $(-)_c^{op}$ (acting on cosheaves), $(-)_c^{op} \dashv (-)_c$. Since adjoints compose, $\mathbb{D}^{op} = (D \circ (-)_c)^{op} = D^{op} \circ (-)_c^{op} \dashv (-)_c \circ D \simeq D \circ (-)_c = \mathbb{D}$. The second statement is clear since we know that $\mathbb{D}$ is also an equivalence in this case, so $\mathbb{D}^{op} \dashv \mathbb{D}$ implies that $\mathbb{D}^{op}$ is its inverse. □

**Proposition 5.1.8.** Given a continuous map $f : X \to Y$ of locally compact Hausdorff spaces, we obtain (apart from the usual direct and inverse image functors) an adjunction $f_! : \ \mathcal{S}h(X; \mathcal{V}) \rightleftarrows \mathcal{S}h(Y; \mathcal{V}) \ : f^!$. The *exceptional direct and inverse image functors* $f_! \dashv f^!$ are defined to fill the respective commutative square in

$$\begin{array}{ccc} \mathcal{S}h(X; \mathcal{V}) & \overset{f^!}{\underset{f_!}{\rightleftarrows}} & \mathcal{S}h(Y; \mathcal{V}) \\ \downarrow{\scriptstyle (-)_c} & & \downarrow{\scriptstyle (-)_c} \\ \mathcal{S}h(X; \mathcal{V}^{op})^{op} & \overset{(f^*)^{op}}{\underset{(f_*)^{op}}{\rightleftarrows}} & \mathcal{S}h(Y; \mathcal{V}^{op})^{op} \end{array}$$

where $f_*, f^*$ denote direct and inverse image of cosheaves (viewed as $\mathcal{V}^{op}$-valued sheaves). In other words for $F \in \mathcal{S}h(X; \mathcal{V}), G \in \mathcal{S}h(Y, \mathcal{V})$,

$$f_!F := (f_*F_c)_c^{op} \ , \quad f^!G := (f^*G_c)_c^{op} \tag{5.7}$$

Equivalently, we could also write $f_! = \mathbb{D}f_*\mathbb{D}^{op}$ and $f^! = \mathbb{D}f^*\mathbb{D}^{op}$.

*Proof.* Since $(-)_c$ is an equivalence, this follows with an analogous argument as in 3.3.5 – be aware that contravariance of the duality functor exchanges the roles of left and right adjoint. The second claim holds because $D$ commutes with $(-)_c$ by 5.1.6; it commutes with $f_*$ as this is just a precomposition, and since $f_* \circ D \cong D \circ f_*$ the left adjoints $D^{op} \circ f^* \cong f^* \circ D^{op}$ also agree. □

*Remark.* In particular, $\Gamma_c(U, f_!F) = \Gamma_c(f^{-1}(U), F)$.

Let us compare the theory we have developed with classical Verdier duality.

**Theorem 5.1.9** ([Lur18b, 2.1.2.2])**.** Let $R$ be an ordinary ring and $X$ any topological space, then there is a canonical equivalence of categories

$$\mathcal{S}h^{hyp}(X; \mathrm{LMod}_{HR}) \simeq \mathcal{S}h^{hyp}(X; D(R)) \simeq D(\mathrm{Sh}(X; R\text{-Mod})) \tag{5.8}$$

where the first equivalence is by the stable Dold-Kan correspondence 1.7.2, and the right side is the derived $\infty$-category of the Grothendieck abelian category of *ordinary* sheaves of ordinary $R$-modules over $X$. Explicitly, this equivalence sends a complex $F$ or ordinary sheaves to the derived sections $R\Gamma(-, F) : \mathrm{Open}(X)^{op} \to D(R)$.

Let us identify $F$ with its image $R\Gamma(-, F)$ in $\mathcal{S}h^{hyp}(X; D(R))$. By construction, the global sections functor $\Gamma(F) = R\Gamma(X, F)$ so its homology groups are sheaf cohomology. For the same reason, $f_*F = R\Gamma(-, F \circ f^{-1})$ agrees with the derived direct image $Rf_*$, so as its adjoint $f^*$ agrees with $Lf^*$.

**Observation 5.1.10.** The composition $\Gamma_*\Gamma^* : \mathrm{LMod}_R \to \mathcal{S}h(X; R) \to \mathrm{LMod}_R$, possibly replacing the middle term with hypercomplete sheaves, is by construction a left exact functor called the *shape* of $\mathcal{S}h(X; R)$. Similarly for sheaves with other coefficients, in particular for $\mathcal{S}$-valued sheaves one obtains a pro-space. In our case, $\Gamma_*\Gamma^*(R) \cong R\Gamma(X, \underline{R})$ calculates the sheaf cohomology of the constant sheaf on $R$, which is an interesting topological invariant of $X$.

For $A \subseteq X$ a closed subset, $\Gamma_A(U, F) = \mathrm{fib}(F(U) \to F(U - A))$ yields sheaf cohomology with support in $A$ since on resolutions, the fiber is quasi-isomorphic to the kernel. A similar argument via resolutions shows that $F_c(U) = R\Gamma_c(U, F)$ calculates (sheaf) cohomology with compact support, so $\mathbb{D}$ agrees with the ordinary Verdier duality functor and $f_!, f^!$ agree with the associated functors $Rf_!, f^!$ on derived categories of sheaves. In the next section, we will also see how $\mathbb{D}$ can also be expressed in terms of the dualizing complex $\omega_X$.

Note that in our setting, we do not talk about resolutions at all to define these functors, allowing us to get rid of extra conditions in many classical theorems that require the existence of (finite) resolutions. In particular, this includes the biduality theorems we discuss in the next sections.

## 5.2 Six-Functor Formalism and Biduality

Let us for simplicity restrict to the case of $\mathcal{V} = \mathrm{LMod}_R$ over a ring spectrum $R$ that is an algebra over a commutative ring spectrum $k$, with duality given by $D(P) := \underline{\mathrm{Hom}}_R(P, R)$. We also omit the *op* on our $D$ as it is always clear from context, and denote $\mathcal{S}h(X, \mathrm{LMod}(R))$ by $\mathcal{S}h(X; R)$ since we will use it a lot. We still fix $X$ to be locally compact Hausdorff, as first goal will be to introduce tensor product and internal Hom that together with $f_*, f^*, f_!, f^!$ form a six-operator formalism.

**Definition 5.2.1.** Let $X, Y$ be topological spaces and $R$ a commutative ring spectrum, then the relative tensor product $\otimes_R : \mathrm{Mod}_R \otimes \mathrm{Mod}_R \to \mathrm{Mod}_R$ of $R$-modules induces a functor

$$\boxtimes : \mathcal{S}h(X; R) \times \mathcal{S}h(Y; R) \to \mathcal{S}h(X \times Y; R) \tag{5.9}$$

sending $F, G$ to the sheaf $F \boxtimes G(U \times V) = F(U) \otimes_R F(V)$ on elementary opens in the product. Similarly for $R$ an arbitrary ring spectrum, or sheaves with values in appropriate bimodules.

To see that this is indeed a sheaf, one uses presentability of $\mathrm{LMod}_R$ to write $\mathcal{S}h(X; R) \simeq \mathcal{S}h(X) \otimes \mathrm{LMod}_R$ which, regarding $\otimes$ as a functor $\mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ since it preservers colimits in both variables, allows us to reduce to the following Lemma.

**Lemma 5.2.2.** For $X, Y$ topological spaces, there is a canonical functor

$$\boxtimes : \mathcal{S}h(X) \times \mathcal{S}h(Y) \to \mathcal{S}h(X \times Y) \tag{5.10}$$

sending $F, G$ to $F \boxtimes G(U \times V) = F(U) \times F(V)$ extended from the product basis to all opens. In fact, if either $X$ or $Y$ is locally compact, this functor induces an equivalence of categories

$$\mathcal{S}h(X \times Y) \simeq \mathcal{S}h(X) \otimes \mathcal{S}h(Y)\ . \tag{5.11}$$

*Proof.* The functor $F \boxtimes G$ is indeed a sheaf since the product $\times : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ preserves limits involved in the descent condition. The stronger statement follows by combining [Lur09a, 7.3.1.11] and [Lur09a, 7.3.3.9]. □

*Remark.* Our definition of $\otimes_R$ is, by the Yoneda Lemma, equivalent to construction [Vol21, 2.2.5]. For more information on the six-functor formalism, in particular for $\mathcal{V}$ that are not necessarily presentable, we refer to the discussion there.

**Definition 5.2.3.** For $X$ any topological space and $\Delta : X \to X \times X$ the diagonal map, we similarly define the *tensor product* of sheaves as the composition

$$\begin{aligned} \otimes_R : \mathcal{S}h(X; R) \otimes \mathcal{S}h(X; R) &\simeq \mathcal{S}h(X) \otimes \mathcal{S}h(X) \otimes \mathrm{LMod}_R \otimes \mathrm{LMod}_R \xrightarrow{\otimes_R} \\ &\to \mathcal{S}h(X \times X) \otimes \mathrm{LMod}_R \xrightarrow{\Delta^*} \mathcal{S}h(X; R) \end{aligned} \tag{5.12}$$

or in other words, $F \otimes_R G := \Delta^*(F \boxtimes_R G)$. Similarly for sheaves over appropriate bimodules.

**Proposition 5.2.4.** For $R$ a commutative ring spectrum, the tensor product we have defined equips $\mathcal{S}h(X;R)$ with a symmetric monoidal structure, with unit the constant sheaf $\underline{R} = \Gamma^*(R)$.

**Definition 5.2.5.** By construction, for a fixed sheaf $F \in \mathcal{S}h(X;R)$ the functor $- \otimes_R F$ preserves colimits, so by the adjoint functor theorem it admits a right adjoint $\underline{\mathrm{Hom}}_R(F,-)$ being itself a contravariant functor in $F$. We obtain a functor

$$\underline{\mathrm{Hom}}_R(-,-) : \mathcal{S}h(X;R)^{op} \times \mathcal{S}h(X;R) \to \mathcal{S}h(X;R) \tag{5.13}$$

that preserves limits in the right, and colimits in $\mathcal{S}h(X;R)$ in the left argument. Also, since $- \otimes_R \underline{R}$ is the identity functor, $\underline{\mathrm{Hom}}(\underline{R},-) \cong \mathrm{Id}$ as well. A similar construction works for sheaves over appropriate bimodules.

**Proposition 5.2.6** ([Vol21, 2.31, 6.12])**.** For $f : X \to Y$ a continuous map between locally compact Hausdorff spaces, $F \in \mathcal{S}h(X;\mathcal{V})$ and $G, G' \in \mathcal{S}h(Y;\mathcal{V})$, the following formulae hold:

$$f^*(G) \otimes_R f^*(G') \cong f^*(G \otimes G') \tag{5.14}$$

$$f_!F \otimes_R G \simeq f_!(F \otimes_R f^*G) \tag{5.15}$$

**Theorem 5.2.7** (Classical Verdier Duality)**.** For $X, Y$ locally compact Hausdorff spaces, $f : X \to Y$ a continuous map, $R$ a commutative ring spectrum and $F \in \mathcal{S}h(X;R), G \in \mathcal{S}h(Y;R)$, there are natural isomorphisms

$$f_*\underline{\mathrm{Hom}}(f^*G, F) \cong \underline{\mathrm{Hom}}(F, f_*G) \tag{5.16}$$

$$f_*\underline{\mathrm{Hom}}(F, f^!G) \cong \underline{\mathrm{Hom}}(f_!F, G) \tag{5.17}$$

*Proof.* Let $E \in \mathcal{S}h(Y;R)$, for the second claim by the Yoneda-Lemma it suffices to show

$$\begin{aligned}\mathrm{Map}(E, f_*\underline{\mathrm{Hom}}_R(F, f^!G)) \cong \mathrm{Map}(f^*E \otimes_R F, f^!G) &\overset{!}{\cong} \\ \cong \mathrm{Map}(E \otimes_R f_!F, G) &\cong \mathrm{Map}(E, \underline{\mathrm{Hom}}(f_!F, G))\end{aligned}$$

This follows from the projection formula $f_!(f^!E \otimes_R F) \cong E \otimes_R f_!F$ above. Similarly, the first claim follows from the first formula above. □

**Definition 5.2.8.** Regard $R$ as a left module over itself, and write $\underline{R}$ for the constant sheaf on $R$. Also, let $t : X \to *$ be the canonical map into the terminal topological space. Then, the *dualizing sheaf* on $X$ is defined as the Verdier dual

$$\omega_X := \underline{R}_c \cong \mathbb{D}\underline{R} = \mathbb{D}t^*R = t^!\mathbb{D}R = t^!R \tag{5.18}$$

where we identify $R$ with its image under $\mathcal{S}h(*;R) \simeq \mathrm{LMod}_R$.

**Example 5.2.9.** If $X$ is an $n$-dimensional topological manifold, we have $\omega_X = \mathcal{O}r[-n]$ where $\mathcal{O}r$ is the *orientation sheaf*. In particular, $\omega_X$ is locally constant.

*Proof.* See [Vol21, 6.18(i)]. In fact, the classical proof using the Poincaré-Lemma as in [Ban07, Proposition 3.5.1] can be transported to the $\infty$-setting without problems. □

**Proposition 5.2.10.** For $F \in \mathcal{S}h(X; R)$, the Verdier duality functor can be rewritten as

$$\mathbb{D}F \cong \underline{\mathrm{Hom}}(F, \omega_X) \ . \tag{5.19}$$

*Proof.* Using that $\mathbb{D}^{op} \dashv \mathbb{D}$ by 5.1.7, we find

$$\mathbb{D}F \cong \underline{\mathrm{Hom}}(\underline{R}, \mathbb{D}F) \cong \underline{\mathrm{Hom}}(F, \mathbb{D}\underline{R}) = \underline{\mathrm{Hom}}(F, \omega_X) \ .$$ □

**Definition 5.2.11.** A sheaf $F \in \mathcal{S}h(X; R)$ on a locally compact Hausdorff space $X$ has

- *perfect stalks*, if for each $x \in X$ the stalk $x^*F$ is a perfect $R$-module.
- *perfect costalks*, if for each $x \in X$ the *costalk* $x^!F$ is a perfect $R$-module.

Similarly for the finitely presented case. Note that we identify $x$ with the associated map $x : * \to X$.

**Theorem 5.2.12.** For $X$ any topological space, the $\infty$-category $\mathcal{S}h(X; R)$ is presentable stable, in particular it has all limits and colimits. If $X$ is locally compact Hausdorff, the full subcategory $\mathcal{S}h^{hyp}_{\mathrm{perf}}(X; R)$ on hypersheaves with perfect stalks and costalks equipped with the quadratic functor

$$\Omega^s_X(F) := \mathrm{map}_{R \otimes_k R}(F \otimes_R F, \omega_X)^{hS_2} \tag{5.20}$$

or its pendant $\Omega^q_X$ is a Poincaré $\infty$-category, with duality functor given by Verdier duality. The same holds when restricting to finitely presented stalks.

*Proof.* For the first claim, $\mathcal{S}h(X; R) = \mathcal{S}h(X) \otimes \mathrm{LMod}_R$ by 1.3.17 since the latter is presentable, and since it is also stable the result will be presentable stable as well. The calculation of the duality functor from $\Omega^s$ is entirely analogous to 2.1.30, in particular:

$$B_{\Omega^s_X}(F, G) = B_{\Omega^q_X}(F, G) = \mathrm{map}_{R \otimes_k R}(F \otimes G, \omega_X)$$
$$D_{\Omega^s_X}(F) = D_{\Omega^q_X}(F) = \underline{\mathrm{Hom}}_R(F, \omega_X) \cong \mathbb{D}F$$

We already know that $D_{\Omega^s}$ is exact, also it preserves the property of having presentable (or finitely presented) stalks and costalks since for $x : \{x\} \to X$ a point,

$$x^*\mathbb{D}F = x^* \circ (-)_c \circ DF = (-)_c \circ x^! DF = Dx^! F \tag{5.21}$$

and vice versa, using that $D$ preserves colimits. Finally, if $F$ is hypercomplete, then $\mathbb{D}F$ as well since by 5.1.7 we know $\mathbb{D}$ is a right adjoint, and those preserve hypercomplete objects by the proof of [Lur09a, 6.5.2.13].

It remains to show that the canonical biduality transformation $\mathrm{Id} \to \mathbb{D}\mathbb{D}$ is an equivalence. In fact, it is enough to see that for each $F \in \mathcal{S}h^{hyp}_{\mathrm{perf}}(X; R)$ the morphism $F \to \mathbb{D}\mathbb{D}F$ is an isomorphism on stalks, since by 1.4.4 it must then be $\infty$-connected, but $\mathbb{D}\mathbb{D}F$ is hypercomplete so it is already an isomorphism. We calculate

$$x^*\mathbb{D}\mathbb{D}F = x^*D(-)_cD(-)_cF \cong x^*DD(-)_c(-)_cF \cong DDx^*F \cong x^*F \tag{5.22}$$

using again that $D$ and $(-)_c$ commute by 5.1.6, that $D$ preserves colimits and that $x^*F$ is perfect. □

*Remark.* This both generalizes [KS13, 3.4.3] and looks a lot more natural.

*Technical Remark.* It would be interesting to know whether this result carries over to $\infty$-categories $\mathcal{V}$ without a symmetric monoidal structure, where we cannot use a tensor product or $\omega_X$. The appropriate bilinear functor in this case would be the end

$$B_{\mathfrak{Q}^s_X}(F, Q) = \mathrm{nat}(F, \mathbb{D}G) = \int_{U \in \mathrm{Open}(X)} \mathrm{map}_{\mathcal{V}}(F(U), DG_c(U)) = \int_{U \in \mathrm{Open}(X)} B_{\mathfrak{Q}}(F(U), G_c(U))$$

but it does not seem obvious how to obtain a quadratic functor from this.

Finally, we will discuss several examples where Verdier duality and the associated functors are particularly simple:

**Proposition 5.2.13.** If $f : X \to Y$ is a proper map, in the sense that the preimage $f^{-1}(K)$ for $K \subseteq Y$ compact is still compact, then $f_! \cong f_*$.

*Proof.* Since in the proof of 5.1.4 we have seen that, given we are working with locally compact Hausdorff spaces, a sheaf is determined by its value on compact subsets, we may reduce to showing that for $F \in \mathcal{S}h(X; \mathcal{V})$ and $K \in K(Y)$, the values $f_*F(K) = F(f^{-1}(K)) \cong f_!F(K)$ are isomorphic, naturally in $F$ and $K$. Since $(-)_c$ is an equivalence with inverse $(-)_c$, we may as well show $(f_*F)_c(K) \cong f_*F_c(K)$. This amounts to

$$(f_*F)_c(K) = \Gamma_K(Y, (f_*F)_c) \overset{(*)}{\cong} \Gamma_K(Y, F \circ f^{-1}) = \mathrm{fib}(F(X) \to F(X - f^{-1}(K)))$$
$$f_*F_c(K) = F_c(f^{-1}(K)) \overset{(*)}{\cong} \Gamma_{f^{-1}(K)}(X, F) = \mathrm{fib}(F(X) \to F(X - f^{-1}(X - K)))$$

where at $(*)$ we use that if $K' \subseteq X$ is compact, then $\Gamma_{K'}(X, F_c) \cong \Gamma_{K'}(X, F)$ by extensive colimit arguments, compare [Lur09a, Section 7.3.4]. □

**Lemma 5.2.14.** Let $X$ be a topological space, and $j : U \hookrightarrow X$ an open subset. Then, the functors $j : \mathrm{Open}(U) \to \mathrm{Open}(X)$ that views an open subset of $U$ as an open subset of $X$ is left adjoint to the functor $j^{-1} : \mathrm{Open}(X) \to \mathrm{Open}(U)$ that sends $W \subseteq X$ to $W \cap U \subseteq U$.

*Proof.* Since the involved categories are posets, all we need to understand is that for $V \in U$ and $W \in X$, we have $j(V) = V \subseteq W$ iff $V \subseteq W \cap U$. □

**Proposition 5.2.15.** If $j : U \hookrightarrow X$ is an open subspace, and $i : Z = X - U \hookrightarrow X$ its closed complement, then $i_*$ and $j^*$ commute with $(-)_c$ so that $i^- = i^!$, $j_+ = j_!$ by 3.3.5. We can thus restate 3.1.5 as saying that

$$\begin{array}{ccccc} & \xleftarrow{\;\; i^* \;\;} & & \xleftarrow{\;\; j_! \;\;} & \\ \mathcal{S}h(Z;\mathcal{V}) & \xrightarrow{\;\; i_* \;\;} & \mathcal{S}h(X;\mathcal{V}) & \xrightarrow{\;\; j^* \;\;} & \mathcal{S}h(U;\mathcal{V}) \\ & \xleftarrow{\;\; i^! \;\;} & & \xleftarrow{\;\; j_* \;\;} & \end{array}$$

is a split Verdier sequence, for $\mathcal{V}$ presentable stable or compactly generated.

*Proof.* First, note that $U, Z$ are again locally compact and Hausdorff. The only non-trivial part in verifying this is the local compactness of $Z$, but for $x \in U \subseteq Z$ open we have $U = V \cap Z$ with $V \subseteq X$ open, so there is a $K \subseteq X$ compact and a $U' \subseteq X$ open with $x \in U' \subseteq K \subseteq V$. But then $x \in U' \cap Z \subseteq K \cap Z$ and $K \cap Z$ is a closed subset of $K$, so it is a compact neighborhood of $x$ inside $U$ as desired.

The map $i$ is proper since a closed subset of a compact set is again compact, so by the previous Proposition 5.2.13 $i_! = (-)_c \circ i_* \circ (-)_c \cong i_*$ proving this case. For $j^*$, we use the fact that the adjunction from 5.2.14 induces an adjoint quadruple $\mathrm{Lan}_{j^{op}} \dashv (- \circ j^{op}) \dashv (- \circ (j^{-1})^{op}) \dashv \mathrm{Ran}_{(j^{-1})^{op}}$ between the presheaf categories $\mathrm{Fun}(\mathrm{Open}(U)^{op}, \mathcal{V})$ and $\mathrm{Fun}(\mathrm{Open}(X)^{op}, \mathcal{V})$, which follows from uniqueness of adjoints and the fact that for a pair of adjoint functors, the associated precomposition functors are also adjoint (but with right and left adjoint exchanged).

This tells us that for $F \in \mathcal{S}h(X; R)$ and an open $V \subseteq U$, the pullback sheaf $j^*F(V) = F(V)$ since the left adjoint to $(- \circ j^{-1})$ on presheaves is $(- \circ j^{op})$, and this functor preserves sheaves by a short calculation or applying the covering lifting property 1.3.13. With this observation, $(-)_c \circ j^* \cong j^* \circ (-)_c$ can be calculated from the very definition of $(-)_c$. □

**Lemma 5.2.16.** In the situation above, for $F, F' \in \mathcal{S}h(Z; R)$ we have

$$j_! F \otimes_R j_! F' \cong j_!(F \otimes_R G) \tag{5.23}$$

*Proof.* Using the second projection formula from 5.2.6, we may write

$$j_! F \otimes_R j_! F' \cong j_!(F \otimes_R j^* j_! F') \ , \tag{5.24}$$

but $j^* j_! = \mathrm{Id}$ by the last Proposition, as $j^*$ is a coreflection right adjoint to $j_!$. □

**Lemma 5.2.17.** In the situation above, we have $i^- \omega_X = \omega_Z$ and $j^* \omega_X = \omega_U$.

*Proof.* Let $t_Z, t_X, t_U$ be the terminal maps from $Z, X, U$; then we may write using the last proposition

$$i^- \omega_X = i^! t_X^! R = t_Z^! R = \omega_Z, \quad j^* \omega_X = j^! t_X^! R = t_U^! R$$

□

**Theorem 5.2.18.** If $X$ is a locally compact Hausdorff space, $U \subseteq X$ an open subset and $Z = X - U$ its closed complement, then the sequence

$$(\mathcal{S}h^{hyp}_{\mathrm{perf}}(Z; R), \Omega^q_Z) \xrightarrow{i_*} (\mathcal{S}h^{hyp}_{\mathrm{perf}}(X; R), \Omega^q_X) \xrightarrow{j^*} (\mathcal{S}h^{hyp}_{\mathrm{perf}}(U; R), \Omega^q_U) \tag{5.25}$$

is a split Poincaré-Verdier sequence, with adjoints as indicated above. Similarly if we work with $\Omega^s_Z, \Omega^s_X$ and $\Omega^s_U$.

*Proof.* Since $\mathrm{LMod}_R$ is compactly generated, this is a stable recollement by 3.1.5 and hence a split Verdier sequence by 3.2.9. The previous proposition shows that $i_*$ and $j^*$ are duality-preserving, so it suffices to show that $\Omega^q_X \circ i_* = \Omega^q_Z$ and $\Omega^q_X \circ j_! = \Omega^q_U$. The symmetric case is analogous.

For the first claim, let $F \in \mathcal{S}h(Z; R)$ and write

$$\Omega^q_X \circ i_*(F) = \mathrm{map}_{R \otimes_k R}(i_* F \otimes_R i_* F, \omega_X)_{hS_2} = \mathrm{map}_{R \otimes_k R}(F \otimes_R F, i^- \omega_X)_{hS_2} = \Omega^q_Z(F) \tag{5.26}$$

using the previous Lemma 5.2.17. Similarly, for $G \in \mathcal{S}h(U; R)$ we have

$$\Omega^q_X \circ j_!(F) = \mathrm{map}_{R \otimes_k R}(j_! F \otimes_R j_! F, \omega_X)_{hS_2} = \mathrm{map}_{R \otimes_k R}(F \otimes_R F, j^* \omega_X)_{hS_2} = \Omega^q_U(F) \tag{5.27}$$

using the same Lemma and $j_! F \otimes_R j_! F \cong j_!(F \otimes_R F)$ by 5.2.16. □

## 5.3 Locally Constant Sheaves

Let $X$ be a topological space and $\mathcal{V}$ a presentable $\infty$-category.

**Definition 5.3.1.** A sheaf $F \in \mathcal{S}h(X; \mathcal{V})$ is called *constant* if it lies in the essential image of the functor $\Gamma^* : \mathcal{V} \to \mathcal{S}h(X; \mathcal{V})$ that is left adjoint to the global sections functor $\Gamma_*$. Explicitly, this means that $F$ is (isomorphic to) the sheafification of a *constant presheaf*, a presheaf that sends every open in $X$ to a fixed object $V \in \mathcal{V}$.

Similarly, a *hyperconstant hypersheaf* is a (hyper-)sheaf in the essential image of the composition $\Gamma^*_{hyp} := (-)^{hyp} \circ \Gamma^*$.

**Definition 5.3.2.** A sheaf $F \in \mathcal{S}h(X;\mathcal{V})$ is called *locally constant* if there is an open cover $(\iota_i : U_i \hookrightarrow X)$ of $X$ such that for each $i$, the restriction $F|_{U_i} := \iota_i^* F$ is locally constant. Denote the full subcategory on locally constant sheaves by $\mathcal{S}h^{lc}(X;\mathcal{V})$.

Similarly, a hypersheaf $F$ is called *locally hyperconstant* if there exists an open cover $(\iota_i : U_i \hookrightarrow X)$ such that the hyperpullbacks $(\iota_i^* F)^{hyp}$ are hyperconstant. Denote the full subcategory on them by $\mathcal{S}h^{hyp,lc}(X;\mathcal{V})$.

*Remark.* These definitions are a special case of [Lur17, A.1.12].

*Warning.* Being a locally constant hypersheaf is *not* equivalent to being locally constant and a hypersheaf.

**Proposition 5.3.3.** If $\mathcal{V}$ is additionally a stable $\infty$-category, then $\mathcal{S}h^{lc}(X;\mathcal{V})$ and $\mathcal{S}h^{hyp,lc}(X;\mathcal{V})$ are stable as well.

*Proof.* We know from 1.3.18 that $\mathcal{S}h(X;\mathcal{V})$ itself is stable, so by 1.5.7 it suffices to show that the full subcategory on locally constant sheaves is closed under finite limits. But given any finite diagram $F : K \to \mathcal{S}h^{lc}(X;\mathcal{V})$ we may choose an open cover $(U_i)$ of $X$ that is closed under intersections such that on each $U_i$, all $F(k)|_{U_i}$ with $k \in K$ are constant. Since pullbacks are left exact, we may calculate our finite limit inside the category of (hyper-)constant sheaves on each $(U_i)$ individually and glue the results together. Hence, without loss of generally, we may reduce to (hyper-)constant sheaves.

But the left adjoint to the global sections functor $\Gamma^* : \mathcal{V} \to \mathcal{S}h(X;\mathcal{V})$ sending $V$ to the associated constant sheaf is left exact, as is the hypercompletion functor $(-)^{hyp}$, so we are finished since $\mathcal{V}$ being stable has finite limits. $\square$

In the classical setting, a well-known result from covering theory can be applied to determine the category of ordinary locally constant sheaves on a good space:

**Theorem 5.3.4.** Let $X$ be a locally path-connected, semi-locally simply connected, path connected topological space and denote by $\mathrm{Cov}(X)$ the category of coverings on it and deck transformations, by $\mathrm{Sh}^{lc}(X)$ the category of locally constant 1-sheaves of sets on $X$ (also called *local systems*), and by $\pi_1(X)$-Set the category of sets with an action of the fundamental group. Then, the following correspondence holds:

$$\mathrm{Cov}(X) \simeq \mathrm{Sh}^{lc}(X) \simeq \pi_1(X)\text{-Set} \simeq \mathrm{Fun}(\pi_{\leq 1}(X), \mathrm{Set}) \tag{5.28}$$

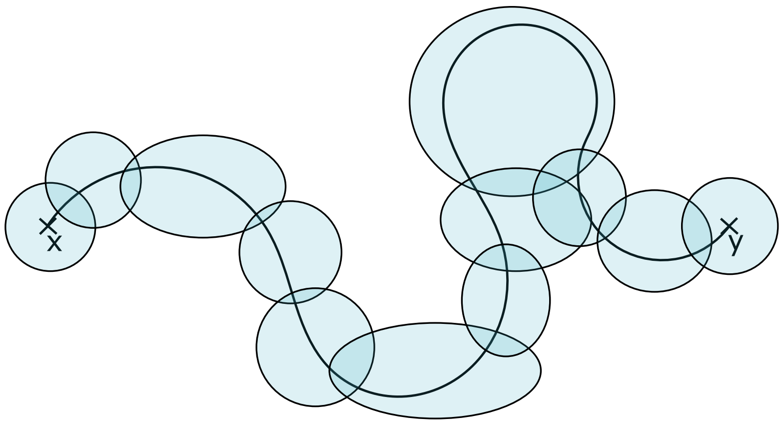
For $X$ not path-connected, the category $\pi(X)$-Set is not equivalent to the rest since it depends on a choice of base point, but the remaining equivalences still hold. Also, we can replace the category of sets with any presentable 1-category.

*Proof Sketch.* The first equivalence is induced by constructing the éspace étalé of a locally constant sheaf, which is always a covering, and inversely taking the sheaf of sections of a covering. The last equivalence is also easy to understand, since for $X$ path-connected, the groupoid $\pi_{\leq 1}(X)$ is also connected. It is therefore equivalent to the one-object category associated to the group $\pi_1(X)$, and functors from it into Set are specified by the image of this object, together with an induced $\pi_1(X)$-action.

Finally, we explain how to associate a monodromy representation $m : \pi_{\leq 1}(X) \to \mathrm{Set}$ to any locally constant sheaf $F$; the converse uses the existence of a universal covering. To each point $x \in X$, we associate the stalk $m(x) := x^*F$; so for each path $\gamma : [0,1] \to X$ from $x$ to a point $y \in X$, we need to find a transport map $m(\gamma) : x^*F \to y^*F$ that is compatible with composition and homotopy invariant. We will mere explain how to construct $m(\gamma)(s_x)$ for a fixed $s_x \in x^*F$.

Choose connected open subsets $(U_i)$ in $X$ that cover $\gamma([0,1])$ such that $F|_{U_i}$ is a constant sheaf for each $i \in I$. Since $[0,1]$ is compact, we can reduce to a finite number $U_0, \ldots, U_N$ of them such that $x \in U_0$ and $y \in U_N$, see the picture. Note that $F|_{U_i}$ is even a constant presheaf since $U_i$ is connected, so we can canonically identify all stalks in $U_i$. This allows us to iteratively transport $s_x$ through all of the finitely many $U_i$ until we reach $U_N$ and a germ $m(\gamma)(s_y) \in F_y$. □

Figure 5.1: Parallel transport from $x$ to $y$ by covering the path with small open sets




Lurie has proven a similar characterization [Lur17, A.4.19] for $\infty$-sheaves, which we state in a slightly generalized version that follows as a special case of [PT22, Theorem 5.17]. The presence of higher homotopies in the context on $\infty$-categories imposes higher local connectivity requirements on our space:

**Definition 5.3.5.** A topological space $X$ is called *locally weakly contractible* if for every $x \in X$, there exists a neighborhood $U \ni x$ such that $\mathrm{Sing}(U)$ is homotopy equivalent to $\Delta^0$ (or equivalently, $\pi_n(U) = 0$ for all $n$).

**Example 5.3.6.** Every topological manifold is locally weakly contractible.

**Theorem 5.3.7** (Monodromy equivalence)**.** For $X$ a locally weakly contractible space and $\mathcal{V}$ a presentable $\infty$-category, there is an equivalence of categories between locally constant hypersheaves and representations of its homotopy type:

$$\mathcal{S}h^{hyp,lc}(X;\mathcal{V}) \simeq \mathrm{Fun}(\mathrm{Sing}(X),\mathcal{V}) \tag{5.29}$$

As indicated in the discussion above, the functor $\mathrm{Sing}(X) \to \mathcal{V}$ associated to a sheaf $F$ sends $x \in X$ to the stalk $x^*F \in \mathcal{V}$. If we assume that $X$ is either

- paracompact Hausdorff of finite covering dimension, so that all sheaves on it are hypercomplete by 1.4.9, and additionally locally weakly contractible; or
- locally of singular shape in the sense of [Lur17, A.4.15], which is a difficult property but holds in particular for topological manifolds and CW complexes,

then the hypercompleteness condition is not necessary and we obtain an equivalence

$$\mathcal{S}h^{lc}(X;\mathcal{V}) \simeq \mathrm{Fun}(\mathrm{Sing}(X),\mathcal{V})\ . \tag{5.30}$$

*Remark.* Since every Kan complex is the homotopy colimit of its points, we can rewrite this as

$$\mathcal{S}h^{lc}(X) = \mathrm{Fun}(\operatorname*{colim}_{\mathrm{Sing}(X)} *,\mathcal{V}) \simeq \lim_{\mathrm{Sing}(X)} \mathcal{V}\ . \tag{5.31}$$

These monodromy equivalences can be interpreted as saying that the category Set acts as a classifying space for covering maps, and the $\infty$-category of spaces $\mathcal{S}$ acts as a classifying space for locally constant $\infty$-sheaves (i.e. local $\infty$-systems). We refer to [Zet23, Section B.5] for a more extensive exposition, connecting this theorem to other Riemann-Hilbert-like correspondences throughout mathematics.

**Corollary 5.3.8.** If $X$ is locally weakly contractible, then since we assume $\mathcal{V}$ is presentable, $\mathcal{S}h^{hyp,lc}(X,\mathcal{V})$ is presentable as well. Similarly, if $\mathcal{V}$ is compactly generated or an $\infty$-topos, then $\mathcal{S}h^{hyp,lc}(X;\mathcal{V})$ has the same property. Under the mentioned extra conditions, this statement translates to $\mathcal{S}h^{lc}(X;\mathcal{V})$.

*Proof.* All of these properties are preserved if we replace $\mathcal{V}$ by $\mathrm{Fun}(\mathrm{Sing}(X),\mathcal{V})$ as long as $\mathrm{Sing}(X)$ is small. □

**Corollary 5.3.9.** The inclusion $i_{lc} : \mathcal{S}h^{hyp,lc}(X;\mathcal{V}) \hookrightarrow \mathcal{S}h^{hyp}(X;\mathcal{V})$ has a left adjoint $L_{lc}$ and a right adjoint $R_{lc}$.

*Proof.* Since the involved categories are presentable as we have just seen, by the Adjoint-Functor-Theorem 1.2.19 it suffices to show that locally constant sheaves are closed under limits and colimits. For colimits, this is [Lur17, A.1.16] and due to the fact that $\Gamma^*$ being a left adjoints commutes with colimits; and for finite (co)limits it follows from our argument in 5.3.3. For general limits, we will see this in 6.3.6. □

**Corollary 5.3.10.** A continuous map $f : X \to Y$ of locally weakly contractible topological spaces induces an adjoint triple on $\infty$-categories os locally constant sheaves, where $f^*$ is given by the usual inverse image:

$$\mathcal{S}h^{hyp,lc}(X;\mathcal{V}) \begin{array}{c} \xrightarrow{\quad f_+^{lc} \quad} \\ \xleftarrow{\quad f^* \quad} \\ \xrightarrow{\quad f_*^{lc} \quad} \end{array} \mathcal{S}h^{hyp,lc}(X;\mathcal{V})$$

*Remark.* Compare this with the case of general sheaves, where there usually are only two functors $f_*$ and $f^*$.

*Proof.* The monodromy equivalence 5.3.7 tells us that $\mathcal{S}h^{lc}(X;\mathcal{V}) = \mathrm{Fun}(\mathrm{Sing}(X), \mathcal{V})$ and similarly for $Y$, so the map $\mathrm{Sing}(f) : \mathrm{Sing}(X) \to \mathrm{Sing}(Y)$ induces an adjoint triple on categories of locally constant sheaves via precomposition, left and right Kan extension (which exist since $\mathcal{V}$ is presentable). It remains to show that under the above equivalence, precomposition with $\mathrm{Sing}(f)$ corresponds to the pullback of sheaves.

Recall from 5.3.7 that for $F$ a locally constant sheaf on $X$, the image of $x \in X$ under the monodromy representation of $F$ is the stalk $x^*F$. But pullbacks of sheaves preserve stalks, so our claim is true on vertices of $\mathrm{Sing}(X)$. On edges and higher simplices, we would need to show that the pullback preserves (higher) parallel transports – instead of doing this explicitly, we refer to the abstract argument of [PT22, 6.8]. □

*Remark.* Pre- and postcomposing pullback and pushforward for general sheaves with the adjoint triple from 5.3.9, we see that $f_*^{lc} = R_{lc} \circ f_* \circ i_{lc}$.

*Remark.* To gain some intuition for these adjoints, consider the classical case. The functor $f^* : \mathrm{Sh}^{lc}(Y) \to \mathrm{Sh}^{lc}(X)$ sends a locally constant sheaf with monodromy representation $M \in \pi_1(Y)$-Set to the restriction of scalars $\pi_1(X) \to \pi_1(Y) \to \mathrm{Aut}(M)$; and $f_+^{lc}, f_*^{lc}$ send a representation $N$ of $\pi_1(X)$ to the induced representation $\mathrm{Ind}_{\pi_1(X)}^{\pi_1(Y)}(N)$ or coinduced representation $\mathrm{CoInd}_{\pi_1(X)}^{\pi_1(Y)}(N)$ of $\pi_1(Y)$.

**Corollary 5.3.11.** For $x \in X$ a point classified by the map $x : * \to X$, and for $t : X \to *$ the terminal map, we obtain adjunctions

$$\begin{array}{ccccc} & \xrightarrow{x_+^{lc}} & & \xrightarrow{C_*} & \\ \mathcal{V} & \xleftarrow{x^*} & \mathcal{S}h^{hyp,lc}(X;\mathcal{V}) & \xleftarrow{\underline{(-)}} & \mathcal{V} \\ & \xrightarrow{x_*^{lc}} & & \xrightarrow{C^*} & \end{array}$$

where $\underline{(-)} = t^*$ and $\Gamma = C^* = t_*^{lc}$ agree with the terminal geometric morphism on sheaves, and $C_* := t_+^{lc}$. As the names suggest, $C_*$ and $C^*$ in the case $\mathcal{V} = D(R)$ for $R$ an ordinary ring calculate the complexes of singular (co)chains with values in a local system.

*Proof.* The adjunctions are a special case of 5.3.10. In particular, this tells us that $t_*$ agrees with the pushforward of sheaves along the terminal map, which calculates sheaf cohomology. In the case of $\mathcal{V} = D(R)$, the (co)limits

$$C^*F = \lim_{\mathrm{Sing}(X)} F\,, \quad C_*F = \operatorname*{colim}_{\mathrm{Sing}(X)} F$$

where we identify $F$ with its monodromy representation can be calculated explicitly from the Čech complex construction in 1.5.18, yielding precisely the singular (co)chain complex with boundary maps twisted by parallel transport along $F$. For definiteness, consider the constant sheaf $\underline{R[0]}$ with monodromy representation the constant functor $\underline{R[0]} : \mathrm{Sing}(X) \to D(R)$, then

$$C_*(\underline{R[0]}) = \operatorname*{colim}_{[n]\in\Delta} \bigoplus_{\mathrm{Sing}(X)_n} R[0] = \left( \cdots \to \bigoplus_{e:|\Delta^1|\to X} R \to \bigoplus_{v:|\Delta^0|\to X} R \right)$$

using the general bar construction. □

*Remark.* Using 3.3.5, conjugating the adjoint triple $f_+^{lc} \dashv f^* \dashv f_*^{lc}$ with Verdier duality induces an adjoint triple $f_!^{lc} \dashv f^! \dashv f_{+!}^{lc}$. In particular for the terminal map, we obtain functors $C^! \dashv \underline{(-)}_c \dashv C_!$ where $C^!$ calculates *compactly supported cohomology* by the discussion after 5.1.9, and $C_!$ is *Borel-Moore-Homology.*

## 5.4 Monodromy

In the case of $\mathcal{V} = \mathrm{LMod}_R$ for $R$ a ring spectrum, there is a more refined version of the monodromy correspondence we have discussed in the last section.

**Definition 5.4.1.** A *generator* of a stable $\infty$-category $\mathcal{C}$ is an element $X \in \mathcal{C}$ such that for any $Y \in \mathcal{C}$, if $\mathrm{Map}_{\mathcal{C}}(X, Y) \simeq \Delta^0$ is contractible, then already $Y = 0$.

**Proposition 5.4.2.** If $X$ is a non-empty connected locally weakly contractible topological space, and $x \in X$ a base point, then $x_+^{lc}(R)$ is a compact generator of $\mathcal{S}h^{hyp,lc}(X; \mathrm{LMod}_R)$, where we regard $R$ as a left module over itself in the canonical way.

*Proof.* Since $X$ is locally weakly contractible, if is in particular locally path connected (since $\pi_0$ vanishes on a neighborhood of each point), so connectedness implies path connectedness.

For $F \in \mathcal{S}h^{lc}(X; \mathrm{LMod}_R)$ be arbitrary, we can write

$$\mathrm{Map}_{\mathcal{S}h^{lc}(X;\mathrm{LMod}_R)}(x_+^{lc}(R), F) \simeq \mathrm{Map}_{\mathrm{LMod}_R}(R, x^*(F)) = x^*F \ .$$

If we assume that this mapping space is contractible, then since $X$ is connected, there exists and isomorphism $x^*F \simeq y^*F \simeq \Delta^0$ for any $y \in X$ induced by parallel transport (to be precise, this follows from the monodromy correspondence). Hence, $F$ must already agree with the terminal sheaf as it admits a canonical map into it which induces isomorphisms on stalks, and both sheaves are hypercomplete.

Finally, $R$ is a compact object of $\mathrm{LMod}_R$ and $x_+^{lc}$ preserves compact objects since it has a left adjoint $x^*$ that preserves colimits, as it is admits a left adjoint $x_*^{lc}$ itself. Therefore $x_+^{lc}R$ is compact. □

**Theorem 5.4.3** ([PT22, Theorem 6.26])**.** For $(X, x)$ a pointed, connected and locally weakly connected topological space, denote by $\Sigma^\infty\Omega(X)$ the associative ring spectrum given as the suspension spectrum of its loop space and define the associative ring spectrum $R' := \Sigma^\infty\Omega(X) \wedge R$. Then, there is a canonical equivalence of categories

$$\mathcal{S}h^{lc}(X; \mathrm{LMod}_R) \simeq \mathrm{LMod}_{R'} \ . \tag{5.32}$$

*Proof.* Since $\mathrm{LMod}_R$ is a presentable stable $\infty$-category, combining the last proposition 5.4.2 with the Schwede-Shipley recognition criterion 1.7.4 we find that it is equivalent to $\mathrm{LMod}_{\mathrm{end}(x_+^{lc}(R))}$, so it remains to calculate this endomorphism ring spectrum:

$$\mathrm{end}(x_+^{lc}(R)) = \mathrm{map}(x_+^{lc}(R), x_+^{lc}(R)) \simeq \mathrm{map}(R, x^*x_+^{lc}(R)) \simeq x^*x_+^{lc}R$$

The pullback square in $\mathcal{S}$

$$\begin{array}{ccc} \mathrm{Sing}(\Omega X) & \xrightarrow{\ t\ } & \Delta^0 \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle x} \\ \Delta^0 & \xrightarrow{\ x\ } & \mathrm{Sing}(X) \end{array}$$

implies that $\mathrm{Sing}\,\Omega X \simeq \{x\} \times_{\mathrm{Sing}(X)} \mathrm{Sing}(X)_{/x}$, so we can calculate

$$x^*x_+^{lc}R = \mathrm{Lan}_x^{\mathrm{Sing}(X)} R(x) = \operatorname*{colim}_{\{x\}\times_{\mathrm{Sing}(X)}\mathrm{Sing}(X)_{/x}} R = R \otimes \Omega X = R \wedge \Sigma^\infty\Omega X$$

using how the tensoring in spaces or spectra is defined. □

If we want to use this theorem to calculate L-groups, we need to find a way to restrict to finitely presented $R$-modules. This involves generalizing the monodromy correspondence to this non-presentable case.

**Proposition 5.4.4.** Let $\mathcal{V}$ be a small stable $\infty$-category, and $X$ a locally weakly contractible space. Then, locally constant hypersheaves in $\mathrm{Ind}(\mathcal{V})$ whose stalks all lie in $\mathcal{V}$ are characterized by their monodromy representation:

$$\mathcal{S}h_{\mathcal{V}}^{hyp,lc}(X;\mathrm{Ind}(\mathcal{V})) \simeq \mathrm{Fun}(\mathrm{Sing}(X),\mathcal{V}) \tag{5.33}$$

*Proof.* Since $\mathrm{Ind}(\mathcal{V})$ is compactly presented, the monodromy correspondence 5.3.7 implies $\mathcal{S}h^{hyp,lc}(X;\mathrm{Ind}(\mathcal{V})) \simeq \mathrm{Fun}(\mathrm{Sing}(X),\mathrm{Ind}(\mathcal{V}))$, so it suffices to show that such a sheaf $F$ has stalks in $\mathcal{V}$ iff for all $x \in X$, the value of its monodromy representation lies in $\mathcal{V} \subseteq \mathrm{Ind}(\mathcal{V})$. But we know that this value is just $x^*F$, so we are finished. □

**Lemma 5.4.5.** If $F_{x,M}$ denotes the skyscraper sheaf at $x \in X$ with value $M \in \mathrm{LMod}_R$ and $X$ is locally weakly contractible, then

$$L_{lc}F_{x,M} \cong x_+^{lc}M\ . \tag{5.34}$$

Similarly if $j : U \subseteq X$ is a weakly contractible open subset with trivial shape containing $x$, then for the sheaf $F_{U,M} := j_!\underline{M}$ with support in $U$,

$$L_{lc}F_{U,M} \cong x_+^{lc}M\ . \tag{5.35}$$

*Proof.* For any locally constant hypersheaf $S$ on $X$,

$$\mathrm{Map}(L_{lc}F_{x,M},S) \simeq \mathrm{Map}(F_{x,M},S) \simeq \mathrm{Map}(M,x^*S) \simeq \mathrm{Map}(x_+^{lc}M,S)$$

hold by applying the appropriate adjunctions, so the result follows from the Yoneda Lemma. Similarly,

$$\mathrm{Map}(L_{lc}F_{U,M},S) \simeq \mathrm{Map}(j_!\underline{M},S) \simeq \mathrm{Map}(\underline{M},S|_U) = \mathrm{Map}(M,\Gamma_*S|_U)$$

which, since $U$ is weakly contractible and $S|_U$ therefore constant, agrees with the (trivial) shape of $U$ applied to the stalk $x^*S$, so the result follows as in the other calculation. □

**Definition 5.4.6.** As in the PL case, we define a hypersheaf $F \in \mathcal{S}h^{hyp}(X;\mathcal{V})$ to be *balanced* if for any $S \in \mathcal{S}h^{hyp,lc}(X;\mathcal{V})$, the mapping space $\mathrm{Map}(F,S)$ is contractible, and denote their full subcategory by $\mathcal{S}h^{\perp hyp,lc}(X;\mathcal{V})$. If $\mathcal{V} = \mathrm{LMod}_R$, this is again equivalent to $L_{lc}F = 0$. Define $\mathcal{S}h_{fp}^{\perp lc}(X;R) := \mathcal{S}h^{\perp lc}(X;R) \cap \mathcal{S}h_{fp}^{hyp}(X;R)$ as the full subcategory of balanced sheaves that have finitely presented stalks and costalks.

**Proposition 5.4.7.** If the underlying space $X$ is a topological manifold or CW complex (for us always locally finite), then a locally constant sheaf $F \in \mathcal{S}h^{lc}(X;R)$ has finitely presented (or perfect) costalks iff it has finitely presented (or perfect) stalks. In fact, we show in 6.4.2 that this is true more generally, but we will for simplicity restrict to the case of topological manifolds in what follows.

**Definition 5.4.8.** As for PL sheaves, we define

$$\mathcal{S}h^{hyp,lc}(X;R)^{(fp)} := \mathcal{S}h^{hyp}_{fp}(X;R) \big/ \mathcal{S}h^{\perp hyp,lc}_{fp}(X;R) \tag{5.36}$$

which as usual agrees with the full subcategory of $\mathcal{S}h^{hyp,lc}(X;R)$ on sheaves of the form $L_{lc}F$ with $F \in \mathcal{S}h^{hyp}_{fp}(X;R)$.

**Theorem 5.4.9.** Let $X$ be a topological manifold. Under the equivalence in 5.4.3, the full subcategories

$$\mathcal{S}h^{hyp,lc}(X;R)^{(fp)} \simeq \mathrm{LMod}^{\mathrm{fp}}_{\Sigma^\infty \Omega X \wedge R} \tag{5.37}$$

are identified. The quadratic functor on the right induced by (twisted) Verdier duality $\mathbb{D} = D_M \circ (-)_c$ is associated to the invertible module $\Sigma^\infty \Omega X \wedge R$ whose involution consists of

- the involution in $M$,
- the loop-reversing involution on $\Omega X$,
- the non-orientability of the orientation sheaf $\omega_X$ along the respective loop adding an extra sign.

*Proof Sketch.* We only prove the first part, the involution can be obtained by carefully going through the proof of 5.4.3, compare [Lur11, Lecture 22]. The sheaves $F_{U,R}$ from 5.4.5 for $U$ charts of $X$ generate $\mathcal{S}h_{fp}(X;R)$ since charts form a basic, so since we can identify $\mathcal{S}h^{lc}(X;R)^{(fp)} \subseteq \mathcal{S}h^{lc}(X;R)$ with the essential image of this category under $L_{lc}$, it suffices to show that smallest stable subcategory of it spanned by the compact generator $x^{lc}_+R$ agrees with the smallest stable subcategory containing all $L_{lc}F_{U,R}$, which is immediate by the mentioned Lemma. □

*Remark.* If we want to prove this theorem in the PL case or on a simplicial complex, we use the sheaves $F^{\tau,V}$ as generators.

**Corollary 5.4.10.** If $M$ is a connected topological manifold, then

$$\mathbb{L}^q(\mathcal{S}h^{lc}(M;R)^{(lc)}) \simeq \mathbb{L}^q(\Sigma^\infty \Omega M \wedge R) \simeq \mathbb{L}^q(\pi_0 R[\pi_1 M]) \tag{5.38}$$

just as in the PL case. If $M$ is not connected but consists of finitely many connected components, it is easy to see that the category of locally constant sheaves orthogonally decomposes into the categories of sheaves on the individual components, and this is compatibly with Verdier-duality in the sense of 3.4.5. Therefore, the total L-group splits as a direct sum.

*Warning.* This result does *not* hold if we replace the term "finitely presented" by "perfect", even in the PL case. There usually is only a fully faithful inclusion

$$\mathcal{S}h^{lc}_{\mathrm{perf}}(X;R)^{(fp)} \subseteq \mathrm{LMod}_R^{\mathrm{perf}} \ . \tag{5.39}$$

Compare the examples in [PT22, 6.30].

## 5.5 Visible L-Groups

It is time to piece together the developments of the last sections. Let $X$ be a topological manifold.

**Proposition 5.5.1.** The left adjoint $L_{lc}$ of the inclusion of $\mathcal{S}h^{lc}(X;\mathrm{LMod}_R)$ into $\mathcal{S}h(X;\mathrm{LMod}_R)$ induces a right split Verdier sequence

$$\mathcal{S}h^{\perp lc}(X;R) \hookrightarrow \mathcal{S}h(X;R) \xrightarrow{L_{lc}} \mathcal{S}h^{lc}(X;R)$$

and the inclusion itself induces a split Verdier sequence

$$\mathcal{S}h^{lc}(X;R) \hookrightarrow \mathcal{S}h(X;R) \xrightarrow{L_{lc}} \mathcal{S}h^{lc}(X;R)^{\perp} \, .$$

*Proof.* As for the PL analogue 4.3.11, this is just an application of 3.2.12 since we know about all required adjoints. □

**Theorem 5.5.2.** The Verdier sequence above reduces to a Poincaré-Verdier sequence

$$\mathcal{S}h^{\perp lc}_{fp}(X;R) \hookrightarrow \mathcal{S}h_{fp}(X;R) \xrightarrow{L_{lc}} \mathcal{S}h^{lc}(X;R)^{(fp)}$$

where all categories are equipped with the respective restrictions of the Verdier duality functor on $\mathcal{S}h(X;R)$. In particular, we obtain a fiber sequence L-spectra

$$\mathbb{L}^q(\mathcal{S}h^{\perp lc}_{fp}(X;R)) \to \mathbb{L}^q(\mathcal{S}h_{fp}(X;R)) \to \mathbb{L}^q(\mathcal{S}h^{lc}(X;R)^{(fp)}) \ . \tag{5.40}$$

*Proof.* This sequence is Verdier by definition, so by 3.3.2 we only need to show that $\mathcal{S}h^{\perp lc}_{fp}(X;R)$ is closed under duality. But since it is the intersection of $\mathcal{S}h^{\perp lc}(X;R)$ and $\mathcal{S}h_{fp}(X;R)$ which are both closed under duality, since locally constant sheaves are because the dualizing complex of a topological manifold is locally constant, we are finished. □

*Remark.* The perfect case is less interesting because 5.4 makes calculations difficult, and even if they were possible we would obtain projective L-groups instead of the L-groups of finitely presented modules we are usually interested in.

Let us define the *visible quadratic L-groups*

$$L_n^{vq}(X;R) := L_n(\pi_0 R[\pi_1 X]) \cong L_n(\mathcal{S}h_{fp}^{lc}(X;R), \Upsilon_X^q) \tag{5.41}$$

of $X$, and $L_n^{vs}(X;R) := L_n(\mathcal{S}h_{fp}^{lc}(X;R), \Upsilon_X^s)$ the *visible symmetric L-groups.* We suspect that the map $\mathbb{L}(\mathcal{S}h_{fp}(X;R), \Upsilon_X^q) \to \mathbb{L}(\mathcal{S}h_{fp}^{lc}(X;R), \Upsilon_X^q)$ induced by $L_{lc}$ is generally not an assembly map, see 6.6.

# 6 L-groups of Stratified Spaces

In the PL setting all of our results generalized without too many problems to the stratified case, and things work out similarly well in the topological world. After developing some of the technical tools, in particular stratified homotopy theory following [DW21] and [Hai18] and the exodromy correspondence following [PT22], we restrict Verdier duality to constructible sheaves making use of the main result in [Vol22] allowing us to define L-groups of constructible sheaves. After showing how these are actually calculable in some examples, we conclude this work by comparing our constructions in the different settings we have considered.

## 6.1 Notions of Stratifications

**Definition 6.1.1.** Let $(P, \leq)$ be a poset. We can equip it with the *Alexandrov topology*, where

- Open subsets are precisely the upwards closed subsets
- Closed subsets are precisely the downwards closed subsets
- Locally closed subsets are precisely the intervals

In particular for $p \in P$, the set $P_{\geq p} = \{q \in P | q \geq p\}$ is open, $P_{\leq p}$ is closed and $\{p\}$ is locally closed.

**Definition 6.1.2.** A *P-stratified space*, usually called filtered space, is a topological space $X$ equipped with a continuous map $f : X \to P$, where $P$ carries the Alexandrov topology. The locally closed subspaces $X_p = f^{-1}(p)$ are called *strata* of $X$, and the closed subspaces $X_{\leq p} = f^{-1}(P_{\leq p})$ are called *closed strata*.

**Example 6.1.3.** An $(\mathbb{N}, \leq)$-stratified space is a topological space $X$, together with a filtration $\bigcup_{i \in \mathbb{N}} X_i = X$ by closed subspaces $X_i$ with $X_i \subseteq X_j$ for $i \leq j$.

**Definition 6.1.4.** A map of stratified spaces $g : (X \to P) \to (Y \to Q)$ consists of a continuous map $X \to Y$ and an order-preserving map $P \to Q$ (equivalently, continuous with respect to the Alexandrov-topology) such that the following square commutes:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ P & \longrightarrow & Q \end{array}$$

We obtain a category $\mathrm{Top}_{/\,\mathrm{Alex}}$, and we will call the isomorphisms *stratified homeomorphisms.* Also, restricting to a fixed poset $P$ and continuous morphisms that cover the identity map on $P$, we obtain a category $\mathrm{Top}_{/P}$.

**Definition 6.1.5.** An *open embedding* $f : (X \to P) \hookrightarrow (Y \to Q)$ of stratified spaces is a map of stratified spaces that induces an open embedding $f : X \hookrightarrow Y$ of topological spaces, as well as open embeddings $f_p : X_p \hookrightarrow Y_{f(p)}$ for each $p \in P$.

**Definition 6.1.6.** For $f : X \to P$ a stratified space, define its *open cone*

$$C(X) := \frac{X \times [0, \infty)}{X \times \{0\}} \tag{6.1}$$

and equip it with its natural stratification by $P^{\triangleleft} := P \cup \{-\infty\}$ that sends $[(x,t)] \mapsto f(x)$ for $t > 0$, and the collapsed cone point to $-\infty$.

**Definition 6.1.7.** A stratified space $f : X \to P$ is called *conically stratified* if for any $p \in P$ and any point $x \in X_p$, there exists a neighborhood $x \in U$ with $f(U) = P_{\geq p}$ such that the space $U$ with its restricted stratification $U \to P_{\geq p}$ is stratified homeomorphic to a space of the form $Y \times C(L)$. Here, $Y$ should be a (trivially stratified) topological space and $L$ a $P_{>p}$-stratified space so that we can identity $P_{\geq p} \cong P^{\triangleleft}_{>p}$.

Being conically stratified implies many useful statements about the (stratified) homotopy type of a space, as we will see later. To capture this definition in a few words, it says that our space should locally look like a cone. There is a similar, even more refined notion we will often use, that mirrors the definition of a topological manifold:

**Definition 6.1.8.** An *n-basic* is inductively defined to be a stratified space of the form $\mathbb{R}^i \times C(L)$, where $i \geq 0$ and its *link* $Z$ is a compact topologically stratified space of dimension $(n - i - 1)$, inductively defined below. To start this induction, the only $(-1)$-dimensional topologically stratified space is $\emptyset \to \emptyset$, and there are no basics of negative dimension.

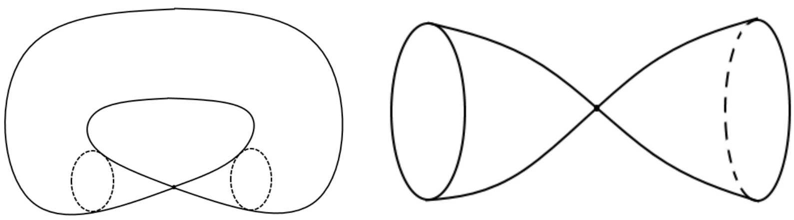
**Definition 6.1.9.** A *topologically stratified space* of dimension $n$, also called $C^0$-stratified space, is a paracompact Hausdorff space that is locally stratified homeomorphic an $n$-basic, in the sense of 6.1.7. We denote the category of them, together with stratified maps, as $\mathrm{Strat}_n^{C^0}$; and if we take only stratified open embeddings of as morphisms, as $\mathrm{Sngl}_n^{C^0}$. Finally, we denote the category of $n$-basics with stratified open embeddings as morphisms by $\mathrm{Bsc}_n^{C^0}$.

*Technical Remark* ([AFT14, 6.2.2]). Since every space in $\mathrm{Sngl}_n^{C^0}$ can be glued from $n$-basics, we can obtain an embedding $\mathrm{Sngl}_n^{C^0} \hookrightarrow \mathcal{PSh}(\mathrm{Bsc}_n^{C^0})$. In fact, open coverings constitute a Grothendieck pretopology on $\mathrm{Bsc}_n^{C^0}$, and the functors of points of topologically stratified spaces are sheaves over it (but not every sheaf is of this form).

**Example 6.1.10.** It follows that the only 0-basic is $C(\emptyset \to \emptyset) = * \to *$, and the only 1-basics are $\mathbb{R}$, $C(* \to *) = (\mathbb{R}_{\geq 0} \to \{0 < 1\})$ and generally $C(\{1, \ldots, k\} \to *) = \mathbb{R}_{\geq 0} \times_{\{0\}} \cdots \times_{\{0\}} \mathbb{R}_{>=0} \to *$.

**Example 6.1.11.**

- Since forming a cone always adds an element to the stratification poset, topologically stratified spaces $(X \to P)$ with $P = *$ must locally look like $\mathbb{R}^i$, so they are precisely topological manifolds. Similarly, one can see that strata of topologically stratified spaces are always topological manifolds.
- Topologically stratified spaces of dimension 0 are disjoint unions of points (with the trivial stratification), and in dimension 1 we obtain undirected graphs stratified over $\{0 < 1\}$ by sending vertices to 0 and edges to 1.
- Let $N \subseteq M$ be an embedded submanifold, and let us stratify $M$ by $\{0 < 1\}$ by sending $N$ to 0 and $M \backslash N$ to 1. This is a topological stratification; an important special case are knots $S^1 \hookrightarrow \mathbb{R}^3$.
- Irreducible complex varieties of pure dimension, with their analytic topology, have a natural topological stratification with only even-dimensional strata.
- The *pinched torus* $S^1 \times S^1 / \{0\} \times S^1$ and the *double cone* $S^1 \times \mathbb{R} / S^1 \times \{0\}$ are topologically stratified of dimension 2; both consist of a singular stratum (the quotient point, with link $S^1 \times S^1$ in both cases) of dimension 0 and a regular stratum.



- The suspension $ST^2 = \frac{[0,1] \times T^2}{\{0,1\} \times T^2}$ of the torus is a topologically stratified space of dimension 3 with two singular points.
- The topological $n$-simplex $|\Delta^n| = \{(x_0, \ldots, x_n) \in [0,1]^{n+1} | x_0 + \cdots + x_n = 1\}$ possesses a natural $\{0 < \cdots < n\}$-stratification, sending $(x_0, \ldots, x_n)$ to the maximal $i$ with $x_i \neq 0$. For example, $|\Delta^1|$ consists of the 0-stratum $\{(1, 0, 0)\}$ and the

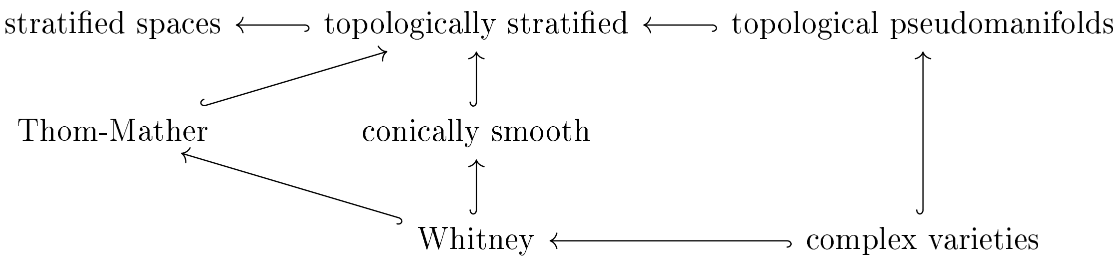
1-stratum given by the remaining half-open line. Alternatively, $|\Delta^n|$ can also be stratified differently by considering it as a manifold with corners.

- Stratifications that are not topological include for example most CW-complexes (stratified by their skeleta); and $\{\frac{1}{n} \,|\, n \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}$ stratified by sending everything to $1 \in \{0 < 1\}$, except for $0 \mapsto 0$.

**Example 6.1.12.** Topological $n$-Manifolds with corners, i.e. spaces that are locally homeomorphic to $\mathbb{R}^{n-i} \times \mathbb{R}^i_{\geq 0}$ for $0 \leq i \leq n$, are topologically stratified over $\{0 < \cdots < n\}$ if we send every corner to its dimension $i$ (i.e. the interior to $n$, the boundary to $n-1$, and so on). This follows from the fact that topologically stratified spaces are closed under forming products.

**Definition 6.1.13.** A topologically stratified space is called a *topological pseudomanifold* if its top-dimensional stratum is dense, and there is no stratum of codimension 1. This allows, for example, the introduction of an orientation class.

Let us capture the most important flavors of stratified spaces in a diagram, where arrows denote an extension in generality. Note that the variants of smooth stratified spaces are always equipped with extra data like an atlas, so the respective arrows are not fully faithful. Conically smooth spaces are slightly non-standard; they were introduced in [AFT14] as a stratified generalization of smooth manifolds that adapts very well to the stratified homotopy theory we develop in the next section.




## 6.2 Exit-Paths

The (weak) homotopy type of a topological space $X$ is described by its singular simplicial set, or fundamental $\infty$-groupoid, $\mathrm{Sing}(X)$ that we defined in 1.1.12. In fact, good topological spaces and $\infty$-groupoids are more or less the same thing according to the homotopy hypothesis.

We want to find a similar simplicial model for the *stratified homotopy type* of a stratified space $(X \to P)$. Since a stratification equips $X$ with a sense of ordering, or direction, we would expect that this model has non-invertible edges, ie. it should not be a Kan complex. In fact, there is a correspondence (akin to the homotopy hypothesis) between

∞-categories and so-called *directed spaces*, which we could regard stratified spaces as a special case of. We will however take a different approach.

Remember that vertices of $\mathrm{Sing}(X)$ are points of $X$, edges are paths, 2-simplices are homotopies and so on. What we would expect for stratified spaces is that vertices of their model $\mathrm{Sing}^P(X)$ should still be points of $X$, but edges should be paths that "move in the direction of the stratification". Let us formalize this:

**Definition 6.2.1.** We introduce a functor $r_{\text{strat}} : \Delta \to \mathrm{Top}_{/\,\mathrm{Alex}}$ that sends $[n]$ to $(|\Delta^n| \to [n])$ with the natural stratification of 6.1.11. Using the fact that $\mathrm{Top}_{/\,\mathrm{Alex}}$ has all colimits, and the nerve-realization paradigm 1.1.7, we obtain an adjunction

$$\mathrm{Top}_{/\,\mathrm{Alex}} \underset{\mathrm{Sing}^{\mathrm{strat}}}{\overset{|-|_{\mathrm{strat}}}{\rightleftarrows}} \mathrm{sSet}$$

We call the simplicial set $\mathrm{Sing}^{\mathrm{strat}}(X)$ with $n$-vertices determined by

$$\mathrm{Sing}^{\mathrm{strat}}(X)_n := \mathrm{Hom}_{\mathrm{Top}_{/\,\mathrm{Alex}}}((|\Delta^n| \to [n]), (X \to P)) \tag{6.2}$$

the *exit-path category* of $X$.

To be more explicit,

- Vertices of $\mathrm{Sing}^{\mathrm{strat}}(X)$ are points in $X$,
- For vertices $x, y \in X$, edges between them in $\mathrm{Sing}^{\mathrm{strat}}(X)$ are paths $|\Delta^1| \to X$ that cover an order-preserving map $[1] \to P$, i.e. *exit-paths* in $X$ that start in a lower stratum and immediately exit into a higher stratum in which they stay,
- 2-simplices are homotopies between exit-paths that, following the stratification of $\Delta^2$, increase in the stratification of $X$,
- Higher simplices are higher homotopies compatible with the stratifications.

In particular, for $p \leq p' \leq p''$ in $P$ and exit-paths $\gamma : x \to y$ starting in $X_p$ and exiting into $X_{p'}$, $\gamma' : y \to z$ starting in $X_{p'}$ and exiting into $X_{p''}$, and $\gamma'' : x \to z$ starting in $X_p$ and exiting into $X_{p''}$, a 2-simplex starting at $\gamma$ and $\gamma'$ and ending at $\gamma''$ is a homotopy between the concatenation $\gamma' * \gamma$ and $\gamma''$ that, apart from beginning and end, completely lies in $X_{p''}$.

*Warning.* Be aware that $\mathrm{Sing}^{\mathrm{strat}}(X)$ generally does *not* have to be an ∞-category, despite the name. The reason is that paths $\gamma'$ and $\gamma$ as above don't necessarily need to have a composite, i.e. a third path $\gamma''$ equipped with a 2-simplex as above. The condition that the homotopy needs to lie in $X_{p''}$ may be to strong.

This construction resolves half of our problem – we can use $\mathrm{Sing}^{\mathrm{strat}}(X)$ as a simplicial model for $X \to P$. What special properties does this simplicial set possess?

**Construction 6.2.2** ([DW21, 2.9])**.** For $P$ a poset, regard it as a thin category and denote by $N(P) \in \mathrm{sSet}$ its nerve, which is an $\infty$-category with no non-trivial isomorphisms. There is a canonical continuous map from the geometric realization $\varphi_P : |N(P)| \to P$: For every non-degenerate simplex of $N(P)$ corresponding to a strictly order-preserving morphism $[n] \to P$, ie. a chain $(p_0 < \cdots < p_n) \subseteq P$, we map the associated simplex $\{(x_0, \ldots, x_n) \in [0,1]^n | \sum x_i = 1\}$ to $P$ via

$$\varphi_P(x_0, \ldots, x_n) := \max\{i \in \{0, \ldots n\} \,|\, t_i \neq 0\}\} \,. \tag{6.3}$$

In particular, $|N(P)|$ is naturally stratified over $P$ and for $P = [n]$, this agrees with the stratification in 6.1.11.

*Remark.* It is a nice exercise to show that this is a well-defined continuous map, understand the stratification in more examples, and to describe the adjoint map $N(P) \to \mathrm{Sing}(P)$.

**Observation 6.2.3.** Postcomposing with, and pulling back along the map $\varphi_P$ induces an adjunction between slice categories:

$$\mathrm{Top}_{/P} \underset{-\times_P |N(P)|}{\overset{\varphi_P \circ -}{\leftrightarrows}} \mathrm{Top}_{/|N(P)|}$$

**Definition 6.2.4.** Given a simplicial set $(K \to N(P)) \in \mathrm{sSet}_{/P}$ equipped with a map to the nerve of $P$, we can form the geometric realization $(|K| \to |N(P)|) \in \mathrm{Top}_{/|N(P)|}$. Together with $\phi_P \circ -$ this yields a composition

$$\mathrm{sSet}_{/N(P)} \underset{\mathrm{Sing}^P}{\overset{|-|}{\rightleftarrows}} \mathrm{Top}_{/|N(P)|} \underset{-\times_P |N(P)|}{\overset{\varphi_P \circ -}{\rightleftarrows}} \mathrm{Top}_{/P}$$

where both functors admit right adjoints. The right adjoint $\mathrm{Sing}^P$ of the geometric realization (which we identify with the composition of both right adjoints) is constructed as the pullback

$$\begin{array}{ccc} \mathrm{Sing}^P(X) & \longrightarrow & \mathrm{Sing}(X) \\ \downarrow & & \downarrow \\ N(P) & \longrightarrow & \mathrm{Sing}(P) \,. \end{array}$$

*Remark.* Both of these statements follow from standard theorems on the interaction of adjoints and slice categories, and pasting in the latter case.

**Proposition 6.2.5.** For $(f : X \to P) \in \mathrm{Top}_{/P}$, the constructions $\mathrm{Sing}^{\mathrm{strat}}(X \to P) \cong \mathrm{Sing}^P(X)$ are naturally isomorphic. Similarly, for $(K \to P) \in \mathrm{sSet}_{/P}$, the underlying topological spaces of $|K|_{\mathrm{strat}}$ and $|K \to P|_P$ agree.

*Proof.* While this can be deduced from abstract nonsense, for the first case this is clear by construction of $\mathrm{Sing}^P$: The pullback in $\mathrm{sSet} = \mathrm{Fun}(\Delta^{op}, \mathrm{Set})$ is computed pointwise, so $\mathrm{Sing}^P(X)$ consists of precisely those simplices $\sigma$ of $\mathrm{Sing}(X)$ that lie over a simplex of $N(P)$, meaning that they can only go in the direction the edges in $N(P)$ point towards, i.e. upwards in the stratification.

For the geometric realizations, note that both of them as well as the slice projections $\mathrm{sSet}_{/P} \to \mathrm{sSet}$, $\mathrm{Top}_{/P} \to \mathrm{Top}$ preserve colimits, so it is enough to show this on $\Delta^n \to P$. But the underlying space of both realizations by definition is just $|\Delta^n|$ in this case. □

*Remark.* We have thus learned that the exit-path category $\mathrm{Sing}^{\mathrm{strat}}(X) \simeq \mathrm{Sing}^P(X)$ is equipped with a canonical map to $P$, and how to calculate the stratified realization.

**Theorem 6.2.6** ([Lur17, A.6.4])**.** If $(X \to P)$ is a conically stratified space, the exit path category $\mathrm{Sing}^P(X) \in \mathrm{sSet}$ is a quasicategory.

**Definition 6.2.7.** A functor $F : \mathcal{C} \to \mathcal{D}$ between $\infty$-categories is called *conservative* if it reflects isomorphisms. This means that if $f$ is a morphism in $\mathcal{C}$ such that $F(f)$ is an isomorphism in $\mathcal{D}$, then $f$ is an isomorphism.

**Definition 6.2.8.** For $P$ a poset, the $\infty$-category $\mathcal{S}_P$ of *abstract stratified homotopy types* over $P$ is the full subcategory of the slice category $\mathcal{C}at_{\infty/N(P)}$ on conservative functors.

**Proposition 6.2.9.** For $(X \to P) \in \mathrm{Top}_{/P}$, the natural map $\mathrm{Sing}^P(X) \to N(P)$ is conservative. In particular, if $(X \to P)$ is conical, $\mathrm{Sing}^P(X) \in \mathcal{S}_P$.

*Proof.* Since $P$ is a poset, the only isomorphisms in $N(P)$ are the identities. Therefore, all we need to check is that for each $p \in P$, the fiber $\mathrm{Sing}^P(X) \times_{N(P)} \{p\}$ is an $\infty$-groupoid. From the definition of $\mathrm{Sing}^P(X)$, we see that morphisms in this fiber are paths in $X$ that stay entirely in $X_p$, without any conditions from the stratification. This means that they are invertible, as we may trace the path in the opposite direction. □

*Remark.* In fact, the same argument applied to higher simplices in $\mathrm{Sing}^P(X)$ shows that $\mathrm{Sing}^P(X) \times_{N(P)} \{p\} \simeq \mathrm{Sing}(X_p)$, see also [Lur17, A.7.5].

**Proposition 6.2.10** ([Hai18, 1.1.9])**.** A morphism $f : K \to L$ in $\mathcal{S}_P$ is an isomorphism iff

- it induces homotopy equivalences of strata $K_p \simeq L_p$ for every $p \in P$, and

- it induces homotopy equivalences between links for all $p < q$ in $P$:

$$\mathrm{Map}_{\mathcal{S}_P}(\{p < q\}, K) \simeq \mathrm{Map}_{\mathcal{S}_P}(\{p < q\}, L) \tag{6.4}$$

After this technical discussion, let us develop some examples. Recall that we always assume CW complexes are locally finite.

**Definition 6.2.11.** A CW complex $X$ is called *regular* iff the inclusions $\phi : D^n \to X$ of $n$-cells into $X$ are homeomorphisms onto their image. For arbitrary CW complexes, this is only true in the interior of $D^n$, and being regular means that this gluing has to be "non-degenerate" along the boundary $\phi_\partial : S^{n-1} \to \mathrm{sk}_{n-1}(X)$ as well.

**Proposition 6.2.12.** If $X$ is a regular CW complex and we denote by $\mathcal{I}_X$ the set of cells in $X$, then

- $\mathcal{I}_X$ carries a natural partial order,
- There is a canonical stratification $X \to \mathcal{I}_X$ sending each point to the unique cell that contains it in its interior (unless the point is a 0-cell itself, in which case it is sent to this 0-cell),
- This stratification is conical (here, we need $X$ to be locally finite),
- The exit-path category $\mathrm{Sing}^{\mathcal{I}_X}(X) \to \mathcal{I}_X$ is equivalent to the identity map, i.e. $\mathrm{Sing}^{\mathcal{I}_X}(X) \simeq \mathcal{I}_X$ as $\infty$-categories.

*Proof.* First, note that a regular CW complex $X$ is in particular normal. This means the set of cells $\mathcal{I}_X$ carries a partial order where $e_1 \leq e_2$ iff, equivalently,

- $e_1$ is contained in the closure $\overline{e_2}$,
- $e_1 \cap \overline{e_2} \neq \emptyset$,

by [TT18, 3.1]. This yields a conical stratification on $X$ by [TT18, 1.7] and a remark in [Lej21, Section 4.2]. To show that the map $\mathrm{Sing}^{\mathcal{I}_X}(X) \to \mathcal{I}_X$ is an equivalence, we proceed by showing it is essentially surjective and fully faithful.

*Essentially surjective:* In the proof of 6.2.9, we saw that the fiber of this map over a cell is just the singular simplicial set of the open cell itself (or, in dimension 0, a point), in particular contractible and non-empty.

*Fully faithful:* Let $e_1$ and $e_2$ be cells in $X$, and $x \in e_1$, $y \in e_2$. If $e_1 \not\leq e_2$, the mapping space $\mathrm{Map}_{\mathrm{Sing}^S(X)}(x, y)$ is also empty since there can't be a path $\gamma : [0,1] \to X$ from $x$ to $y$ that lies over the arrow $e_1 \to e_2$ in $S$, as it would have to somehow jump from $e_1$ to $e_2$ even though $e_1 \cap \overline{e_2} = \emptyset$, violating continuity.

If $e_1 \leq e_2$, so $e_1$ lies in the boundary of $e_2$, we need to show that $\mathrm{Map}_{\mathrm{Sing}^{\mathcal{I}_X}(X)}(x, y)$ is contractible. As in the proof of [Lur17, A.6.10], we can identify this with $\mathrm{Sing}(P_{x,y})$ with $P_{x,y}$ the space of paths $\gamma : [0,1] \to X$ from $x$ to $y$ such that $\gamma((0,1]) \subseteq e_2$. This only works because we know the stratification is conical so $\mathrm{Sing}^{\mathcal{I}_X}(X)$ is an $\infty$-category. However $\gamma([0,1]) \subseteq \overline{e_2}$, the image of the gluing map $D^n \to X$ of $e_2$, which by regularity is a homeomorphism onto its image.

Thus, we can identify $P_{x,y}$ with the space of maps $\gamma : [0,1] \to \mathbb{R}^n$ such that $\gamma(0) = y'$ for some fixed $y'$ with $|y'| = 1$ that corresponds to $y$, $\gamma(1) = x'$, and $|\gamma(t)| < 1$ for all $0 < t \leq 1$. This can clearly be contracted to the linear path, since the open unit ball is convex. □

**Corollary 6.2.13.** If $K$ is a simplicial complex and we stratify its geometric realization $|K|$ by the poset $\mathcal{I}_K$ of simplices, the exit-path category $\mathrm{Sing}^{\mathcal{I}_K}(|K|) \to \mathcal{I}_K$ is equivalent to the identity $\mathcal{I}_K \to \mathcal{I}_K$.

*Proof.* By definition of a simplicial complex and its geometric realization, $|K|$ is a regular CW-complex with poset of cells $\mathcal{I}_K$. □

**Example 6.2.14.**

- For a trivially stratified space $X \to \Delta^0$, the exit-path category agrees with the homotopy type $\mathrm{Sing}^{\Delta^0}(X) = \mathrm{Sing}(X)$.
- Using the same argument as in 6.2.12, one shows $\mathrm{Sing}^{[1]}(\mathbb{R}_{\geq 0}) \simeq \Delta^1$.
- As a right adjoint, $\mathrm{Sing}^{\mathrm{strat}}$ commutes with products.
- By [AFR15, 3.3.12], if $r : (X \to P) \to (X \to Q)$ is a *refinement*, i.e. a map of stratified spaces determined by the identity on $X$ and an order-preserving surjection $P \to Q$, then the induced functor $\mathrm{Sing}^P(X) \to \mathrm{Sing}^Q(X)$ is a localization (loc. cit. only works in the conically smooth case, but this should hold more generally).
- For $D^n \to [1]$ stratified as a manifold with boundary, choose the triangulation $\Delta^n$. Then,
$$\mathrm{Sing}^{[1]} D^n \simeq (\mathrm{Sing}^{[n]} \Delta^n)[W^{-1}] \simeq \mathcal{P}(\{1, \dots, n\})[W^{-1}] \tag{6.5}$$
where $W$ is the class of face inclusions in $\Delta^n$ that do not involve the interior, and $\mathcal{P}$ denotes the power set ordered by inclusion.
- By [AFT14, 6.1.4], $\mathrm{Sing}^P(C(X)) \simeq \mathrm{Sing}^P(X)^{\triangleleft}$ in the conically smooth case. In particular, the exit-path category of a basic is
$$\mathrm{Sing}^{P^{\triangleleft}}(\mathbb{R}^i \times C(L)) \simeq \mathrm{Sing}^{\Delta^0} \mathbb{R}^i \times \mathrm{Sing}^{P^{\triangleleft}}(C(L)) \simeq (\mathrm{Sing}^P(L))^{\triangleleft} . \tag{6.6}$$
- By [Vol22], the exit-path category of a compact conically smooth stratified space is equivalent to a finite $\infty$-category.

## 6.3 Constructible Sheaves

Recall that on a topological space $X$, we have defined special classes of sheaves with values in a presentable $\infty$-category $\mathcal{V}$ that locally do not change. If $\Gamma^* : \mathcal{V} \to \mathcal{S}h(X, \mathcal{V})$ denotes the left adjoint of the global sections functor, sheaves of the form $\Gamma^*(V)$ for any $V \in \mathcal{V}$ were called constant. Also, if there is an open cover $(U_i)_{i \in I}$ of $X$ such that $F|_{U_i}$ is constant for every $i$, we called $F$ locally constant.

**Definition 6.3.1.** For $X \to P$ a stratified space, we call a sheaf $F \in \mathcal{S}h(X, \mathcal{V})$ *constructible* if for each $p \in P$, the restriction $F|_{X_p}$ to the respective pure stratum is locally constant. The full subcategory on constructible sheaves will be denoted by $\mathcal{S}h^{cbl}(X; \mathcal{V})$ when the stratification is clear from the context.

Similarly, we define *constructible hypersheaves* as hypersheaves $F \in \mathcal{S}h^{hyp}(X; \mathcal{V})$ whose restrictions to strata are locally constant hypersheaves after hypercompleting them. In other words, for each $p \in P$ there must exist an open cover $(U_i^{(p)})_{i \in I_p}$ of $X_p$ such that all

$$\left( (F|_{X_p})^{hyp}|_{U_i^{(p)}} \right)^{hyp} = \left( F|_{U_i^{(p)}} \right)^{hyp} \tag{6.7}$$

can be written as $(\Gamma^* V_i^{(p)})^{hyp}$ for some $V_i^{(p)} \in \mathcal{V}$.

*Warning.* As in the locally constant case, be aware that being a constructible hypersheaf is not equivalent to being constructible and hypercomplete.

**Definition 6.3.2.** A partially ordered set $P$ is called *noetherian* if it satisfies the ascending chain condition, i.e. there exists no infinite chain of elements $p_0 < p_1 < p_2 < \dots$ in $P$. Equivalently, any subset of $P$ has a maximum.

There are some useful criteria to check whether a given sheaf is locally constant or constructible. In the following, assume that either

- $\mathcal{V}$ is presentable stable or the tensor product of a compactly generated $\infty$-category and an $\infty$-topos, and $P$ is noetherian, or
- $\mathcal{V}$ is compactly generated.

We will call this the *joint conservativity assumption.*

**Theorem 6.3.3** ([PT22] 5.22)**.** Let $X \to P$ be a conically stratified space and $\mathcal{V}$ satisfy the joint conservativity assumption. Then, for a sheaf $F \in \mathcal{S}h(X; \mathcal{V})$, the following are equivalent:

- $F$ is a constructible hypersheaf

- For all open subsets $U \subseteq V \subseteq X$ such that the induced map $\mathrm{Sing}^P(U) \to \mathrm{Sing}^P(V)$ is an equivalence, the restriction $F(V) \to F(U)$ is also an equivalence
- For each conical neighborhood $Z \times C(Y)$ in $X$, any open subsets $U' \subseteq V' \subseteq Z$ such that $U, V$ are weakly contractible, and all $0 < \epsilon < \epsilon'$, application ot $F$ to the inclusions

$$U \times C(Y) \subseteq V \times C(Y)$$
$$Z \times C_{<\epsilon}(Y) \subseteq Z \times C_{<\epsilon'}(Y)$$

yields isomorphisms. Here, $C_{<\epsilon}(Y)$ denotes the open subset of the cone where the real parameter is $< \epsilon$.

*Remark.* If $X$ is topologically stratified, every sheaf is hypercomplete by 1.4.9, so this becomes a characterization of constructible sheaves.

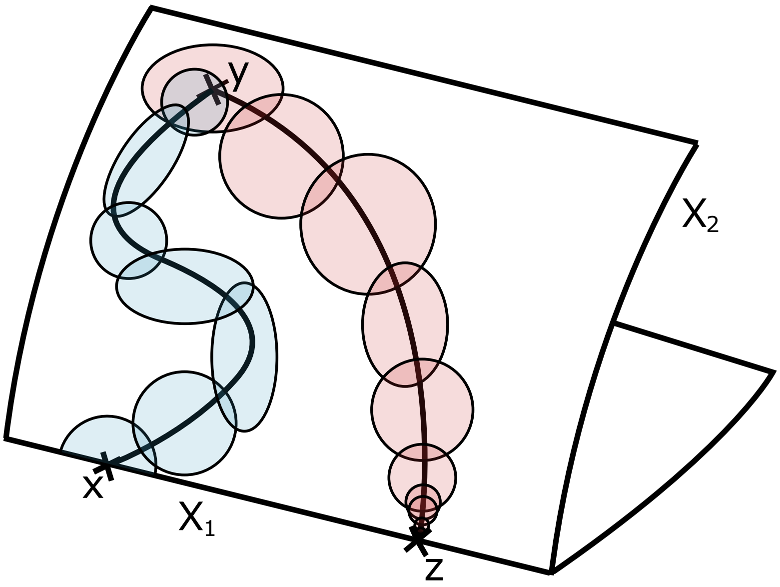
*Remark.* In the cases of conically smooth stratified spaces and topological manifolds, one can give a more refined characterization. If $\mathcal{V}$ is a presentable $\infty$-category and $F \in \mathcal{S}h(M; \mathcal{V})$ on a topological manifold $M$, then $F$ is locally constant iff it sends inclusions of charts into each other to isomorphisms. This follows from the monodromy correspondence 5.3.7 combined with the fact [Lur17, 5.4.5.2] that $\mathrm{Sing}(M) \simeq \{\mathbb{R}^n\} \times_{\mathrm{Mfd}_n} (\mathrm{Mfd_n})_{/M}$, with $\mathrm{Mfd}_n$ the $\infty$-category of topological $n$-manifolds with morphism spaces given by the spaces of open embeddings $\mathrm{Emb}(M, N)$ equipped with the compact-open topology. We only need $\mathcal{V}$ to be presentable in this case, since the criterion on $\mathcal{V}$ in [PT22, 5.17] is satisfied because $M$ only has a single stratum.

Our next goal is to generalize the monodromy correspondence 5.3.7 to constructible sheaves on stratified spaces. Since the abstract homotopy type of a stratified space is described by its exit-path category, which is (on conically stratified spaces) an $\infty$-category and not a Kan complex like $\mathrm{Sing}(X)$, we expect that there is some directionality involved in the notion of parallel transport that classifies a constructible sheaf. In the classical setting, remember how the monodromy correspondence between locally constant sheaves and representations of the fundamental groupoid was proven by using the local constancy to transport germs along paths inside small open subsets.

Now, suppose we are given an exit path $\gamma : [0, 1] \to X$ such that $x := \gamma(0) \in X_1$ and $\gamma((0, 1]) \subseteq X_2$, as well as a constructible 1-sheaf $F \in \mathrm{Sh}^{cbl}(X)$ and a germ $s \in x^*\mathcal{F}$. By definition of the stalk, there is a small open neighborhood $U_0$ around $y$ such that $s$ stems from a section of $F(U_0)$, meaning that we can parallel transport $s$ from $y$ to any point in this neighborhood, in particular to some $\gamma(\epsilon)$ with $\epsilon > 0$. From here on, we can work with $F|_{X_2}$ which is locally constant and parallel transport further until we reach $\gamma(1) =: y$, as indicated by the blue open sets in the picture below.

If our path however starts at $y$ and ends in the lower stratum $X_1$ at $z$, we might run into a problem as shown by the red sets. Since we can only parallel transport a germ inside of open sets where the respective sheaf is constant, we might never reach $X_1$ as there

Figure 6.1: Parallel transport is only possible from lower to higher strata




does not have to be an open neighborhood around $z$ where $F|_{X_{\leq 2}}$ is constant. We realize that constructible sheaves can only be transported along exit paths, an idea leading us to the exodromy correspondence:

**Theorem 6.3.4** (Topological Exodromy, [Lur17, A.9.3] and [PT22])**.**
Let $(X \to P)$ be a conically stratified such that any stratum $X_p$ is locally weakly contractible, and $\mathcal{V}$ satisfy the joint conservativity condition, then constructible hypersheaves are specified by their exodromy representation:

$$\mathcal{S}h^{hyp,cbl}(X;\mathcal{V}) \simeq \mathrm{Fun}(\mathrm{Sing}^P(X),\mathcal{V}) \ . \tag{6.8}$$

The functor $\mathrm{Sing}^P(X) \to \mathcal{V}$ associated to a sheaf $F$ sends $x \in X$ to the stalk $x^*F$. In case all sheaves on $X$ are hypercomplete (e.g. $X$ is paracompact Hausdorff of finite covering dimension), the *hyp* can of course be dropped.

We may instead require that $(X \to P)$ is a paracompact Hausdorff conically stratified space that is locally of singular shape, $P$ satisfies the ascending chain condition and $\mathcal{V}$ the joint conservativity condition, then we can characterize constructible sheaves by

$$\mathcal{S}h^{cbl}(X;\mathcal{V}) \simeq \mathrm{Fun}(\mathrm{Sing}^P(X),\mathcal{V}) \ . \tag{6.9}$$

*Remark.* The term *exodromy* stems from applications of this concept to study étale sheaves in algebraic geometry, see [BGH18]. However, the original (topological) statement for ordinary constructible sheaves is due to unpublished work by MacPherson.

*Technical Remark.* We may rewrite this in analogy with 5.31 as

$$\mathcal{S}h^{cbl}(X;\mathcal{V}) \simeq \mathrm{Fun}(\underset{\mathrm{Sing}^P(X)}{\mathrm{laxcolim}}\Delta^0,\mathcal{V}) \simeq \underset{\mathrm{Sing}^P(X)}{\mathrm{laxlim}}\,\mathcal{V}\,. \tag{6.10}$$

**Proposition 6.3.5.** Let $\mathcal{V}$ be a small stable $\infty$-category, and $(X \to P)$ satisfy the conditions for the exodromy correspondence. Then, constructible hypersheaves in $\mathrm{Ind}(\mathcal{V})$ whose stalks all lie in $\mathcal{V}$ are characterized by their exodromy representation in $\mathcal{V}$:

$$\mathcal{S}h^{hyp,cbl}_{\mathcal{V}}(X;\mathrm{Ind}(\mathcal{V})) \simeq \mathrm{Fun}(\mathrm{Sing}^P(X),\mathcal{V}) \tag{6.11}$$

*Proof.* Since the exodromy representation of a sheaf $F$ sends a point $x \in X$ to the stalk $F_x$, we can apply it to $\mathrm{Ind}(\mathcal{V})$ and restrict to the above full subcategories. In particular, $\mathrm{Ind}(\mathcal{V})$ is by definition compactly generated so we do not always need to assume that $P$ is noetherian. □

This allows us to extend several statements that we have used for locally constant sheaves to constructible sheaves. Let $(X \to P)$, $(Y \to P)$ be stratified spaces and $\mathcal{V}$ an $\infty$-category such that the conditions of the exodromy correspondence are satisfied on both of them. Depending on which situation we work with, the hypercompleteness assumptions in the following discussion may be dropped.

**Proposition 6.3.6** ([PT22, 5.20])**.** The full subcategory $\mathcal{S}h^{hyp,cbl}(X;\mathcal{V})$ on constructible hypersheaves in $\mathcal{S}h^{hyp}(X;\mathcal{V})$ is closed under limits and colimits. Using the Adjoint Functor Theorem, we obtain an adjoint triple:

$$\mathcal{S}h^{hyp,cbl}(X;\mathcal{V}) \; \begin{smallmatrix} \xleftarrow{\;L_{cbl}\;} \\ \hookrightarrow \\ \xleftarrow{\;R_{cbl}\;} \end{smallmatrix} \; \mathcal{S}h^{hyp}(X;\mathcal{V})$$

*Proof.* By the exodromy correspondence and our conditions on $\mathcal{V}$, both categories are presentable, so the Adjoint Functor Theorem can indeed be applied. Constructible hypersheaves are closed under colimits since locally constant sheaves are by 5.3.9, and the pullback functors to the strata preserve colimits. The case of limits follows from the construction of the exodromy correspondence: By the given reference, there is a fully faithful functor $\Psi^{hyp}_{X,P} : \mathrm{Fun}(\mathrm{Sing}^P(X),\mathcal{V}) \to \mathcal{S}h^{hyp}(X;\mathcal{V})$ that admits a right adjoint and restricts to the exodromy correspondence. □

**Proposition 6.3.7** ([PT22, 6.13])**.** Let $f : (X \to P) \to (Y \to P)$ be a map of stratified spaces. This induces an adjoint triple

$$\mathcal{S}h^{hyp,cbl}(X;\mathcal{V}) \; \begin{smallmatrix} \xrightarrow{\;f^{cbl}_{+}\;} \\ \xleftarrow{\;f^{*}\;} \\ \xrightarrow{\;f^{cbl}_{*}\;} \end{smallmatrix} \; \mathcal{S}h^{hyp}(X;\mathcal{V})$$

*Proof.* As in the case of locally constant hypersheaves, all we have to show is that precomposing the exodromy correspondence with $f$ agrees with the pullback by $f$ on sheaves. This is difficult and we can not develop the necessary background, see 6.7 and 6.8 in loc. cit. □

**Example 6.3.8.** As in the locally constant case, for $x \in X$ a point and $t : X \to *$ the terminal map, we obtain functors

$$\begin{array}{ccccc}
 & \xrightarrow{\;x^{lc}_{+}\;} & & \xrightarrow{\;C_*\;} & \\
\mathcal{V} & \xleftarrow{\;x^*\;} & \mathcal{S}h^{hyp,cbl}(X;\mathcal{V}) & \xleftarrow{\;\underline{(-)}\;} & \mathcal{V} \\
 & \xrightarrow{\;x^{lc}_{*}\;} & & \xrightarrow{\;C^*\;} &
\end{array}$$

including variations of singular (co)homology replacing local systems by constructible sheaves. In fact, it agrees with sheaf cohomology by [Vol22, 3.16].

## 6.4 Verdier Duality for Constructible Sheaves

The goal of this section is to show that Verdier duality restricts to the full subcategory of constructible (hyper)sheaves in good cases. Assume throughout this section that $(X \to P)$ is a topologically stratified space.

**Proposition 6.4.1** ([Vol22, 4.1])**.** If $(X \to P)$ is a topologically stratified space, then $\omega_X$ is constructible.

**Proposition 6.4.2.** Let $x \in X$ and $F \in \mathcal{S}h(X; R)$ a sheaf, and assume that there exists a basic neighborhood $\mathbb{R}^i \times C(L)$ around $x$ such that the exit-path category $\mathrm{Sing}^P(L)$ is a finite $\infty$-category (i.e. Joyal-equivalent to a simplicial set consisting of finitely many non-degenerate simplices). Then, the stalk $x^*F$ is finitely presented (or perfect) iff the costalk $x^!F$ is.

*Remark.* This condition is satisfied at all points for topological manifolds with corners, (locally finite) regular CW complexes, and for conically smooth stratified spaces by [Vol22, 2.13]; we will refer to such spaces as having *exit-finite links*. In particular, it is true for Whitney stratified spaces and complex varieties.

*Proof.* This argument is due to the proof of [Vol22, 4.2]. Let $U = X - \{x\}$ which is open since $X$ is Hausdorff, and $j : U \hookrightarrow X$, then we obtain a fiber sequence $x_*x^*F \to F \to j_*j^*F$ by 3.2.7, and applying the exact (since right adjoint) global sections functor $\Gamma$ yields another fiber sequence

$$x^!F = \Gamma(X, x_*x^!F) \longrightarrow \Gamma(X, F) \longrightarrow \Gamma(U, j^*F) \tag{6.12}$$

exhibiting $x^! \cong \mathrm{fib}(F(X) \to F(U))$ as the *relative cohomology* of $F$ at $x$.

Now, use the fact that $X$ is topologically stratified to choose a basic $B = \mathbb{R}^i \times C(L)$ around $x$. Since $X = U \cup_{B-\{x\}} B$ and by the sheaf condition, the diagram

$$\begin{array}{ccc} F(X) & \longrightarrow & F(U) \\ \downarrow & & \downarrow \\ F(B) & \longrightarrow & F(B-\{x\}) \end{array}$$

is a pullback square, so the above fiber is isomorphic to $x^! \cong \mathrm{fib}(F(B) \to F(B - \{x\}))$. But just as in the case of manifolds, $x^*F \cong \Gamma(B)$ as indicated in [Vol22, 3.5], so all we have to show is that $F(B - \{x\})$ is finitely presented as it measures the difference between fiber and cofiber. Use the fact from 6.3.8 that global sections of a constructible sheaf can be calculated as a limit over its exodromy representation, then it suffices to show that $\mathrm{Sing}^P(X)$ is finite (replacing by a Joyal-equivalent simplicial set does not change the limit). By the Seifert-van-Kampen theorem for exit-path categories [Lur17, A.7.1], the diagram

$$\begin{array}{ccc} \mathrm{Sing}^P((\mathbb{R}^n - \{0\}) \times \mathbb{R}_{>0} \times L) & \longrightarrow & \mathrm{Sing}^P(\mathbb{R}^n \times \mathbb{R}_{>0} \times L) \\ \downarrow & & \downarrow \\ \mathrm{Sing}^P((\mathbb{R}^n - \{0\}) \times C(L)) & \longrightarrow & \mathrm{Sing}^P(B - \{x\}) \end{array}$$

is a pushout square, so it suffices to show the other involved exit-path categories are finite. But the exit-path category is compatible with cones and products, and $\mathrm{Sing}^P(L)$ is finite by assumption. □

**Theorem 6.4.3** (Verdier duality for constructible sheaves, [Vol22, 4.2])**.** If $X$ is a topologically stratified space with exit-finite links, then the Verdier-duality functor $\mathbb{D}$ restricts to an equivalence

$$\mathbb{D} : \mathcal{S}h^{cbl}_{fp}(X; R)^{op} \simeq \mathcal{S}h^{cbl}_{fp}(X; R) \tag{6.13}$$

on constructible sheaves with finitely presented stalks (and costalks, by the last proposition). We could also have chosen perfect stalks; in fact the argument even works in an arbitrary closed symmetric monoidal stable bicomplete $\infty$-category as indicated in the reference.

*Proof Sketch.* First, we show that $F \mapsto F_c$ restricts to constructible (co)sheaves. If $F$ is locally constant, then by passing to an open cover, we may assume it is actually constant so $F = t^*M$ for some $M \in \mathrm{LMod}_R$ and $t : X \to *$ the terminal morphism. But then $F_c = (-)_c \circ t^*M_c = t^!M \cong t^*M \otimes t^!R = \underline{M} \otimes \omega_X$, using [Vol21, 6.16], is constructible as $\omega_X$ is. The general case for $F$ locally constant follows by passing to basics, we leave it to the reference. By the last proposition, the stalk $x^*F_c = (-)_c \circ x^*F_c = x^!F$ of $F_c$

is finitely presented iff the stalk of $F$ is, so $(-)_c$ in fact induces an equivalence between $\mathcal{S}h^{cbl}_{fp}(X;R)$ and $co\mathcal{S}h^{cbl}_{fp}(X;R)$.

Since Verdier duality is given by the composition $(-)_c \circ D_R$, it suffices to show that $D_R$ induces a similar equivalence between sheaves and cosheaves. But on exodromy representations, $D_R$ acts by postcomposition since $x^*(D_R F) = D_R x^* F$ on stalks. Since $D_R$ itself is an equivalence on perfect, in particular finitely presented modules, we are finished. □

**Definition 6.4.4.** Again, we define a category $\mathcal{S}h^{\perp cbl}(X;R)$ of *constructibly balanced* sheaves as the left orthogonal to the full subcategory $\mathcal{S}h^{cbl}(X;R) \subseteq \mathcal{S}h(X;R)$. Equivalently, it is the kernel of $L_{cbl}$. We also define

$$\mathcal{S}h^{\perp cbl}_{fp}(X;R) := \mathcal{S}h^{\perp cbl}(X;R) \cap \mathcal{S}h_{fp}(X;R) \tag{6.14}$$

as the subcategory on those sheaves with finitely presented stalks.

**Theorem 6.4.5.** Let $(X \to P)$ be a topologically stratified space, then the inclusion of the full subcategory of constructibly balanced sheaves and the left adjoint $L$ from 6.3.6 form a right split Verdier sequence

$$\mathcal{S}h^{\perp cbl}(X;R) \hookrightarrow \mathcal{S}h(X;R) \overset{L_{cbl}}{\to} \mathcal{S}h^{cbl}(X;R)\ , \tag{6.15}$$

adjoint to the split Verdier sequence

$$\mathcal{S}h^{cbl}(X;R) \hookrightarrow \mathcal{S}h(X;R) \to \mathcal{S}h^{\perp cbl}(X;R)\ . \tag{6.16}$$

*Proof.* As always, this follows from 3.2.12 since we know about the involved adjoints. □

**Definition 6.4.6.** Define the stable $\infty$-category of *finitely presented constructible sheaves* as the Verdier quotient

$$\mathcal{S}h^{cbl}(X;R)^{(fp)} := \mathcal{S}h_{fp}(X;R) \big/ \mathcal{S}h^{\perp cbl}_{fp}(X;R)\ . \tag{6.17}$$

As usual, this is the full subcategory of $\mathcal{S}h^{cbl}(X;R)$ spanned by sheaves of the form $L_{cbl}F$ for $F \in \mathcal{S}h_{fp}(X;R)$.

**Theorem 6.4.7.** Let $(X \to P)$ be a topologically stratified space, then there is a Poincaré-Verdier sequence

$$(\mathcal{S}h^{\perp cbl}_{fp}(X;R), Ϙ^q_{\mathbb{D}}) \hookrightarrow (\mathcal{S}h_{fp}(X;R), Ϙ^q_{\mathbb{D}}) \to (\mathcal{S}h^{cbl}(X;R)^{(fp)}, Ϙ^q_{\mathbb{D}})\ . \tag{6.18}$$

and similarly for symmetric L-theory, where all quadratic functors are restricted from Verdier-duality on $\mathcal{S}h(X;R)$.

*Proof.* As in the previous sections, we only need to show that the left subcategory is closed under duality. This however follows from the proof of 6.4.3. □

**Corollary 6.4.8.** The split Poincaré-Verdier sequence above induces a fiber sequence of spectra

$$\mathbb{L}^q(\mathcal{S}h_{fp}^{\perp cbl}(X;R),\mathbb{D}) \to \mathbb{L}^q(\mathcal{S}h_{fp}(X;R),\mathbb{D}) \to \mathbb{L}^q(\mathcal{S}h^{cbl}(X;R)^{(fp)},\mathbb{D}) \tag{6.19}$$

and similarly for symmetric L-theory. We obtain a long exact sequence of L-groups:

$$\begin{aligned}\dots \to L_1^q(\mathcal{S}h_{fp}^{\perp cbl}(X;R),\mathbb{D}) &\longrightarrow L_1^q(\mathcal{S}h_{fp}(X;R),\mathbb{D}) \longrightarrow L_1^q(\mathcal{S}h^{cbl}(X;R)^{(fp)},\mathbb{D}) \\ \hookrightarrow L_0^q(\mathcal{S}h_{fp}^{\perp cbl}(X;R),\mathbb{D}) &\longrightarrow L_0^q(\mathcal{S}h_{fp}(X;R),\mathbb{D}) \longrightarrow L_0^q(\mathcal{S}h^{cbl}(X;R)^{(fp)},\mathbb{D}) \\ \hookrightarrow L_{-1}^q(\mathcal{S}h_{fp}^{\perp cbl}(X;R),\mathbb{D}) &\to L_{-1}^q(\mathcal{S}h_{fp}(X;R),\mathbb{D}) \to L_{-1}^q(\mathcal{S}h^{cbl}(X;R)^{(fp)},\mathbb{D}) \to \dots\end{aligned}$$

*Proof.* Combine 6.4.7 with 3.3.9. □

## 6.5 L-Groups of Constructible Sheaves

**Lemma 6.5.1.** For $(f : X \to P)$ a filtered space, and $(P_-, P_+)$ a slicing of $P$, the inverse images $X_- := f^{-1}(P_-)$ and $X_+ := f^{-1}(P_+)$ are a decomposition of $X$ into a closed and an open subset.

*Proof.* By definition of a slicing, $P_-$ is downward and $P_+$ upward closed and $P_- \cup P_+ = P$. Thus, $X_- \cup X_+ = f^{-1}(P_- \cup P_+) = X$ and by definition of the Alexandrow topology on $P$, the subset $P_-$ is closed and $P_+$ is open, so their inverses under a continuous function are so as well. □

**Theorem 6.5.2.** Let $(X \to P)$ be a topologically stratified space, and $(P_-, P_+)$ a slicing of $P$ with inverse images $X_-, X_+ \subseteq X$. Then, there is a square diagram

$$\begin{array}{ccccc} \mathcal{S}h^{\perp cbl}(X_-;R) & \longrightarrow & \mathcal{S}h(X_-;R) & \longrightarrow & \mathcal{S}h^{cbl}(X_-;R) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{S}h^{\perp cbl}(X;R) & \longrightarrow & \mathcal{S}h(X;R) & \longrightarrow & \mathcal{S}h^{cbl}(X;R) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{S}h^{\perp cbl}(X_+;R) & \longrightarrow & \mathcal{S}h(X_+;R) & \longrightarrow & \mathcal{S}h^{cbl}(X_+;R) \end{array}$$

where all rows are right split and columns are split Verdier sequences. In fact, this holds for any $\mathcal{V}$ satisfying the conditions of 3.1.5.

*Proof.* The middle column is a split Verdier sequence since $\mathrm{LMod}_R$ is presentable stable, see 3.1.5. Also, by 6.4.5, the rows are split Verdier sequences. Hence, we are finished if we can show that the right column is split Verdier, applying the 9-lemma 3.3.8. The factorization condition we need for this Lemma follows as in 4.6.5 by pulling a constructibly balanced sheaf back to $X_-$, since this preserves the property of being constructibly balanced as pushforward along a closed immersion preserves constructibility as we now show.

Write $i : X_- \hookrightarrow X$ and $j : X_+ \hookrightarrow X$ for the respective inclusions; we need to show that $i_*, i^*, i^*, j_*$ preserve constructible sheaves, since their restrictions are in this case still pairwise adjoint, and respectively fully faithful or jointly conservative and left exact (since the full subcategory of constructible sheaves is closed under limits). For the pullback functors $i^*$ and $j^*$ this is clear: With $F \in \mathcal{S}h^{cbl}(X; \mathcal{V})$ and $i_p : X_p \hookrightarrow X$ the inclusions of strata, the restrictions $F|_{X_p} = i_p^* F$ are all locally constant. But then, $(i^*F)|_{X_p} = (i_p|^{X_-})^* i^* F = i_p^* F$ for $p \in P_-$ are locally constant so that $i^*F$ is constructible, and similarly for $j^*F$.

Now, let $G_+ \in \mathcal{S}h^{cbl}(X_+; \mathcal{V}), G_- \in \mathcal{S}h^{cbl}(X_-; \mathcal{V})$ and $p \in P$; we need to show that $i_p^* j_* G_+$ and $i_+^* i_* G_-$ are locally constant for each $p \in P$.

- If $p \in P_+$, the inclusion $i_p = j \circ i_p|^{X_+}$ factors through $j$, so using $j^* i_* = 0$ on sheaves we see that $i_p^* i_* G_-$ is the constant zero sheaf. Also, $i_p^* G_+ = i_p|^* j^* j_* G_+ = i_p|^* G_+$ is locally constant by definition of $G_+$, using $j^* j_* \cong \mathrm{Id}$.
- If $p \in P_-$, factoring $i_p = i \circ i_p|_{X_-}$, the case of $G_-$ follows analogously to the case of $G_+$ above. It remains to show that $i_p|^* i^* j_* G_+$ is locally constant, which is tricky without further preparations.

See [PT22, 6.35, 6.37] for a full and abstract proof. □

*Remark.* As follows from the reference given at the end of the proof, this result holds more generally if $X$ is locally weakly contractible, working with constructible hypersheaves instead.

**Corollary 6.5.3.** The family of decompositions $(\mathcal{S}h^{hyp}(X_-; \mathcal{V}), \mathcal{S}h^{hyp}(X_+; \mathcal{V}))$ of $\mathcal{S}h^{hyp}(X; \mathcal{V})$ for $(P_-, P_+)$ any slicing of $P$ form a (stable) $P$-slicing, which for finite $P$ is equivalent to other decomposition data of this stable $\infty$-category over $P$ by 3.5.8. Similarly for $\mathcal{S}h^{\perp hyp,cbl}(X; \mathcal{V})$ and $\mathcal{S}h^{hyp,cbl}(X; \mathcal{V})$. For $\mathcal{V}$ compactly generated, one can get a stronger statement in the case of infinite $P$ using 3.5.9.

**Theorem 6.5.4.** Let $(X \to P)$ be a topologically stratified space and $(P_-, P_+)$ a slicing of $P$ with inverse images $X_-, X_+ \subseteq X$. Then, one obtains a square diagram

$$\begin{array}{ccccc}
\mathcal{S}h^{\perp cbl}_{fp}(X_-;R) & \longrightarrow & \mathcal{S}h_{fp}(X_-;R) & \longrightarrow & \mathcal{S}h^{cbl}(X_-;R)^{(fp)} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{S}h^{\perp cbl}_{fp}(X;R) & \longrightarrow & \mathcal{S}h_{fp}(X;R) & \longrightarrow & \mathcal{S}h^{cbl}(X;R)^{(fp)} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{S}h^{\perp cbl}_{fp}(X_+;R) & \longrightarrow & \mathcal{S}h_{fp}(X_+;R) & \longrightarrow & \mathcal{S}h^{cbl}(X_+;R)^{(fp)}
\end{array}$$

where all rows and columns are Poincaré-Verdier sequences with respect to Verdier duality restricted from $\mathcal{S}h(X;R)$.

*Proof.* All rows are Poincaré-Verdier sequences by 6.4.7, so using the 9-Lemma 3.3.8 it suffices to show that the first two columns are Poincaré-Verdier, since the factorization condition can be verified by restricting a sheaf to $X_-$ as in 4.6.5. For the middle sequence, this is the statement of 5.2.18. The left sequence fully faithfully embeds into it and the left sequence of 6.5.2, in fact it is their intersection, and its quadratic functor is restricted from both. Unwinding the definitions, as in 4.6.6 this shows our claim. □

**Corollary 6.5.5.** The pairs $(\mathcal{S}h^{cbl}(X_-;R)^{fp}, \mathcal{S}h^{cbl}(X_+;R)^{fp})$ associated to any slicing $(P_-, P_+)$ of $P$ form a Poincaré $P$-slicing of $\mathcal{S}h^{cbl}(X;R)^{(fp)}$. Similarly, we obtain Poincaré $P$-slicings for $\mathcal{S}h^{\perp cbl}_{fp}(X;R)$ and $\mathcal{S}h_{fp}(X;R)$. In particular, on L-spectra

$$\mathbb{L}^q(\mathcal{S}h^{cbl}(X;R)^{(fp)};\mathbb{D}) = \operatorname{fib}\left(\mathbb{L}^q(\mathcal{S}h^{cbl}(X_+;R)^{(fp)};\mathbb{D}) \longrightarrow \mathbb{L}^q(\mathcal{S}h^{cbl}(X_-;R)^{(fp)};\mathbb{D})[1]\right) \tag{6.20}$$

where the map is following 3.3.10 constructed by pushing a Verdier self-dual sheaf $F$ forward from $X_+$ to $X$, and forming the cofiber $\operatorname{cofib}(F \to \mathbb{D}F)$ which by assumption vanishes on $X_+$, yielding a Verdier self-dual sheaf on $X_-$ with a different shift.

Let us apply this to a few examples.

**Example 6.5.6.** For $M$ a topological manifold equipped with the trivial stratification,

$$\mathcal{S}h^{lc}(M;R)^{(fp)} \simeq \mathcal{S}h^{cbl}(M;R)^{(fp)} \tag{6.21}$$

as there is only a single stratum (note that the finiteness conditions are defined in the same way). In particular, the L-groups with respect to Verdier duality agree as well, which means that if $M$ is connected,

$$\mathbb{L}^q(\mathcal{S}h^{cbl}(M;R)^{(fp)}, \mathbb{D}) \simeq \mathbb{L}^q(\pi_0 R[\pi_1 M]) \tag{6.22}$$

as discussed in 5.4.9. Similarly for manifolds with several connected components.

Generally, the L-groups of constructible sheaves also depend on the L-groups of the fundamental groups of the strata, but they can be glued together in a non-trivial way since the above Poincaré-Verdier sequences need not be orthogonal.

**Example 6.5.7.** Let $M \to [1]$ be a topological manifold with boundary equipped with its canonical stratification, and let the interior $\mathring{M}$ and the boundary $\partial M$ be connected. Then, there is a Poincaré-Verdier sequence

$$\mathcal{S}h^{lc}(\partial M; R)^{(fp)} \to \mathcal{S}h^{cbl}(M; R)^{(fp)} \to \mathcal{S}h^{lc}(\mathring{M}; R)^{(fp)} \tag{6.23}$$

inducing a fiber sequence of L-spectra

$$\mathbb{L}^q(\pi_0 R[\pi_1 \partial M]) \to \mathbb{L}^q(\mathcal{S}h^{cbl}(M; R)^{(fp)}; \mathbb{D}) \to \mathbb{L}^q(\pi_0 R[\pi_1 \mathring{M}])\ . \tag{6.24}$$

Using $\pi_1(M) \simeq \pi_1(\mathring{M})$ we can calculate the L-group of constructible sheaves as the fiber

$$\mathbb{L}^q(\mathcal{S}h^{cbl}(M; R)^{(fp)}; \mathbb{D}) = \mathrm{fib}\,(\mathbb{L}^q(\pi_0 R[\pi_1 M]) \longrightarrow \mathbb{L}^q(\pi_0 R[\pi_1 \partial M])[1]) \tag{6.25}$$

of the map described in 3.3.10: After we have lifted a locally constant sheaf from the interior to all of $M$, which is automatic in above formula because the fundamental groups agree, we take the cofiber of the canonical map into its Verdier dual, which leaves us with a sheaf that is pushed forward from the boundary, where it is locally constant. We do not know how this construction is related to the push-pull $i^* j_*$ that, as discussed in [Ban07, 8.2], also sends self-dual sheaves to self-dual sheaves up to a shift – we doubt they are the same, since informally speaking we are thinking about Verdier-duality (related to the tensor hermitian structure), while the bordism theory of self-dual sheaves is about Poincaré-Lefschetz type dualities (related to the cotensor hermitian structure). Again, a generalization to multiple connected components of interior or boundary is evident.

**Example 6.5.8.** A similar calculation can be carried out for finite simplicial complexes and finite regular CW complexes – alternatively, one can also use the results from Section 4.6 that can be adapted without change the CW case using our knowledge about the exit-path category from 6.2.12. Explicitly, can build a finite simplicial complex $K$ by starting with its 0-simplices, and iteratively gluing higher simplices to it that are maximal in the stratification of the simplicial complex we have built up to that point. At every step, we apply the fiber sequence of L-groups 6.5.5 to the slicing where the added simplex makes up the open part. While we do not know enough about the boundary map we glue along to give a general result, this procedure can in principle be used to find $\mathbb{L}^q(\mathcal{S}h^{cbl}(K; R)^{(fp)})$ starting from $\mathbb{L}^q(R)$.

*Remark.* While we were not able to show it, we suspect that for a topologically stratified space without strata of odd codimension (for example a complex variety), the fiber sequence of spectra in 6.5.5 splits, exhibiting

$$\mathbb{L}^q(\mathcal{S}h^{cbl}(X; R)^{(fp)}) \cong \bigoplus_{p \in P} \bigoplus_{\nu \in V_p} \mathbb{L}^q(\pi_0 R[\pi_1 X_{p,\nu}]) \tag{6.26}$$

where $V_p$ parametrizes the connected components $X_{p,\nu}$ of $X_p$. In other words, the total L-group of constructible sheaves splits into the L-groups of locally constant sheaves on the individual strata, glued together in a trivial way. Compare the discussion in [SW20].

## 6.6 Conclusion

Let us compare our results in the topological case to the large diagram we had obtained for a compact PL space, or a simplicial complex, in 4.5:

$$\begin{array}{ccccc}
\mathbb{L}^q\left(\mathcal{S}h_{fp}^{\perp cbl}(X;R)\right) & \longrightarrow & \mathbb{L}^q\left(\mathcal{S}h_{fp}^{comb}(X;R)\right) & \longrightarrow & \mathbb{L}^q\left(\mathcal{S}h^{cbl}(X;R)^{(fp)}\right) \\
\uparrow & & \| & & \downarrow \\
\mathbb{L}^q\left(\mathcal{S}h_{fp}^{\perp lc}(X;R)\right) & \longrightarrow & \mathbb{L}^q\left(\mathcal{S}h_{fp}^{comb}(X;R)\right) & \xrightarrow{A} & \mathbb{L}^q\left(\mathcal{S}h^{lc}(X;R)^{(fp)}\right) \\
 & & \downarrow\cong & & \downarrow\cong \\
 & & \Sigma^\infty X_+ \wedge \mathbb{L}^q(\pi_0 R) & \xrightarrow{A} & \mathbb{L}^q(\pi_0 R[\pi_1 X])
\end{array}$$

On a topological manifold, we had in 5.5.2 derived the sequence of L-groups:

$$\mathbb{L}^q(\mathcal{S}h_{fp}^{\perp lc}(X;R)) \to \mathbb{L}^q(\mathcal{S}h_{fp}(X;R)) \to \mathbb{L}^q(\mathcal{S}h^{lc}(X;R)^{(fp)}) \tag{6.27}$$

In 6.4.5, we obtained the following fiber sequence of L-groups on a topologically stratified space:

$$\mathbb{L}^q(\mathcal{S}h_{fp}^{\perp cbl}(X;R)) \hookrightarrow \mathbb{L}^q(\mathcal{S}h_{fp}(X;R)) \to \mathbb{L}^q(\mathcal{S}h^{cbl}(X;R)^{(fp)}) \tag{6.28}$$

We can not join the first two sequences together like this, since Verdier duality on a topologically stratified space will generally not preserve the class of locally constant sheaves. In fact, this only works in the latter case because of the somewhat naive way we have incorporated Verdier duality in the combinatorial setting without using the dualizing sheaf.

In the topological case, $\mathbb{L}^q(\mathcal{S}h^{lc}(X;R)^{(fp)})$ was still identified with the L-groups of the group ring $\mathbb{L}^q(\pi_0 R[\pi_1 X])$. However, we suspect that $\mathbb{L}^q(\mathcal{S}h_{fp}(X;R))$ does not satisfy excision, since the $\infty$-category of all sheaves is too big and acted on by arbitrary homeomorphisms of $X$, which is also a fairly big group. In particular, the map between them, while inducing an equivalence on the point, is probably not an assembly map; but for trivial reasons the assembly map for the L-groups of the group ring factors over this map.

Our decomposition results for L-groups bear great similarity to the decomposition of *Browder-Quinn L-groups* in [Wei94, p. 129] for the PL case and *loc. cit.* p. 134 for the topological case (compare [AP17, Section 7.1] for a more extensive discussion). For $X_0$ the minimal stratum in a Whitney-stratified space $X$, there is a fiber sequence

$$\mathbb{L}^{BQ}(\overline{X - X_0}\,\mathrm{rel}\,\partial) \to \mathbb{L}^{BQ}(X) \to \mathbb{L}(\mathbb{Z}[\pi_1 X_0]) \tag{6.29}$$

that allows us to inductively calculated the Browder-Quinn L-spectrum of $X$ from the strata and the relative groups on the right where $X_0$ is removed, assuming that we understand the boundary map which is given by a form of transfer to a collar of $X_0$. This sounds somewhat similar to our fiber sequence

$$\mathbb{L}^q(\mathcal{S}h^{cbl}(X_-;R)^{(fp)};\mathbb{D}) \to \mathbb{L}^q(\mathcal{S}h^{cbl}(X;R)^{(fp)};\mathbb{D}) \to \mathbb{L}^q(\mathcal{S}h^{cbl}(X_+;R)^{(fp)};\mathbb{D}) \tag{6.30}$$

and description of the involved map in 6.5.5. Note however that the open and closed stratum have changed sides, which is very peculiar. Of course, both groups agree on topological manifolds as they specialize to $\mathbb{L}^q(\pi_0 R[\pi_1 X])$ in that case.

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**Erklärung:**

Hiermit erkläre ich, dass ich diese Arbeit selbstständnig verfasst und nur die angegeben Hilfsmittel und Quellen verwendet habe. Weiterhin bestätige ich, dass die übermittelte elektronische Version in Inhalt und Wortlaut der gedruckten Fassung entspricht. Mit einer universitätsinternen Prüfung anhand einer Plagiatssoftware bin ich einverstanden.

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Ort, Datum

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